Variational Calculus - Week 6 - The General Noether Theorem & the Hamiltonian Formalism

Antonio León Villares

October 2022

Contents

Based on the notes by Jelle Hartong, Section 7.3 & Sections 8.1-8.4

1 Generalising Noether's Theorem

1.1 From Noether Theorem I to General Noether's Theorem

- **What additional generalisations can be made to Noether's Theorem?**
  - Noether's Theorem deals with **continuous symmetries** of the **Lagrangian**, given changes in $\underline{x}$
  - this allows 2 further generalisations:
    1. **Time Change**: we can enforce that a **diffeomorphism** warps both **space** $\underline{x}$ and **time** t
    2. **Lagrangian Uniqueness**: recall, Lagrangians are **unique** up to a total time derivative; that is, given 2 Lagrangians

       $$L(x, \dot{x}, t) \qquad L'(x, \dot{x}, t)$$

       related by

       $$L'(x, \dot{x}, t) = L(x, \dot{x}, t) + \frac{d}{dt} F(x, t)$$

       They generate the **same** Euler-Lagrange Equations:

       $$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L'}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{x}^i}\right) = 0$$

       This means that for a **diffeomorphism** to be a **symmetry**, we just require that iy leaves the **Lagrangian** invariant **up to a total time derivative**

- **How are symmetries defined in terms of the action?**
  - we previously defined a **symmetry** as a **diffeomorphism** family which left the Lagrangian **invariant**
  - this is **too restrictive**
  - a more **general** statement is that a **symmetry** leads to **invariance** in the **action**
  - if this is the case, then making the **Lagrangian** invariant is just a special case

1.2 Theorem: General Noether's Theorem

Let:

$$I[\underline{x}] = \int_0^1 L(\underline{x}, \dot{\underline{x}}, t)dt$$

*be an **action** for **regular curves**:*

$$\underline{x} : [0, 1] \to \mathbb{R}^n$$

*and let L be **invariant** under a **one-parameter** family of **diffeomorphisms**:*

$$\mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n \times \mathbb{R}$$

$$(\underline{x}, t) \mapsto (\bar{\underline{x}}(\underline{x}, t, s), \bar{t}(\underline{x}, t, s))$$

Moreover, define:

$$\zeta^j = \left.\frac{\partial \bar{x}^j}{\partial s}\right|_{s=0} \qquad \tau = \left.\frac{\partial \bar{t}}{\partial s}\right|_{s=0} \qquad K_s = \frac{d}{ds}F_s$$

for some function F_s.
*Then, the **Noether charge**:*

$$N(\underline{x}, \dot{\underline{x}}, t) = \left(L - \sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right)\tau + \sum_k \frac{\partial L}{\partial \dot{x}^k}\zeta^k - K_0$$

*is **conserved** along extremals of I; that is, along curves obeying the Euler-Lagrange equations:*

$$\frac{dN}{dt} = 0$$

Proof. Consider the diffeomorphism family:

$$(\underline{x}, t) \mapsto (\bar{\underline{x}}(\underline{x}, t, s), \bar{t}(\underline{x}, t, s))$$

This family will be a symmetry if it leaves the action invariant. That is, if $t \in [0, 1], \bar{t} \in [a, b]$, we want:

$$I = \int_a^b L(\bar{\underline{x}}, \dot{\bar{\underline{x}}}, \bar{t})d\bar{t} = \int_0^1 L(\underline{x}, \dot{\underline{x}}, t)dt$$

where we abuse notation and define:

$$\dot{\bar{\underline{x}}} \equiv \frac{d\bar{\underline{x}}}{d\bar{t}}$$

Notice, we can think of $\bar{\underline{x}}, \dot{\bar{\underline{x}}}, \bar{t}$as functions of $\underline{x}, \dot{\underline{x}}, t, s$. Thus, we can apply a change of varaibles, and write the barred action as:

$$I = \int_0^1 L(\bar{\underline{x}}, \dot{\bar{\underline{x}}}, \bar{t})\frac{d\bar{t}}{dt}dt$$

(notice, this makes sense: since we are changing t, we expect there to be a normalisation factor accounting for this warp within the action integral)

But then we have that:

$$\int_0^1 L(\underline{\bar{x}}, \underline{\dot{\bar{x}}}, \bar{t})\frac{d\bar{t}}{dt}dt = \int_0^1 L(\underline{x}, \underline{\dot{x}}, t)dt$$

so by the FTC:

$$L(\underline{\bar{x}}, \underline{\dot{\bar{x}}}, \bar{t})\frac{d\bar{t}}{dt} = L(\underline{x}, \underline{\dot{x}}, t)$$

Moreover, since Lagrangians are identical up to a total time derivative, we will have:

$$L(\underline{\bar{x}}, \underline{\dot{\bar{x}}}, \bar{t})\frac{d\bar{t}}{dt} = L(\underline{x}, \underline{\dot{x}}, t) + \frac{d}{dt}F_s(\underline{x}, t)$$

for some function F_s. This is our new requirement for $(\underline{x}, t) \mapsto (\underline{\bar{x}}, \bar{t})$ to be a symmetry.

Now, from Lie Group Theory (diffeomorphisms are an example of a Lie group, since they are a continuous symmetry), to "understand" the symmetry requirement it will be sufficient to differentiate with respect to s, and evaluate the result at $s = 0$.

Before doing this we note again that from Lie algebras we have that the diffeomorphisms are **analytic**; that is, their Taylor series are well-defined. Expanding at $s = 0$, and recalling that

$$\zeta^j = \left.\frac{\partial \bar{x}^j}{\partial s}\right|_{s=0} \qquad \tau = \left.\frac{\partial \bar{t}}{\partial s}\right|_{s=0}$$

we obtain:

$$\begin{aligned} \bar{x}^j &= \bar{x}^j(s=0) + \zeta^j(t,x)s + \mathcal{O}(s^2) = x^j + \zeta^j(t,x)s + \mathcal{O}(s^2) \\ \bar{t} &= \bar{t}(s=0) + \tau(t,x)s + \mathcal{O}(s^2) = t + \tau(t,x)s + \mathcal{O}(s^2) \end{aligned}$$

where we have used the fact that a diffeomorphism family produces the identity at $s = 0$.

Using this, we can compute the derivative with respect to s. Noting that $L(\underline{x}, \underline{\dot{x}}, t)$ doesn't depend on s thus gives us:

$$\begin{aligned} &\frac{d}{ds}\left(L(\underline{\bar{x}}, \underline{\dot{\bar{x}}}, \bar{t})\frac{d\bar{t}}{dt}\right) = \frac{d}{ds}\left(L(\underline{x}, \underline{\dot{x}}, t) + \frac{d}{dt}F_s(\underline{x}, t)\right) \\ \implies &\left(\left[\sum_k \frac{\partial L}{\partial \bar{x}^k}\frac{\partial \bar{x}^k}{\partial s}\right] + \left[\sum_k \frac{\partial L}{\partial \dot{\bar{x}}^k}\frac{\partial \dot{\bar{x}}^k}{\partial s}\right] + \frac{\partial L}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial s}\right)\frac{d\bar{t}}{dt} + L\frac{\partial d}{\partial s}\frac{d\bar{t}}{dt} = \frac{dK_s}{dt}, \qquad K_s = \frac{dF_s}{ds} \end{aligned}$$

We now evaluate at $s = 0$ noting that the diffeomorphisms will become the identity mappings:

$$\left(\left[\sum_k \frac{\partial L}{\partial x^k}\zeta^k\right] + \left.\left[\sum_k \frac{\partial L}{\partial \dot{x}^k}\frac{\partial \dot{\bar{x}}^k}{\partial s}\right]\right|_{s=0} + \frac{\partial L}{\partial t}\tau\right)\left.\frac{d\bar{t}}{dt}\right|_{s=0} + L\left.\frac{\partial d}{\partial s}\frac{d\bar{t}}{dt}\right|_{s=0} = \frac{dK_0}{dt}$$

We thus need to compute the following quantities:

$$\left.\frac{\partial \dot{\bar{x}}^k}{\partial s}\right|_{s=0} \qquad \left.\frac{d\bar{t}}{dt}\right|_{s=0} \qquad \left.\frac{\partial d}{\partial s}\frac{d\bar{t}}{dt}\right|_{s=0}$$

① $\left.\frac{\partial \dot{\bar{x}}^k}{\partial s}\right|_{s=0}$

We start by determining:

$$\frac{d\bar{x}^k}{d\bar{t}} = \frac{d\bar{x}^k}{dt}\frac{dt}{d\bar{t}}$$

This is valid, in the sense that $\bar{x}^k$ will ultimately be a function of t, and whilst $\bar{t}$ is defined explicitly as a function of t, the fact that we have a diffeomorphism means that in particular we have a bijection. Thus, we can also think of t as a function of $\bar{t}$. In particular, due to the bijection, we have the following relation:

$$\frac{d\bar{t}}{dt} = \frac{1}{\frac{dt}{d\bar{t}}}$$

We have that $\bar{t} = t + \tau s + \mathcal{O}(s^2)$ so:

$$\frac{d\bar{t}}{dt} = 1 + s\frac{d\tau}{dt} + \mathcal{O}(s^2) \implies \frac{dt}{d\bar{t}} = \frac{1}{1 + s\frac{d\tau}{dt} + \mathcal{O}(s^2)}$$

Using the expansion $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$ we can rewrite this as:

$$\frac{d\bar{t}}{dt} = \sum_{i=0}^{\infty}\left(-s\frac{d\tau}{dt} - \mathcal{O}(s^2)\right)^i = 1 - s\frac{d\tau}{dt} + \mathcal{O}(s^2)$$

Moreover, since $\bar{x}^k = x^k + \zeta^k s + \mathcal{O}(s^2)$:

$$\frac{d\bar{x}^k}{dt} = \frac{dx^k}{dt} + s\frac{d\zeta^k}{dt} + \mathcal{O}(s^2)$$

Hence:

$$\begin{aligned}\frac{d\bar{x}^k}{d\bar{t}} &= \left(\frac{dx^k}{dt} + s\frac{d\zeta^k}{dt} + \mathcal{O}(s^2)\right)\left(1 - s\frac{d\tau}{dt} + \mathcal{O}(s^2)\right)\\ &= \frac{dx^k}{dt} - s\frac{d\tau}{dt}\frac{dx^k}{dt} + s\frac{d\zeta^k}{dt} + \mathcal{O}(s^2)\end{aligned}$$

Now, we get:

$$\frac{\partial \dot{\bar{x}}^k}{\partial s} = \frac{\partial}{\partial s}\frac{d\bar{x}^k}{d\bar{t}} = -\frac{d\tau}{dt}\frac{dx^k}{dt} + \frac{d\zeta^k}{dt} + \mathcal{O}(s)$$

and evaluating at $s = 0$:

$$\left.\frac{\partial \dot{\bar{x}}^k}{\partial s}\right|_{s=0} = -\frac{d\tau}{dt}\frac{dx^k}{dt} + \frac{d\zeta^k}{dt}$$

② $\left.\frac{d\bar{t}}{dt}\right|_{s=0}$

We already computed that:

$$\frac{d\bar{t}}{dt} = 1 + s\frac{d\tau}{dt} + \mathcal{O}(s^2)$$

so:

$$\left.\frac{d\bar{t}}{dt}\right|_{s=0} = 1$$

③ $\left.\frac{\partial d}{\partial s}\frac{d\bar{t}}{dt}\right|_{s=0}$

We already computed that:

$$\frac{d\bar{t}}{dt} = 1 + s\frac{d\tau}{dt} + \mathcal{O}(s^2)$$

so:

$$\frac{\partial d}{\partial s}\frac{d\bar{t}}{dt}\bigg|_{s=0} = \frac{d\tau}{dt}$$

Hence, using:

$$\frac{\partial \dot{\bar{x}}^k}{\partial s}\bigg|_{s=0} = -\frac{d\tau}{dt}\frac{dx^k}{dt} + \frac{d\zeta^k}{dt} \qquad \frac{d\bar{t}}{dt}\bigg|_{s=0} = 1 \qquad \frac{\partial d}{\partial s}\frac{d\bar{t}}{dt}\bigg|_{s=0} = \frac{d\tau}{dt}$$

we get that:

$$\begin{aligned}
&\left(\left[\sum_k \frac{\partial L}{\partial x^k}\zeta^k\right] + \left[\sum_k \frac{\partial L}{\partial \dot{x}^k}\frac{\partial \dot{\bar{x}}^k}{\partial s}\right]\bigg|_{s=0} + \frac{\partial L}{\partial t}\tau\right)\frac{d\bar{t}}{dt}\bigg|_{s=0} + L\frac{\partial d}{\partial s}\frac{d\bar{t}}{dt}\bigg|_{s=0} = \frac{dK_0}{dt} \\
\implies &\left(\left[\sum_k \frac{\partial L}{\partial x^k}\zeta^k\right] + \left[\sum_k \frac{\partial L}{\partial \dot{x}^k}\left(\frac{d\zeta^k}{dt} - \frac{d\tau}{dt}\frac{dx^k}{dt}\right)\right] + \frac{\partial L}{\partial t}\tau\right) + L\frac{d\tau}{dt} = \frac{dK_0}{dt} \\
\implies &\left[\sum_k \frac{\partial L}{\partial x^k}\zeta^k\right] + \left[\sum_k \frac{\partial L}{\partial \dot{x}^k}\left(\frac{d\zeta^k}{dt} - \dot{x}^k\frac{d\tau}{dt}\right)\right] + \frac{\partial L}{\partial t}\tau + L\frac{d\tau}{dt} = \frac{dK_0}{dt}
\end{aligned}$$

Now, if we can write the above expression as some total time derivative, then the expression being differentiated will be our conserved Noether charge.

It is useful to split the terms in the RHS based on whether there is a τ or a ζ_k:

$$\begin{aligned}
&\left[\sum_k \frac{\partial L}{\partial x^k}\zeta^k\right] + \left[\sum_k \frac{\partial L}{\partial \dot{x}^k}\left(\frac{d\zeta^k}{dt} - \dot{x}^k\frac{d\tau}{dt}\right)\right] + \frac{\partial L}{\partial t}\tau + L\frac{d\tau}{dt} \\
&= \left[\sum_k \frac{\partial L}{\partial x^k}\zeta^k + \frac{\partial L}{\partial \dot{x}^k}\frac{d\zeta^k}{dt}\right] + \left[\frac{\partial L}{\partial t}\tau + L\frac{d\tau}{dt} - \sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\frac{d\tau}{dt}\right] \\
&= \left[\sum_k \frac{\partial L}{\partial x^k}\zeta^k + \frac{\partial L}{\partial \dot{x}^k}\frac{d\zeta^k}{dt}\right] + \left[\frac{\partial L}{\partial t}\tau + \frac{d\tau}{dt}\left(L - \sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right)\right]
\end{aligned}$$

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(1) ζ_k

Now, notice that:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\zeta^k\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\zeta_k + \frac{\partial L}{\partial \dot{x}^k}\frac{d\zeta^k}{dt}$$

Hence:

$$\begin{aligned}
\frac{\partial L}{\partial x^k}\zeta^k + \frac{\partial L}{\partial \dot{x}^k}\frac{d\zeta^k}{dt} &= \frac{\partial L}{\partial x^k}\zeta^k + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\zeta^k\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\zeta_k \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\zeta^k\right) + \zeta_k\left(\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)
\end{aligned}$$

(2) τ

Notice that:

$$\frac{d}{dt}\left(\left(L-\sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right)\tau\right)$$
$$=\left(\frac{dL}{dt}+\left[\sum_k \frac{\partial L}{\partial x^k}\dot{x}^k+\cancel{\frac{\partial L}{\partial \dot{x}^k}\ddot{x}^k}\right]-\left[\sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\dot{x}^k+\cancel{\frac{\partial L}{\partial \dot{x}^k}\ddot{x}^k}\right]\right)\tau+\left(L-\sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right)\frac{d\tau}{dt}$$
$$=\frac{dL}{dt}\tau+\tau\left[\sum_k \dot{x}^k\left(\frac{\partial L}{\partial x^k}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)\right]+\left(L-\sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right)\frac{d\tau}{dt}$$

Hence:

$$\frac{\partial L}{\partial t}\tau+\frac{d\tau}{dt}\left(L-\sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right)$$
$$=\frac{d}{dt}\left(\left(L-\sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right)\tau\right)-\tau\left[\sum_k \dot{x}^k\left(\frac{\partial L}{\partial x^k}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)\right]$$
$$=\frac{d}{dt}\left(L\tau-\tau\sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right)-\tau\left[\sum_k \dot{x}^k\left(\frac{\partial L}{\partial x^k}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)\right]$$

Thus, we finally write:

$$\left[\sum_k \frac{\partial L}{\partial x^k}\zeta^k+\frac{\partial L}{\partial \dot{x}^k}\frac{d\zeta^k}{dt}\right]+\left[\frac{\partial L}{\partial t}\tau+\frac{d\tau}{dt}\left(L-\sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right)\right]=\frac{dK_0}{dt}$$
$$=\left[\sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\zeta^k\right)+\zeta_k\left(\frac{\partial L}{\partial x^k}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)\right]$$
$$+\left[\frac{d}{dt}\left(L\tau-\tau\sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right)-\tau\left[\sum_k \dot{x}^k\left(\frac{\partial L}{\partial x^k}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)\right]\right]=\frac{dK_0}{dt}$$
$$=\frac{d}{dt}\left(L\tau-\tau\sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right)+\left[\sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\zeta^k\right)\right]+\left[\sum_k(\zeta_k-\dot{x}^k\tau)\left(\frac{\partial L}{\partial x^k}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)\right]=\frac{dK_0}{dt}$$

Now, if we assume our curve obeys the Euler-Lagrange Equations, this reduces to:

$$\frac{d}{dt}\left(L\tau-\tau\sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right)+\left[\sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\zeta^k\right)\right]+\left[\sum_k(\zeta_k-\dot{x}^k\tau)\underbrace{\left(\frac{\partial L}{\partial x^k}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\right)}_{=0}\right]=\frac{dK_0}{dt}$$
$$\frac{d}{dt}\left(\tau\left[L-\sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right]+\left[\sum_k \frac{\partial L}{\partial \dot{x}^k}\zeta^k\right]-K_0\right)=0$$

So the Noether Charge which gets conserved aong extremals satisfying the EL equations is:

$$N=\tau\left[L-\sum_k \frac{\partial L}{\partial \dot{x}^k}\dot{x}^k\right]+\left[\sum_k \frac{\partial L}{\partial \dot{x}^k}\zeta^k\right]-K_0$$

as required.

Here, the term proportional to τ is the contribution due to change in ***time****;* $\sum_k \frac{\partial L}{\partial \dot{x}^k}\zeta^k$ *is our standard Noether charge when we only shift by* $\underline{x}$*;* K_0 *is the contribution given by the invariance of the Lagrangian.*

□

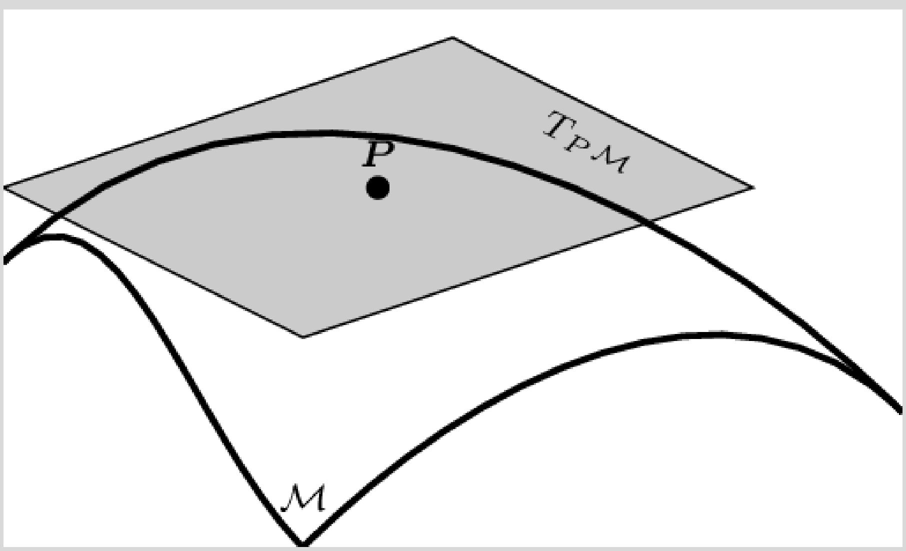
2 1-forms

I recommend reading this article by Terence Tao, which builds up great intuition for understanding 1-forms.

2.1 Definition: The Tangent Space

Let $U \subset \mathbb{R}^n$ be open, and consider a point $\underline{a} \in U$.
The ***tangent space*** *to U at $\underline{a}$ is a n-dimensional* ***real vector space*** *$T_{\underline{a}}U$, spanned by:*

$$\left\{ \left. \frac{\partial}{\partial x^i} \right|_a \right\}$$



(Theorem E.1)

The notation $\frac{\partial}{\partial x^i}$ is meant to represent a unit vector, which will be tangent to our point $\underline{a}$ in the direction of $\underline{e}_i \in U$.

2.2 Definition: 1-Forms

*Let $U \subset \mathbb{R}^n$. A **1-form** at $\underline{a} \in U$ is a **linear map**:*

$$\alpha : T_{\underline{a}}U \to \mathbb{R}$$

*mapping elements in the **tangent space** to some real value via:*

$$\alpha(x + cy) = \alpha(x) + c\alpha(y), \qquad x, y \in T_{\underline{a}}U, \quad c \in \mathbb{R}$$

2.3 Definition: Dual Vector Space

*The **dual vector space** is the **vector space** of **all** 1-forms at $\underline{a} \in U$. We denote the **dual vector space** as:*

$$T_{\underline{a}}^{*}U$$

*The **dual vector space** is a vector space of linear mappings, which assign a real value to a tangent vector in **tangent space** $T_{\underline{a}}U$.*

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*A **basis** for the **dual vector space** $T_{\underline{a}}^{*}U$ is given by the set of 1-forms which we denote as:*

$$\{ dx^i\big|_a \}$$

which are defined such that (rememebr dx^i are functions which take tangent vectors as inputs!):

$$dx^j\big|_a \left(\frac{\partial}{\partial x^i}\bigg|_a \right) = \delta_{ij}$$

Such that if $y \in T_{\underline{a}}U$:

$$y = \sum_{i=1}^{n} y^i \left.\frac{\partial}{\partial x^i}\right|_a$$

Then:

$$dx^i\big|_a (y) = \sum_{j=1}^{n} dx^i\big|_a \left(y^j \left.\frac{\partial}{\partial x^j}\right|_a \right) = \sum_{j=1}^{n} y^j \; dx^i\big|_a \left(\left.\frac{\partial}{\partial x^j}\right|_a \right) = y_i$$

2.4 Definition: Differential 1-Forms

*A **differential 1-form** on U is a map α which assigns to each $\underline{a} \in U$ a 1-form in $T^*_{\underline{a}}U$.*
In particular, if $\alpha_i : U \to \mathbb{R}$:

$$\alpha = \sum_{i=1}^{n} \alpha_i dx^i$$

2.5 Definition: Differential/Exterior Derivative

*The **differential/exterior derivative** of a **scalar field** f is the **dual vector**:*

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$$

*In particular, a 1-form α is **exact** if $\exists f$:*

$$\alpha = df$$

If we now pick $\underline{a} \in U$ and $\underline{y} \in T_{\underline{a}}U$, then:

$$df(\underline{a})(\underline{y}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\underline{a}) \; dx^i\big|_a (\underline{y}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\underline{a}) y_i = \left\langle \nabla f, \underline{y} \right\rangle = Df(\underline{a})(\underline{y})$$

*Hence, the **total derivative** is nothing but a particular element of the dual vector space $T^*_{\underline{a}}U$.*

3 Hamilton's Canonical Formalism

3.1 The Hamiltonian in 1 Dimension

- **How can we express the Euler-Lagrange Equations as first order ODEs?**
  - consider a 1-dimensional Lagrangian:

    $$L(x, \dot{x}, t)$$

    which gives rise to the **Euler-lagrange equation**:

    $$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$$

  - we can convert this into an equivalent system of first order ODEs via:

    $$\frac{dx}{dt} = v \qquad \frac{d}{dt}\frac{\partial L}{\partial v} = \frac{\partial L}{\partial x}$$

- **What is Hamilton's canonical formalism?**
  - the system
  $$\frac{dx}{dt} = v \qquad \frac{d}{dt}\frac{\partial L}{\partial v} = \frac{\partial L}{\partial x}$$
  doesn't look **symmetric**
  - **Hamilton's formalism** seeks to use **canonical variables** to restore symmetry

3.1.1 Definition: One-Dimensional Hamiltonian

Let L be a ***Lagrangian****, with* ***Euler-Lagrange*** *equations written as the system:.*

$$\frac{dx}{dt} = v \qquad \frac{d}{dt}\frac{\partial L}{\partial v} = \frac{\partial L}{\partial x}$$

Let p be the ***momentum conjugate*** *to x:*

$$p = \frac{\partial L}{\partial v}$$

Then the ***Hamiltonian*** *is:*

$$H(x, p) = pv - L(x, v)$$

where we think of v as a function of p.
The ***Euler-Lagrange Equations*** *then become* ***Hamilton's Equations****:*

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

as expected.

Notice, by introducing the ***Hamiltonian****, we can express the EL equations in a much more symmetric form. The set of coordinate (x, p) define what is known as* ***phase space****.*

3.1.2 Example: Hamiltonian

Consider the Lagrangian:

$$L(x, v) = \frac{1}{2}mv^2 - V(x)$$

where V is some potential. The momentum conjugate to x is:

$$p = \frac{\partial L}{\partial v} = mv$$

So we can write $v = \frac{p}{m}$. Hence, the Hamiltonian will be:

$$H(x, p) = p\frac{p}{m} - \frac{1}{2}m\frac{p^2}{m^2} - V(x) = \frac{p^2}{2m} + V(x)$$

We now verify Hamilton's Equations:

$$\frac{\partial H}{\partial p} = \frac{p}{m} = v = \frac{dx}{dt}$$
$$\frac{\partial H}{\partial x} = \frac{dV}{dx} = -\frac{\partial L}{\partial x}$$

3.1.3 The Legendre Transform: Hamiltonian and Lagrangian

Lagrangians and Hamiltonians seem to be very symmetrical:

$$L(x,v,t) \iff H(x,p,t)$$
$$p = \frac{\partial L}{\partial v} \iff v = \frac{\partial H}{\partial p}$$
$$H = pv - L \iff L = vp - H$$

In particular, we say that H is the ***Legendre Transform*** *of L*

3.2 Definition: The General Hamiltonian

Consider a general ***Lagrangian****:*

$$L(\underline{x}, \underline{v}, t), \qquad \underline{x}, \underline{v} \in \mathbb{R}^n$$

Define the ***conjugate momentum*** *to $\underline{x}$ as $\underline{p}$, such that:*

$$p_i = \frac{\partial L}{\partial v^i}$$

The ***Hamiltonian*** *is then given by:*

$$H(\underline{x}, \underline{p}, t) = \langle \underline{v}, \underline{p} \rangle - L(\underline{x}, \underline{v}, t) = \sum_{i=1}^{n} v^i p_i - L(\underline{x}, \underline{v}, t)$$

where $v^i = v^i(\underline{x}, \underline{p}, t)$.

3.2.1 Regular Lagrangian

Notice, here we are assuming that the velocity is defined ***implicitly*** *in terms of momentum. For this, we require the* ***implicit function theorem****, which tells us that the solution to:*

$$p_i = \frac{\partial L}{\partial v^i}$$

is a graph:

$$v^i = v^i(\underline{x}, \underline{p}, t)$$

at points where the Jacobian:

$$\frac{\partial^L}{\partial v^i \partial v^j}$$

is ***inveritble****. If L satisfies this invertibility requirement for all $(\underline{x}, \underline{v}, t)$,then L is* ***regular****; otherwise, L is a* ***singular*** *Lagrangian. Hence, if L is regular, this guarantees that we can locally write $\underline{v}$ as a function* *of $\underline{x}, \underline{p}, t$, and so, our definition for the Hamiltonian will be valid. However, this doesn't mean that if L is singular, it doesn't have a Hamiltonian description. Indeed, $L = \|\underline{v}\|$, which gives rise to the arclength functional, is a **singular** Lagrangian, which has a perfectly well defined Hamiltonian - we just need to make sure we pick a different, but equivalent, Lagrangian, whose Hamiltonian will be well-defined.*

3.3 Theorem: General Hamilton's Equations

***Hamilton's Equations** are:*

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}$$

*These are **canonical/Hamiltonian** form of the **first-order** version of the **Euler-Lagrange Equations**:*

$$\frac{dx^i}{dt} = v^i \qquad \frac{d}{dt}\frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial x^i}$$

(Equation 8.3)

Proof. We shall use **differentials**. By linearity:

$$dH = d\left(\left[\sum_{i=1}^{n} v^i p_i\right] - L(\underline{x}, \underline{v}, t)\right) = \left[\sum_{i=1}^{n} d(v^i p_i)\right] - d(L(\underline{x}, \underline{v}, t))$$

Then we can write:

$$\begin{aligned} dH &= \sum_{i=1}^{n}(dp_i v^i + p_i dv^i) - \left(\frac{\partial L}{\partial t}dt + \left[\sum_{i=1}^{n}\frac{\partial L}{\partial x^i}dx^i\right] + \left[\sum_{i=1}^{n}\frac{\partial L}{\partial v^i}dv^i\right]\right) \\ &= \left[\sum_{i=1}^{n} dp_i v^i\right] + \cancel{\left[\sum_{i=1}^{n}\frac{\partial L}{\partial v^i}dv^i\right]} - \left(\frac{\partial L}{\partial t}dt + \left[\sum_{i=1}^{n}\frac{\partial L}{\partial x^i}dx^i\right] + \cancel{\left[\sum_{i=1}^{n}\frac{\partial L}{\partial v^i}dv^i\right]}\right) \\ &= \left[\sum_{i=1}^{n} dp_i v^i\right] - \frac{\partial L}{\partial t}dt - \left[\sum_{i=1}^{n}\frac{\partial L}{\partial x^i}dx^i\right] \end{aligned}$$

The 1-forms form a basis, so the components with differnt differentials will be independent. This then allows us to see that:

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$
$$\frac{\partial H}{\partial p_i} = v^i$$
$$\frac{\partial H}{\partial x^i} = -\frac{\partial L}{\partial x^i}$$

which converts:

$$\frac{dx^i}{dt} = v^i \qquad \frac{d}{dt}\frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial x^i}$$

into:

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}$$

as required.

□

3.4 Theorem: Non-Uniqueness of the Hamiltonian

2 ***Hamiltonians*** *differing by a* ***partial time derivative*** *give rise to the* ***same*** *set of* ***Hamilton's Equations****:*

$$H'(Q,P,t) = H(q,p,t) - \frac{\partial}{\partial t}F(q,t)$$

where:

$$q^i = Q^i \qquad p_i = P_i - \frac{\partial F(Q,t)}{\partial Q^i}$$

In fact, if:

$$L'(q,\dot{q},t) = L(q,\dot{q},t) + \frac{d}{dt}F(q,t)$$

then L' ***Legendre transforms*** *into* H'*, and* L ***Legendre transforms*** *into* H*.*
(Equation 8.8)

Proof. □

3.5 Poisson Brackets

3.5.1 Definition: The Poisson Bracket

The ***Poisson bracket*** *is a* ***binary operation*** *on functions defined by:*

$$[f,g] = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i}\right)$$

(Equation 8.9)

3.5.2 Lemma: Properties of the Poisson Bracket

1. ***Bilinearity***:
$$[\alpha f, \beta g] = \alpha\beta[f, g]$$
2. ***Antisymmetry***:
$$[f, g] = -[g, f]$$
3. ***"Distributivity"***:
$$[f, gh] = [f, g]h + g[f, h]$$
4. ***Jacobi Identity***:
$$[f, [g, h]] = [[f, g], h] + [g, [f, h]]$$

---

*These properties mean that any **smooth** (infinitely differentiable) functions on **phase space** (x, p) form a **Lie algebra under the Poisson bracket**; in fact, alongside commutative multiplication of functions, these functions form a **Poisson algebra**, which appear in the context of **deformations of algebraic structures** (of which **quantum mechanics** are one instance).*

3.5.3 Theorem: Conserved Quantity from Poisson Bracket with Hamiltonian

*Let $\Phi(\underline{x}, \underline{p})$ be a function defined on **phase space**, where $\underline{x}, \underline{p}$ obey **Hamilton's Equations**:*

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}$$

Then:

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + [\Phi, H]$$

*In particular, Φ is **conserved** if:*

$$\frac{\partial \Phi}{\partial t} = -[\Phi, H] = [H, \Phi]$$

*If Φ doesn't depend **explicitly** on time, then:*

$$\frac{d\Phi}{dt} = 0 \iff [\Phi, H] = 0$$

*and we say that Φ **Poisson-commutes with** H.*
(Equation 8.10)

Proof. We compute directly:

$$\begin{aligned}\frac{d\Phi}{dt} &= \frac{\partial \Phi}{\partial t} + \left[\sum_{i=1}^{n} \frac{\partial \Phi}{\partial x^i}\frac{dx^i}{dt}\right] + \left[\sum_{i=1}^{n} \frac{\partial \Phi}{\partial p^i}\frac{dp^i}{dt}\right] \\ &= \frac{\partial \Phi}{\partial t} + \left[\sum_{i=1}^{n} \frac{\partial \Phi}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial \Phi}{\partial p^i}\frac{\partial H}{\partial x^i}\right] \\ &= \frac{\partial \Phi}{\partial t} + [\Phi, H]\end{aligned}$$

where we have used **Hamilton's Equations** to get from the first line to the second line.

□

3.5.4 Corollary: Generating New Conserved Quantities from Old

*Let Φ_1, Φ_2 be **conserved** functions in **phase space**. Then:*

$$[\Phi_1, \Phi_2]$$

is also conserved.

The beauty of this result is that from 2 conserved quantities, we can get another one for free by using the Poisson brackets!

Proof. Since Φ_1, Φ_2 are conserved in phase space, then:

$$[\Phi_1, H] = [\Phi_2, H] = 0$$

But then, by the **Jacobi identity**:

$$[[\Phi_1, \Phi_2], H] = -[[H, \Phi_1], \Phi_2] - [\Phi_1, [H, \Phi_2]] = 0$$

by bilinearity of the Poisson bracket. Hence, $[\phi_1, \Phi_2]$ is also conserved. □

3.6 Theorem: A Partial Converse to Noether's Theorem

Let $\Phi(\underline{x}, \underline{p}, t)$ be a ***conserved quantity****:*

$$\frac{\partial \Phi}{\partial t} = [-\Phi, H] = [H, \Phi]$$

Then, the continuous transformations defined by:

$$\frac{dx^i}{ds} = [x^i, \Phi] = \frac{\partial \Phi}{\partial p_i} \qquad \frac{dp_i}{ds} = [p_i, \Phi] = -\frac{\partial \Phi}{\partial x^i}$$

leave the ***Hamiltonian*** *invariant, up to a* ***partial time-derivative****:*

$$\frac{dH}{ds} = \frac{\partial \Phi}{\partial t}$$

That is, the continuous transformations are ***continuous symmetries*** *of* ***Hamilton's Equations****, taking solutions to solutions.*
(Equations 8.11 & 8.12)

Proof. It is easy to see that:

$$[x^i, \Phi] = \sum_{j=1}^{n} \left(\frac{\partial x^i}{\partial x^j} \frac{\partial \Phi}{\partial p_j} - \frac{\partial x^i}{\partial p_j} \frac{\partial \Phi}{\partial x^j} \right) = \frac{\partial \Phi}{\partial p_i}$$

$$[p_i, \Phi] = \sum_{j=1}^{n} \left(\frac{\partial p_i}{\partial x^j} \frac{\partial \Phi}{\partial p_j} - \frac{\partial p_i}{\partial p_j} \frac{\partial \Phi}{\partial x^j} \right) = -\frac{\partial \Phi}{\partial x_i}$$

The **Hamiltonian vector field** is the **vector field** on the **phase space** $\mathbb{R}^{2n}$ defined by:

$$X_\Phi = [\Phi, -] = \frac{\partial \Phi}{\partial x^i} \frac{\partial}{\partial p_i} - \frac{\partial \Phi}{\partial p_i} \frac{\partial}{\partial x^i}$$

$$\underline{a} \mapsto \begin{pmatrix} \frac{\partial \Phi}{\partial \underline{p}}(\underline{a}) \\ -\frac{\partial \Phi}{\partial \underline{x}}(\underline{a}) \end{pmatrix}, \qquad \frac{\partial \Phi}{\partial \underline{p}}(\underline{a}), -\frac{\partial \Phi}{\partial \underline{p}}(\underline{a}) \in \mathbb{R}^n$$

Now, define a curve in phase space $(\underline{x}(s), \underline{p}(s))$ as a solution to the system:

$$\frac{dx^i}{ds} = [x^i, \Phi] = \frac{\partial \Phi}{\partial p_i} \qquad \frac{dp^i}{ds} = [p_i, \Phi] = -\frac{\partial \Phi}{\partial x^i}$$

This has a solution, by the existence and uniqueness theorem (given some initial condition $(\underline{x}(0), \underline{p}(0))$.

Furthermore, along these solutions:

$$\begin{aligned}
\frac{dH}{ds} &= \left[\sum_{i=1}^{n} \frac{\partial H}{\partial x^i}\frac{dx^i}{ds}\right] + \left[\sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{dp_i}{ds}\right] \\
&= \left[\sum_{i=1}^{n} \frac{\partial H}{\partial x^i}\frac{\partial \Phi}{\partial p_i}\right] - \left[\sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial \Phi}{\partial x^i}\right] \\
&= \left[\sum_{i=1}^{n} \frac{\partial H}{\partial x^i}\frac{\partial \Phi}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial \Phi}{\partial x^i}\right] \\
&= [H, \Phi] \\
&= \frac{\partial \Phi}{\partial t}
\end{aligned}$$

That is, the Hamiltonian changes by a partial time-derivative. But we saw that Hamiltonians which differ by a partial time-derivative produce the same set of equations of motion. Hence, we can think of these continuous transformation as leaving the Hamiltonian invariant, and thus, are symmetries of the Hamiltonian on phase space.

□

3.6.1 Worked Example: Angular Momentum Conservation

The theorem above can be thought of as a converse to Noether's Theorem: from a conserved quantity in phase space, we are capable of deriving the continuous symmetry whcih generates it.

Last week we considered the following problem:

Consider the Lagrangian:

$$L = \frac{1}{2}m\|\underline{\dot{x}}\|^2 - V(\underline{x})$$

for plane curves:

$$\underline{x} : [0,1] \to \mathbb{R}^2$$

Assume that the potential V only depends on $\|\underline{x}\|$. Show that L is invariant under the one-parameter symmetry group:

$$\varphi_s : \mathbb{R}^2 \to \mathbb{R}^2$$

defined by:

$$\varphi_s(\underline{x}) = \begin{pmatrix} x^1\cos(s) - x^2\sin(s) \\ x^1\sin(x) + x^2\cos(s) \end{pmatrix}$$

We now work in reverse.

The conjugate momentum is:

$$p_i = \frac{\partial L}{\partial v} = mv^i \implies v^i = \frac{p_i}{m}$$

Thus, we write the Hamiltonian as:

$$H(\underline{x}, \underline{p}) = \frac{\|\underline{p}\|^2}{2m} + V(\|\underline{x}\|)$$

We have that:

$$\Phi = J = x^1 p_2 - x^2 p_1$$

is the **angular momentum**, which is conserved

We can now compute the Poisson brackets with the canonical variables:

$$[x^1, J] = \frac{\partial J}{\partial p_1} = -x^2$$

$$[x^2, J] = \frac{\partial J}{\partial p_2} = x^1$$

$$[p_1, J] = -\frac{\partial J}{\partial x_1} = -p_2$$

$$[p_2, J] = -\frac{\partial J}{\partial x_2} = p_1$$

which gives the system of ODEs:

$$\frac{dx^1}{ds} = -x^2 \qquad \frac{dx^2}{ds} = x^1 \qquad \frac{dp_1}{ds} = -p_2 \qquad \frac{dp_2}{ds} = p_1$$

We could solve the ODEs by writing it as a system, but it is easier to notice that:

$$\frac{dx^1}{ds} = -x^2 \implies \frac{d^2x^1}{ds^2} = -\frac{dx^2}{ds} = -x^1$$

This is a standard second-order ODE with characteristic polynomial $r^2 = -1$, so:

$$x^1(s) = A\cos(s) + B\sin(s)$$

Doing this for all the remaining ODEs and using initial conditions we get:

$$x^1(s) = x^1(0)\cos(s) - x^2(0)\sin(s)$$

$$x^2(s) = x^2(0)\cos(s) + x^1(0)\sin(s)$$

$$p_1(s) = p_1(0)\cos(s) - p_2(0)\sin(s)$$

$$p_2(s) = p_2(0)\cos(s) + p_1(0)\sin(s)$$

which are indeed the initial transformations that we had.