Variational Calculus - Week 9 - Holonomic Constraints

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Contents

Based on the notes by Jelle Hartong, Section 10

1 Constrained Extremisation of Functionals via Functions

- **How do functions impose constraints on functionals?**
  - we saw that we can **extremise** a **functional** $J[y]$ subject to a **constraint** from another functional $I[y] = 0$
  - we can also **extremise** $J[x] = \int K(x, \dot{x}, t)dt$ subject to a **function** constraint $G(x, \dot{x}, t) = 0$
- **How do rheonomic constraints differ from scleronomic constraints?**
  - a **scleronomic** constraint depends **explicitly** on **time** t
  - a **rehonomic** constriant doesn't depend **explicitly** on t
- **What is a holonomic constraint?**
  - a **holonomic** constraint doesn't depend **explicitly** on the velocities $\underline{\dot{x}}$ (in general it will only relate to positions)
  - a **non-holonomic** constraint is one which deosn't depend explicitly on velocity, **and** which can't be integrated into a **holonomic** constraint
  - for example:

    $$\dot{x}^1 + \dot{x}^2 = 0$$

    isn't **non-holonomic**, since if we integrate with respect to time:

    $$x^1 + x^2 = c$$

    we get a **holonomic** constraint
  - on the other hand:

    $$(\dot{x}^1)^2 = 0$$

    is **non-holonomic**

2 Theorem: Euler-Lagrange Equations for Holonomic Constraints

Let:

$$\underline{x} : [0,1] \to \mathbb{R}^n, \quad \underline{x} \in C^2(\mathbb{R}^n)$$

*be **extrema** for the **functional**:*

$$J[\underline{x}] = \int_0^1 L(\underline{x}, \dot{\underline{x}}, t) dt$$

*subject to the **holonomic constraint**:*

$$g(\underline{x}(t), t) = 0$$

and with:

$$\forall t \in [0,1], \quad \nabla g(\underline{x}(t), t) \neq \underline{0}$$

*Then, $\exists \lambda : [0,1] \to \mathbb{R}$, such that $\underline{x}$ satisfies the **Euler-Lagrange Equations** of:*

$$L - \lambda(t) g$$

(Theorem 10.1)

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*Notice, the main difference, beyond g being just a function, is that λ is no longer a **constant**: it is an explicit function of **time**.*

Proof. We prove this for:

$$\underline{x} : [0,1] \to \mathbb{R}$$

where $\underline{x}$ is constrained by $g(\underline{x}, t) = 0$ and:

$$\begin{pmatrix} \frac{\partial g}{\partial x^1} \\ \frac{\partial g}{\partial x^2} \end{pmatrix} \neq \underline{0}$$

Say that:

$$\underline{x}(0) = \underline{x}_0 \qquad \underline{x}(1) = \underline{x_1}$$

Since g constrains $\underline{x}$, we must have that:

$$g(\underline{x}_0, 0) = 0 \qquad g(\underline{x}_1, 1) = 0$$

Without loss of generality, assume that $\nabla g \neq \underline{0}$ because:

$$\frac{\partial g}{\partial x^2} \neq 0$$

It could be the case that $\frac{\partial g}{\partial x^1} \neq 0$ instead, or that $\frac{\partial g}{\partial x^2} \neq 0$ on $[t_0, 1]$ whilst $\frac{\partial g}{\partial x^1} \neq 0$ on $[0, t_0]$. In this case, we coud split the action integral into 2 based on t_0, and perform the same analysis as we shall perform below. Ultimately, we will reach the same conclusion as by assuming that $\frac{\partial g}{\partial x^2} \neq 0$ always, so it is simpler to work under this assumption.

As we always do, we shall apply a variation:

$$\underline{\hat{x}} = \underline{x} + s\underline{\varepsilon}$$

For $\underline{\varepsilon}$ to be an **admissible variation**, we shall require:

- $\underline{\varepsilon} \in C^1$
- $\underline{\hat{x}}(0) = \underline{x}_0$
- $\underline{\hat{x}}(1) = \underline{x}_1$
- $\forall t \in [0, 1]$ if s is close to 0, then $g(\underline{\hat{x}}, t) = 0$

In particular, for s close to 0, admissible variations must satisfy:

$$\underline{\varepsilon}(0) = \underline{\varepsilon}(1) = 0$$

Moreover, our constraint becomes:

$$g(\underline{\hat{x}}, t) = g(\underline{x} + s\underline{\varepsilon}, t) = 0$$

By the **Implicit Function Theorem**, and since $\nabla g \neq 0$ (in particular since $\frac{\partial g}{\partial x^2} \neq 0$) tells us that $g(\underline{x} + s\underline{\varepsilon}, t) = 0$ allows us to write ε^2 as a function of ε^1 and s, whenever s is close to 0.

We now vary our functional. If $\underline{x}$ is an extrema of $J[\underline{x}]$, then:

$$\frac{d}{ds} J[\underline{\hat{x}}] \bigg|_{s=0} = \frac{d}{ds} J[\underline{x} + s\underline{\varepsilon}] \bigg|_{s=0} = 0$$

Hence:

$$\begin{aligned}
0 &= \frac{d}{ds} J[\underline{x} + s\underline{\varepsilon}] \bigg|_{s=0} \\
&= \int_0^1 \frac{d}{ds} \Big(L(\underline{\hat{x}}, \underline{\dot{\hat{x}}}, t) \Big) \, dt \bigg|_{s=0} \\
&= \int_0^1 \left(\frac{\partial L}{\partial \hat{x}^1} \frac{\partial \hat{x}^1}{\partial s} + \frac{\partial L}{\partial \dot{\hat{x}}^1} \frac{\partial \dot{\hat{x}}^1}{\partial s} + \frac{\partial L}{\partial \hat{x}^2} \frac{\partial \hat{x}^2}{\partial s} + \frac{\partial L}{\partial \dot{\hat{x}}^2} \frac{\partial \dot{\hat{x}}^2}{\partial s} \right) dt \Bigg|_{s=0} \\
&= \int_0^1 \left(\frac{\partial L}{\partial x^1} \varepsilon^1 + \frac{\partial L}{\partial \dot{x}^1} \dot{\varepsilon}^1 + \frac{\partial L}{\partial x^2} \varepsilon^2 + \frac{\partial L}{\partial \dot{x}^2} \dot{\varepsilon}^2 \right) dt
\end{aligned}$$

Applying integration by parts:

$$\int_0^1 \frac{\partial L}{\partial \dot{x}^i} \dot{\varepsilon}^i dt = \left[\frac{\partial L}{\partial \dot{x}^i} \varepsilon^i \right]_0^1 - \int_0^1 \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) \varepsilon^i dt = \int_0^1 \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) \varepsilon^i dt$$

where we have used that $\underline{\varepsilon}$ is an admissible variation, so $\varepsilon^i(0) = \varepsilon^i(1) = 0$.

Hence:

$$\int_0^1 \left[\left(\frac{\partial L}{\partial x^1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right)\right)\varepsilon^1 + \left(\frac{\partial L}{\partial x^2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^2}\right)\right)\varepsilon^2 \right] dt = 0$$

Now, recall that we can regard ε^2 as a function of ε^1, so ε^2 isn't some arbitrary C^1 function, and so, we can't immediately apply the Fundamental Lemma. Notice, the relation of ε^1 and ε^2 arises from the constraint $g(\underline{\hat{x}}, t) = 0$; if we differentiate this with respect to s and evaluate at $s = 0$:

$$0 = \frac{d}{ds} g(\underline{\hat{x}}, t)\bigg|_{s=0} = \frac{\partial g}{\partial x^1}\varepsilon^1 + \frac{\partial g}{\partial x^2}\varepsilon^2$$

Since $\frac{\partial g}{\partial x^2} \neq 0$, we can divide through to obtain:

$$\varepsilon^2 = -\frac{\frac{\partial g}{\partial x^1}}{\frac{\partial g}{\partial x^2}}\varepsilon^1$$

Now, **define** (!) a function $\lambda(t)$, such that it satisfies:

$$\frac{\partial L}{\partial x^2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^2} = \lambda(t)\frac{\partial g}{\partial x^2}$$

Notice, L is **smooth** by definition and $\underline{x} \in C^2$, so $\frac{\partial L}{\partial x^2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^2}$ is a **continuous** function of t; similarly, $\frac{\partial g}{\partial x^2}$ will also be continuous, so $\lambda(t)$ is a **continuous** function.

Plugging this back into our expression:

$$\begin{aligned} 0 &= \int_0^1 \left[\left(\frac{\partial L}{\partial x^1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right)\right)\varepsilon^1 + \left(\frac{\partial L}{\partial x^2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^2}\right)\right)\varepsilon^2 \right] dt \\ &= \int_0^1 \left[\left(\frac{\partial L}{\partial x^1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right)\right)\varepsilon^1 + \left(\lambda(t)\frac{\partial g}{\partial x^2}\right)\left(-\frac{\frac{\partial g}{\partial x^1}}{\frac{\partial g}{\partial x^2}}\varepsilon^1\right) \right] dt \\ &= \int_0^1 \left[\left(\frac{\partial L}{\partial x^1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right) - \lambda(t)\frac{\partial g}{\partial x^1}\right)\varepsilon^1 \right] dt \end{aligned}$$

ε^1 is now an arbitrary C^1 function, so by the Fundamental Lemma, we must have that:

$$\frac{\partial L}{\partial x^1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right) - \lambda(t)\frac{\partial g}{\partial x^1} = 0$$

which are precisely the Euler-Lagrange equations of $L - \lambda g$, as required.

In general, if we had more constraints, we'd just have to add more Lagrange multipliers $\lambda_i(t)$.

□

- **What is the constraint force?**
  - recall, $\frac{\partial L}{\partial x^i}$ is the **force** on the system defined by L
  - similarly, we call the term:

    $$-\lambda\frac{\partial g}{\partial x^i}$$

    the **constraint force**: it is the force exerted on a system, such that the **constraint** is enforced
  - finding λ thus allows us to see the forces enforcing the constraint g

3 Constraint Examples

3.1 Holonomic Constraints: Geodesics Over Surfaces

Say we have a **regular surface** (i.e non-zero gradient) $\Sigma \subset \mathbb{R}^3$ defined implicitly by:

$$g(x^1, x^2, x^3) = 0$$

To find a **geodesic**, we seek to minimise the **arclength**:

$$S[\underline{x}] = \int_0^1 \|\dot{\underline{x}}\| dt$$

subject to the constraint $g(\underline{x}) = 0$ (i.e that $\underline{x}$ lies on Σ). This is a **holonomic constraint**, since it purely depends on the position.

If we define:

$$L = \|\dot{\underline{x}}\| - \lambda g(\underline{x})$$

we can compute the Euler-Lagrange Equations:

$$\frac{\partial L}{\partial x^i} = \frac{\partial g}{\partial x^i}$$

$$\frac{\partial L}{\partial \dot{x}^i} = \frac{\dot{x}^i}{\|\dot{\underline{x}}\|}$$

Thus, the Euler-Lagrange Equations are:

$$\frac{d}{dt}\left(\frac{\dot{x}^i}{\|\dot{\underline{x}}\|}\right) = \lambda \frac{\partial g}{\partial x^i}$$

If we multiply both sides by $\frac{\dot{x}^i}{\|\dot{\underline{x}}\|}$ and add over all i, we get:

$$\begin{aligned}
\sum_{i=1}^n \frac{\dot{x}^i}{\|\dot{\underline{x}}\|} \frac{d}{dt}\left(\frac{\dot{x}^i}{\|\dot{\underline{x}}\|}\right) &= \lambda \sum_{i=1}^n \frac{\partial g}{\partial x^i} \frac{\dot{x}^i}{\|\dot{\underline{x}}\|} \\
\implies \frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^n \frac{\dot{x}^i \dot{x}^i}{\|\dot{\underline{x}}\|}\right) &= \lambda \sum_{i=1}^n \frac{\partial g}{\partial x^i} \frac{\dot{x}^i}{\|\dot{\underline{x}}\|} \\
\implies 0 = \lambda \left\langle \nabla g, \frac{\dot{\underline{x}}}{\|\dot{\underline{x}}\|} \right\rangle
\end{aligned}$$

If $\underline{x}$ lies on the surface, this is clearly true, since then $g = 0 \implies \nabla g = 0$. In other words, the **tangent** vectors $\dot{x}^i$ of the **geodesic** will be **orthogonal** to the direction of the **constraint** g, which itself is the **normal** of Σ. In other words, **geodesics** will be tangent to the surface Σ.

Using this, if we define $\|\underline{x}\| = 1$, we can prove that geodesics of **spheres** must be **great circles**: circles obtained by planes cutting the sphere through the origin.

3.2 Non-Holonomic Constraints: Pfaffian Constraints

A **Pfaffian constraint** is a **non-holonomic constraint**, such that:

$$g(\underline{x}, \underline{\dot{x}}, t) = 0$$

is **at most** linear in the velocities

Using **Pfaffian constraints**, we can find the Euler-Lagrange equations for Lagrangians which depend on derivatives of x beyond just $\dot{x}$. Whilst this can be done by using additional variations, Pfaffian constraints ensure that we can find teh EL equation using the tools which we already have.

Consider $L(x, \dot{x}, \ddot{x}, t)$, where $x \in C^3$. Extremising the action involving L is equivalent to extremising $L(x, \dot{x}, \dot{y}, t)$ subject to the constraint $y = \dot{x}$.

If we define:

$$M = L - \lambda(y - \dot{x})$$

then:

$$\frac{\partial M}{\partial x} = \frac{\partial L}{\partial x}$$

$$\frac{\partial M}{\partial \dot{x}} = \frac{\partial L}{\partial \dot{x}} + \lambda$$

$$\frac{\partial M}{\partial y} = -\lambda$$

$$\frac{\partial M}{\partial \dot{y}} = \frac{\partial L}{\partial \dot{y}}$$

Thus, the EL equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} + \lambda\right) = \frac{\partial L}{\partial x}$$

and:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = -\lambda$$

Combining them, we get that:

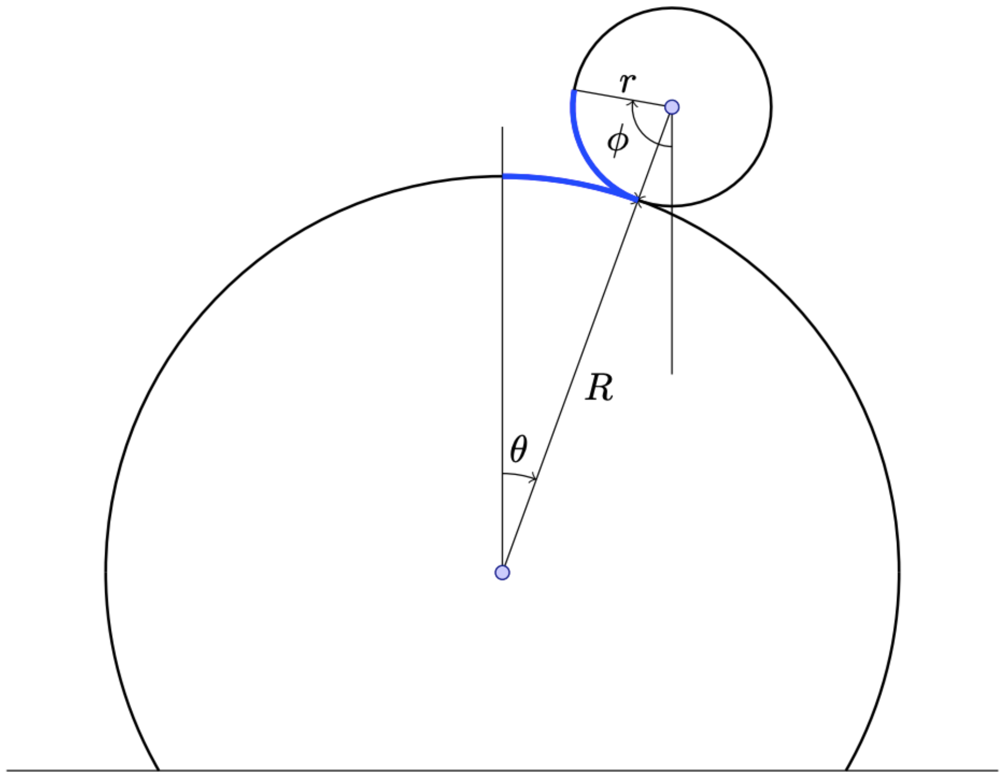
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)\right) = \frac{\partial L}{\partial x} \implies \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) = 0$$

where we have used the constraint $\dot{y} = \ddot{x}$.

3.3 Holonomic Constraints: Rolling Without Slipping

Consider the following setup: we have a cylinder of radius R, on top of which there is a **hoop** (hollow ring) of radius r. The hoop starts at the top of the cylinder, and begins to **roll without slipping** (i.e each time the hoop completes one full rotation, it has traversed one circumference-worth of distance atop the cylinder). The question is: at what angle θ does the hoop fall off the cylinder?

It helps to draw a diagram:



We let θ denote the angle travelled by the hoop with respect to the surface. Similarly, ϕ corresponds to the angle between a vertical line through the hoop, and the original position of the hoop. We seek to find θ at the point where the hoop stops rolling, and falls from the cylinder.

The first constraint is that if ρ is the distance between the centres of the cylinders, and the hoop is rolling atop the cylinder, then:

$$\rho = R + r$$

This is a **holonomic** constraint.

We have a second constraint: the hoop must roll without slipping. What this means is that the distance travelled by the hoop on the cylinder must be equal to the amount of circumference of the hoop which has been in contact with the cylinder. In other words:

$$r(\phi - \theta) = R\theta$$

This can be obtained from the diagram: there is a z angle, which tells us that the angle traversed by the hoop is $\phi - \theta$. This just tells us that the blue lines in the diagram must be of equal length.

We now need to find a **Lagrangian** defining this problem. We can take this to be $T - V$, where T is the kinetic energy, and V is the potential energy.

The hoop falls under the influence of gravity. The centre of mass of the hoop is at its centre, and gravity acts vertically through this centre of mass. Since $V(x)$ is defined up to a constant, we can define the origin (where $V = 0$) as the horizontal line going through the centre of the cylinder. Then:

$$V = mgh = mg(\rho \cos \theta)$$

The kinetic energy is a bit more complicated: there is kinetic energy associated with the translation of the hoop (as a result of the movement of its centre of mass), but also due to rotation:

$$T = \underbrace{\frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2)}_{\text{centre of mass movement}} + \underbrace{\frac{1}{2}mr^2\dot{\phi}^2}_{\text{rotation movement}}$$

Thus, our Lagrangian, subject to the constraints, becomes:

$$\begin{aligned} L = & \underbrace{\frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2)}_{\text{centre of mass movement}} \\ & + \underbrace{\frac{1}{2}mr^2\dot{\phi}^2}_{\text{rotation movement}} \\ & - \underbrace{mg(\rho\cos\theta)}_{\text{potential energy}} \\ & - \underbrace{\lambda(\rho - r - R)}_{\text{hoop in contact with cylinder}} \\ & - \underbrace{\mu(r\phi - (R+r)\theta)}_{\text{rolling without slipping}} \end{aligned}$$

The variables defining the EL equations will be θ, ϕ, ρ, whilst R, r, λ, μ are constants (notice, we can't jsut take ρ to be constant: we first have to derive equations using EL, and then we can apply the constraints).

Thus:

$$\begin{aligned} \frac{\partial L}{\partial \dot{\rho}} &= m\dot{\rho} & \frac{\partial L}{\partial \rho} &= m\rho\dot{\theta}^2 - mg\cos\theta - \lambda \\ \frac{\partial L}{\partial \dot{\theta}} &= m\rho^2\dot{\theta} & \frac{\partial L}{\partial \theta} &= mg\rho\sin\theta + \mu(R+r) \\ \frac{\partial L}{\partial \dot{\phi}} &= mr^2\dot{\phi} & \frac{\partial L}{\partial \phi} &= -\mu r \end{aligned}$$

alongside the constraints:

$$\rho = R + r \qquad r(\phi - \theta) = R\theta$$

For ρ:

$$m\ddot{\rho} - m\rho\dot{\theta}^2 + mg\cos\theta + \lambda = 0$$

For θ:

$$m\frac{d}{dt}(\rho^2\dot{\theta}) - mg\rho\sin\theta - \mu(R+r) = 0$$

For ϕ:

$$mr^2\ddot{\phi} + \mu r = 0$$

Since $\rho = r + R$ must be constant $\ddot{\rho} = 0$, the equation for ρ becomes:

$$-m(R+r)\dot{\theta}^2 + mg\cos\theta + \lambda = 0$$

and the equation for θ becomes:

$$m(R+r)^2\ddot{\theta} - mg(R+r)\sin\theta - \mu(R+r) = 0 \implies m(R+r)\ddot{\theta} - mg\sin\theta - \mu = 0$$

We can use the EL for ϕ to determine μ:

$$mr^2\ddot{\phi} + \mu r = 0 \implies \mu = -mr\ddot{\phi}$$

and using the no-slip constraint:

$$r(\phi - \theta) = R\theta \implies r(\ddot{\phi - \theta}) = R\ddot{\theta} \implies r\ddot{\phi} = \ddot{\theta}(R+r)$$

so:

$$\mu = -m(R+r)\ddot{\theta}$$

and thus, the equation for θ becomes:

$$m(R+r)\ddot{\theta} - mg\sin\theta - \mu = 0 \implies 2m(R+r)\ddot{\theta} - mg\sin\theta = 0$$

If we multiply through by $\dot{\theta}$:

$$2m(R+r)\ddot{\theta}\dot{\theta} - \dot{\theta}mg\sin\theta = 0$$

where notice that:

$$\frac{d}{dt}(\dot{\theta}^2) = 2\dot{\theta}\ddot{\theta}$$

$$\frac{d}{dt}(mg\cos\theta) = -\dot{\theta}mg\sin\theta$$

so integrating we get that:

$$2m(R+r)\dot{\theta}^2 = -mg\cos\theta + C$$

Now, if $\theta = 0$ (and assuming that $\dot{\theta} = 0$ - the hoop is put into movement by applying a negligible force) this forces $C = mg$. Thus, we have that:

$$m(R+r)\dot{\theta}^2 = mg(1-\cos\theta)$$

Finally, we go back to the ρ equation to get:

$$-m(R+r)\dot{\theta}^2 + mg\cos\theta + \lambda = 0 \implies -mg(1-\cos\theta) + mg\cos\theta + \lambda = 0$$

so that:

$$\lambda = mg(1 - 2\cos\theta)$$

Recall, λ is used to define the constraint force $-\lambda\frac{\partial g}{\partial x^i}$. When $\lambda = 0$, the constraint force no longer acts; that is, without a constraint force, the hoop will no longer remain in contact with the cylinder. Hence, when $\theta = \frac{\pi}{3}$, $\lambda = 0$, and so, the hoop falls from the cylinder.

Notice, the force will act normally to the hoop: it is an outward force which acts to oppose the gravitational force. In fact, if we compute the total force acting on the hoop radially (along ρ):

$$\frac{\partial L}{\partial \rho} = -mg\cos\theta + m(r+R)\dot{\theta}^2 - \lambda = 0$$

by using the constraints.

However, along θ the forces won't cancel:

$$\frac{1}{\rho}\frac{\partial L}{\partial \theta} = mg\sin\theta + \mu = m(r+R)\ddot{\theta}$$

where $\mu = -\frac{1}{2}mg\sin\theta$ is the **friction force**, derived by the constraint that we must have rolling without slipping.

The above framework is very generalisable, and can apply to a multitude of shapes rolling on different surfaces. All that changes will be the kinetic energy terms, which depend on how the objects rotate. for instance, if instead of a hooop we had a solid cylinder, the rotational kinetic energy will be:

$$\frac{1}{4}mr^2\dot{\phi}^2$$