Variational Calculus - Week 7 - Canonical Transformations

Antonio León Villares

November 2022

Contents

Based on the notes by Jelle Hartong, Section 8.5-8.7

1 Bargmann Algebra from the Free Particle

- **What is the Lagrangian defining the motion of a free particle?**
  - recall, a **free particle** is a particle moving via:
  $$m\ddot{\underline{x}} = 0$$
  - the associated **Lagrangian** is:
  $$L(\underline{x}, \dot{\underline{x}}, t) = \frac{1}{2}m\|\dot{\underline{x}}\|^2$$
  (the Euler-Lagrange equations give us that:
  $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0 \implies m\ddot{\underline{x}} = 0$$
)
- **What is the Hamiltonian of the free particle?**
  - defining:
  $$p_i = \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i \implies \dot{x}^i = \frac{p_i}{m}$$
  - so the **Hamiltonian** will be (using Einstein summation notation):
  $$H = p_i\dot{x}^i - L = \frac{p_i^2}{2m} - \frac{1}{2}m\left\|\frac{\underline{p}}{m}\right\|^2 = \frac{\|\underline{p}\|^2}{2m}$$
- **Is energy conserved for a free particle in phase space?**
  - since we have a free particle, in particular it is not affected by external forces, so no **potential** acts on it, so $V(\underline{x}) = 0$
  - hence, the **total energy** of the particle will be:
  $$E = \frac{1}{2}m\|\dot{\underline{x}}\|^2 = \frac{\|\underline{p}\|^2}{2m} = H$$
  - hence, the **Hamiltonian** represents the total energy of the free particle
  - it is clear that:
  $$[H, H] = 0$$
  and since H is time independent, this implies that:
  $$[H, H] = 0 \iff \frac{dH}{dt} = \frac{dE}{dt} = 0$$
  so **energy** is **conserved** in phase space (of course, we already knew this)
- **Are angular momentum and linear momentum conserved for a free particle in phase space?**
  - the **linear momentum** is just:
  $$P_i = p_i$$
  and:
  $$[H, P_i] = \frac{\partial H}{\partial x^i} = 0$$

  - since P_i Poisson commutes with H, the **momenta** are also **conserved** in **phase space**
  - the **angular momentum** is given (in 2 dimensions) by:

$$J = x^1 p_2 - x^2 p_1$$

    and:

$$[H, J] = -\frac{\partial H}{\partial p_1}\frac{\partial J}{\partial x^1} - \frac{\partial H}{\partial p_2}\frac{\partial J}{\partial x^2} = \frac{p_1}{m}p_2 - \frac{p_2}{m}(-p_1) = 0$$

  - since J Poisson commutes with H, the **angular momentum** is also **conserved** in **phase space**

- **What is the Noether charge associated with a Galilean boost?**

  - consider the **Galilean boost**:

$$\varphi_s(\underline{x}) = \underline{x} + \underline{v}ts \qquad \varphi_s(t) = t$$

    where $\underline{v}$ is some constant vector
  - this is a **symmetry** of the Lagrangian, since if we define $\underline{y}(s,t) = \varphi_s(\underline{x})$ then:

$$\begin{aligned}
L(\underline{y}, \underline{\dot{y}}, t) &= \frac{1}{2}m\|\underline{\dot{y}}\|^2 \\
&= \frac{1}{2}m\|\underline{\dot{x}} + \underline{v}s\|^2 \\
&= \frac{1}{2}m\left\langle \underline{\dot{x}} + \underline{v}s, \underline{\dot{x}} + \underline{v}s\right\rangle \\
&= \frac{1}{2}m\left(\langle \underline{\dot{x}}, \underline{\dot{x}}\rangle + 2s\langle \underline{\dot{x}}, \underline{v}\rangle + s^2\langle \underline{v}, \underline{v}\rangle\right) \\
&= \frac{m}{2}\|\underline{\dot{x}}\|^2 + ms\langle \underline{v}, \underline{\dot{x}}\rangle + \frac{m}{2}s^2\|\underline{v}\|^2
\end{aligned}$$

    In particular, since Lagangians are "the same" up to a total time derivative, we have a symmetry if:

$$\frac{d}{ds}L(\underline{y}, \underline{\dot{y}}, t) \times \frac{d\varphi_s(t)}{dt} = \frac{d}{dt}K_s(\underline{x}, t) = \frac{d}{dt}\frac{dF_s}{ds}$$

    (this is what we used to derive the generalised Noether Theorem)
    Since $\frac{d\varphi_s(t)}{dt} = 1$, we compute:

$$\frac{d}{ds}L(\underline{y}, \underline{\dot{y}}, t) = m\langle \underline{v}, \underline{\dot{x}}\rangle + ms\|\underline{v}\|^2$$

    Now, notice that:

$$m\frac{d}{dt}\langle \underline{v}, \underline{x}\rangle = m\langle \underline{v}, \underline{\dot{x}}\rangle$$

$$m\frac{d}{dt}\langle \underline{v}, \underline{v}ts\rangle = ms\langle \underline{v}, \underline{v}\rangle$$

    Hence:

$$\frac{d}{ds}L(\underline{y}, \underline{\dot{y}}, t) = m\langle \underline{v}, \underline{\dot{x}}\rangle + ms\|\underline{v}\|^2 = \frac{d}{dt}\left(m\langle \underline{v}, \underline{x} + \underline{v}st\rangle\right)$$

    and so:

$$K_s(\underline{x}) = m\langle \underline{v}, \underline{x} + \underline{v}st\rangle$$

    Thus, L is **invariant under Galilean boosts** (up to a total time derivative)

    Recall, the general Noether Charge is given by:

$$N(\underline{x}, \underline{\dot{x}}, t) = \left(L - \sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right)\tau + \sum_k \frac{\partial L}{\partial \dot{x}^k}\zeta^k - K_0$$

φ_s is the identity transformation when changing time, so:

$$\tau = \left.\frac{\partial \varphi_s(t)}{\partial s}\right|_{s=0} = 0$$

and:

$$\zeta^k = \left.\frac{\partial y^j}{\partial s}\right|_{s=0} = v^k t$$

so:

$$N = \sum_k m\dot{x}^k v^k t - m\langle \underline{v}, \underline{x}\rangle = m\langle \underline{v}, \underline{\dot{x}}t - \underline{x}\rangle$$

- we can indeed check that this is conserved, since trajectories of the particle obey $m\underline{\ddot{x}} = 0$ and:

$$\frac{dN}{dt} = m\langle \underline{v}, \underline{\ddot{x}}t + \underline{\dot{x}} - \underline{\dot{x}}\rangle = 0$$

- **How can we express the Noether charge in terms of phase space variables?**
  - notice $m\langle \underline{v}, \underline{\dot{x}}t - \underline{x}\rangle$ will be conserved for any element v^i in $\underline{v}$
  - as such, what must be conserved will be:

$$G_i = m\dot{x}^i t - mx^i$$

  - using **phase space coordinates**:

$$G_i = tp_i - mx^i$$

  - we can then express the **Noether charge** as:

$$\Phi = \sum_k v^i G_i$$

  - moreover, since G_i depends explicitly on time and is conserved:

$$\frac{\partial G_i}{\partial t} + [G_i, H] = 0$$

*Notice, if we find the **integral curves** for the Hamiltonian Vector Field in $\mathbb{R}^2$ for the Noether charge Φ:*

$$\frac{dx^i}{ds} = [x^i, \Phi] = \frac{\partial \Phi}{\partial p_i} = tv^i$$

$$\frac{dp_i}{ds} = [p_i, \Phi] = -\frac{\partial \Phi}{\partial x^i} = mv^i$$

If we integrate, we find:

$$x^i(s) = x^i(0) + stv^i \qquad x^i(s) = x^i(0) + stv^i$$

which is precisely the Galilean boost which generated Φ in the first place.

- **What symmetries correspond to the conserved charges described above?**

– we have found 4 conserved charges corresponding to the free particle:

1. H: the **energy**, corresponding to **invariance under time shifts**

*$\Phi = H = \frac{\|\underline{p}\|^2}{2m}$ is **time-independent**. Let's assume that p_i, x^i are functions dependent on s, t. Then, finding integral curves to the Hamiltonian vector field gives us, by using Hamilton's Equations:*

$$\frac{\partial x^i}{\partial s} = [x^i, \Phi] = \frac{\partial \Phi}{\partial p_i} = \frac{\partial x^i}{\partial t}$$

$$\frac{dp_i}{ds} = [p_i, \Phi] = -\frac{\partial \Phi}{\partial x^i} = \frac{\partial p_i}{\partial t}$$

Now, consider a change of variables:

$$u = s - t \qquad v = s + t$$

so that:

$$x^i(s,t) = x^i(u(s,t), v(s,t))$$

Then:

$$\frac{\partial x^i}{\partial s} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial s} = \frac{\partial x}{\partial u} + \frac{\partial x}{\partial v}$$

$$\frac{\partial x^i}{\partial t} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial t} = -\frac{\partial x}{\partial u} + \frac{\partial x}{\partial v}$$

But then:

$$\frac{\partial x^i}{\partial s} = \frac{\partial x^i}{\partial t} \implies \frac{\partial x}{\partial u} = 0$$

In other words, x^i only depends (explicitly) on $v = s + t$. The same can be applied to p_i, and so:

$$x^i(s,t) = x^i(t+s) \qquad p_i(s,t) = p_i(t+s)$$

so x is symmetric with respect to time shifts.

2. P_i: the **momentum** (conjugate to x^i), corresponding to **invariance under spatial shifts**

$\Phi = \sum a^i P_i$ gives us:

$$\frac{dx^i}{ds} = [x^i, \Phi] = \frac{\partial \Phi}{\partial p_i} = a^i$$

$$\frac{dp_i}{ds} = [P_i, \Phi] = -\frac{\partial \Phi}{\partial x^i} = 0$$

So if we integrate:

$$x^i(s) = x^i(0) + sa^i \qquad p_i(s) = p_i(0)$$

which corresponds to shifts in position.

3. J: the **angular momentum** ($J = x^1 p_2 - x^2 p_1$), corresponding to **invariance under rotation** (we already showed this last week)
4. G_i: the **Galilean boost charge**, corresponding to **invariance under Galilean Boosts**

- **What is the Bargmann algebra?**
  - the **algebra** derived by taking the **Poisson brackets** of the **conserved charges** H, G_i, P_i, J corresponding to the **free particle**
  - we have the **trivial** brackets corresponding to **conservation**:
    * $$[H, H] = 0$$
    * $$[H, J] = 0$$
    * $$[H, P_i] = 0$$
    * $$[H, G_i] = \frac{\partial G_i}{\partial t} = \frac{d}{dt}(tp_i - mx^i) = p_i$$
  - and the "cross brackets":
    * $$[P_i, G_j] = -\frac{\partial G_j}{\partial x^i} = m\delta_{ij}$$
    * $$[P_i, J] = -\sum_{j=1}^{2} \frac{\partial J}{\partial x^i} = -\sum_{j=1}^{2} \varepsilon_{ij} P_j$$

      where $\varepsilon_{ij} = -\varepsilon_{ji}$ is the 2 dimensional Levi-Civita symbol, and $\varepsilon_{12} = 1$
    * $$[G_i, J] = \sum_{j=1}^{2} \frac{\partial G_i}{\partial x^j}\frac{\partial J}{\partial p_j} - \frac{\partial G_i}{\partial p_j}\frac{\partial J}{\partial x^j} = \left(m\delta_{i1}x^2 - t\delta_{i1}p_2\right) + \left(-m\delta_{i2}x^1 + t\delta_{i2}p_1\right) = \delta_{i2}G_1 - \delta_{i1}G_2$$

2 Canonical Transformations

2.1 Definition: Canonical Conjugate Variables

Say $\underline{x}, \underline{p} \in \mathbb{R}^n$
*$\underline{x}, \underline{p}$ are **canonical conjugate variables** (or **Darboux coordinates**) if:*

1. *The x^i **Poisson commute***

$$[x^i, x^j] = 0$$

2. *The p_i **Poisson commute**:*

$$[p_i, p_j] = 0$$

3. *The x^i, p_j **Poisson commute** when $i \neq j$:*

$$[x^i, p_j] = \delta_{ij}$$

2.2 Definition: Canonical Transformation

*Consider a **change of variables**:*

$$(x^i, p_i) \mapsto (X^i(\underline{x}, \underline{p}), P_j(\underline{x}, \underline{p}))$$

*This is a **canonical transformation** or **symplectomorphism** if $(\underline{X}, \underline{P})$ are **canonical conjugate variables**.*
That is:

$$[X^i, X^j] = 0 \qquad [P_i, P_j] = 0 \qquad [X^i, P_j] = \delta_{ij}$$

Moreover, we can also write $(\underline{x}, \underline{p})$ in terms of $(\underline{X}, \underline{P})$.

- **What canonical transformation have we already seen?**
  - recall, when we showed that Hamiltonians differing by a partial time derivative lead to the same set of Hamilton's equations, we used a change of variables:

$$X^i = x^i \qquad P_i = p_i + \frac{\partial F(x^i, t)}{\partial x^i}$$

    where $\frac{dF}{dt}$ was a total time derivative between 2 Lagrangians
  - we also saw that this transformation was invertible:

$$x^i = X^i \qquad p_i = P_i - \frac{\partial F(X^i, t)}{\partial X^i}$$

  - this is indeed a **canonical transformation**:

$$[X^i, X^j] = [x^i, x^j] =$$

$$
\begin{aligned}
[P^i, P^j] &= \left[p_i + \frac{\partial F(x^i,t)}{\partial x^i}, p_j + \frac{\partial F(x^j,t)}{\partial x^j}\right] \\
&= [p_i, p_j] + \left[p_i, \frac{\partial F(x^j,t)}{\partial x^j}\right] + \left[\frac{\partial F(x^i,t)}{\partial x^i}, p_j\right] + \left[\frac{\partial F(x^i,t)}{\partial x^i}, \frac{\partial F(x^j,t)}{\partial x^j}\right] \\
&= -\frac{\partial}{\partial x^i}\frac{\partial F(x^i,t)}{\partial x^i} + \frac{\partial}{\partial x^i}\frac{\partial F(x^i,t)}{\partial x^i} \\
&= 0
\end{aligned}
$$

where we have used the fact that:

$$
\frac{\partial}{\partial p_k}\left(\frac{\partial F(x^i,t)}{\partial x^i}\right) = 0 \qquad \frac{\partial p_i}{\partial x^k} = \frac{\partial p_j}{\partial x^k} = 0
$$

2.3 Lemma: Canonical Transformations Preserve Hamilton's Equations

*Let $(\underline{X}, \underline{P})$ be **canonically transformed** variables of $(\underline{x}, \underline{p})$, and let $K(\underline{X}, \underline{P})$ be a **Hamiltonian** for $(\underline{X}, \underline{P})$:*

$$
H(\underline{x}, \underline{p}) = K(\underline{X}(\underline{x}, \underline{p}), \underline{P}(\underline{x}, \underline{p}))
$$

*Then, $(\underline{X}, \underline{P})$ **preserve Hamilton's equations**:*

$$
\dot{X}^i = [X^i, H] = \frac{\partial K}{\partial P_i} = [X^i, K] \qquad \dot{P}_i = [P_i, H] = -\frac{\partial K}{\partial X^i} = [P_i, K]
$$

(recall, if Φ is defined on phase space, then $\frac{d\Phi}{dt} = [\Phi, H]$ if Φ doesn't depend explicitly on time).
*Here, the **Poisson brackets** with H are computed with respect to $(\underline{x}, \underline{p})$, whilst the **Poisson brackets** with K are computed with respect to $(\underline{X}, \underline{P})$*

Proof. We compute directly in the case $n = 1$, using the fact that:

$$
H(x,p) = K(X(x,p), P(x,p))
$$

Indeed:

$$
\begin{aligned}
\dot{X} &= [X, H] \\
&= \frac{\partial X}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial X}{\partial p}\frac{\partial H}{\partial x} \\
&= \frac{\partial X}{\partial x}\left(\frac{\partial K}{\partial X}\frac{\partial X}{\partial p} + \frac{\partial K}{\partial P}\frac{\partial P}{\partial p}\right) - \frac{\partial X}{\partial p}\left(\frac{\partial K}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial K}{\partial P}\frac{\partial P}{\partial x}\right) \\
&= \frac{\partial X}{\partial x}\frac{\partial K}{\partial P}\frac{\partial P}{\partial p} - \frac{\partial X}{\partial p}\frac{\partial K}{\partial P}\frac{\partial P}{\partial x} \\
&= \frac{\partial K}{\partial P}[X, P] \\
&= \frac{\partial K}{\partial P}
\end{aligned}
$$

since X, P are canonical conjugates, so $[X, P] = 1$.

Similarly:

$$\begin{aligned}\dot{P} &= [P, H] \\ &= \frac{\partial P}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial H}{\partial x} \\ &= \frac{\partial P}{\partial x}\left(\frac{\partial K}{\partial X}\frac{\partial X}{\partial p} + \frac{\partial K}{\partial P}\frac{\partial P}{\partial p}\right) - \frac{\partial P}{\partial p}\left(\frac{\partial K}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial K}{\partial P}\frac{\partial P}{\partial x}\right) \\ &= \frac{\partial P}{\partial x}\frac{\partial K}{\partial X}\frac{\partial X}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial K}{\partial X}\frac{\partial X}{\partial x} \\ &= \frac{\partial K}{\partial X}[P, X] \\ &= -\frac{\partial K}{\partial X}[X, P] \\ &= -\frac{\partial K}{\partial X}\end{aligned}$$

as required.

□

3 Integrable Systems

- **What can we consider an "easy-to-solve" Hamiltonian system?**
  - we have seen that applying a **canonical transformation** won't affect the derived **Hamilton's Equations**
  - it is natural to thus find a set of **canonical variables**, such that $K(\underline{X}, \underline{P})$ leads to an "easy-to-solve" system
  - the **easiest** class of **Hamiltonian systems** arise from:

    $$H = f(\|\underline{P}\|)$$

    where:

    $$f(\|\underline{P}\|) = \frac{\|\underline{P}\|^2}{2m} \quad or \quad f(\|\underline{P}\|) = v\|\underline{P}\|, \qquad m, v \in \mathbb{R}$$

  - with these Hamiltonians, we get that Hamilton's Equations reduce to:

    $$\dot{X}^i = \frac{\partial H}{\partial P^i} = \frac{P^i}{m} \qquad \dot{P}^i = -\frac{\partial H}{\partial X^i} = 0$$

    But if $\dot{P}^i = 0$, P^i is a constant, and so, $\dot{X}^i$ is a constant, implying that:

    $$\underline{X}(t) = \underline{X}(0) + t\dot{X}(0)$$

3.1 Definition: Functions in Involution

> *A set of* ***functions*** *are said to be* ***in involution*** *if they* ***Poisson-commute*** *amongst themselves.*

- **Why was the Hamiltonian $H = f(\|\underline{P}\|)$ easy to solve?**
  - notice the set of momenta P_i Poisson commutes with themselves:
  $$[P_i, P_j] = 0$$
  - since H is a function of $\underline{P}$ alone, understandably:
  $$[H, P_i] = -\frac{\partial H}{\partial X^i} = 0$$
  - in particular, if the **Hamiltonian** has a lot of conserved quantities (such as the P_i), then they will be **in involution**, which in turn heavily reduces the system (since most of our computations will reduce to 0s)

3.2 Theorem: Liouville Theorem & Integrable Systems

Liouville's Theorem *(formally) states that:*
"The density of states in an ensemble of many identical states with different initial conditions is constant along every trajectory in phase space."

What this means is that, if a ***Hamiltonian****:*

$$H(\underline{x}, \underline{p}) \in \mathbb{R}^{2n}$$

admits n ***independent, conserved*** *quantities* ***in involution****, then there is a* ***canonical transformation*** *to so-called* ***action/angle variables****:*

$$(\underline{X}, \underline{P})$$

such that ***Hamilton's Equations*** *say that:*

- $\underline{P}$ *is* ***constant***
- *thus:*
$$\underline{X}(t) = \underline{X}(0) + t\underline{\dot{X}}(0)$$

Such a system is said to be ***integrable****.*

3.2.1 Example: Integrable System via Harmonic Oscillator

We saw that the **harmonic oscillator** (in 1 dimension) has Lagrangian:

$$L(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

where $\frac{1}{2}kx^2$ is the **elastic potential**.

The momentum is:

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

so the Hamiltonian will be:

$$H(p,x) = \frac{p^2}{m} - L = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

(which is indeed the total energy of the system)

This is not "easy-to-solve". However, if we define $\omega^2 = \frac{k}{m}$ we can rewrite the Hamiltonian as:

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right)$$

With this new formulation, if we can find $f(P)$ such that:

$$p(X,P) = f(P)\cos(X) \qquad x(X,P) = \frac{f(P)}{m\omega}\sin(X)$$

is a **canonical transformation**, then the Hamiltonian, in terms of X, P becomes:

$$K(X,P) = \frac{1}{2m}\left(f^2(P)\cos^2(X) + f^2(P)\sin^2(X)\right) = \frac{f^2(P)}{2m}$$

which is "easy-to-solve"; and since we used a canonical transformation, this will have the same set of Hamilton's equations as $H(x,p)$.

To find $f(P)$, we can enforce the fact that we have a canonical transformation, and so require:

$$[x,x] = [p,p] = 0 \qquad [x,p] = 1$$

(where the Poisson brackets are with respect to X, P)

The first 2 are immediate:

$$[x,x] = \frac{\partial x}{\partial X}\frac{\partial x}{\partial P} - \frac{\partial x}{\partial P}\frac{\partial x}{\partial X} = 0$$

$$[p,p] = \frac{\partial p}{\partial X}\frac{\partial p}{\partial P} - \frac{\partial p}{\partial P}\frac{\partial p}{\partial X} = 0$$

The last Poisson bracket gives:

$$\begin{aligned}[x,p] &= \frac{\partial x}{\partial X}\frac{\partial p}{\partial P} - \frac{\partial x}{\partial P}\frac{\partial p}{\partial X} \\ &= \frac{f(P)}{m\omega}\cos(X)f'(P)\cos(X) + \frac{f'(P)}{m\omega}\sin(X)f(P)\sin(X) \\ &= \frac{f(P)f'(P)}{m\omega}\end{aligned}$$

Hence, we require:

$$f(P)f'(P) = m\omega$$

Notice:

$$\frac{d}{dP}f^2(P) = 2f(P)f'(P)$$

so we convert the ODE into:

$$\frac{d}{dP}f^2(P) = 2m\omega$$

which we integrate with respect to P to obtain:

$$f^2(P) = 2m\omega P + C$$

Since we just need a particular solution, we pick:

$$f(P) = \sqrt{2m\omega P}$$

So our canonical transformation will be given by:

$$p(X, P) = \sqrt{2m\omega P}\cos(X) \qquad x(X, P) = \frac{\sqrt{2m\omega P}}{m\omega}\sin(X)$$

which leads to the Hamiltonian:

$$K(X, P) = \omega P$$

By Liouville's Theorem, P is **conserved**, and:

$$\dot{X} = [X, K] = \frac{\partial K}{\partial P} = \omega$$

such that if $\alpha = X(0)$ we get:

$$X(t) = \alpha + t\omega$$

and thus coming back to x, p we get:

$$p(X, P) = \sqrt{2mK}\cos(\alpha + t\omega)$$

$$x(X, P) = \sqrt{\frac{2K}{m\omega^2}}\sin(\alpha + t\omega)$$