Variational Calculus - Week 5 - Introduction to Noether's Theorem

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Contents

Based on the notes by Jelle Hartong, Chapter 7

1 Mapping Between Functionals

1.1 Definition: Diffeomorphisms

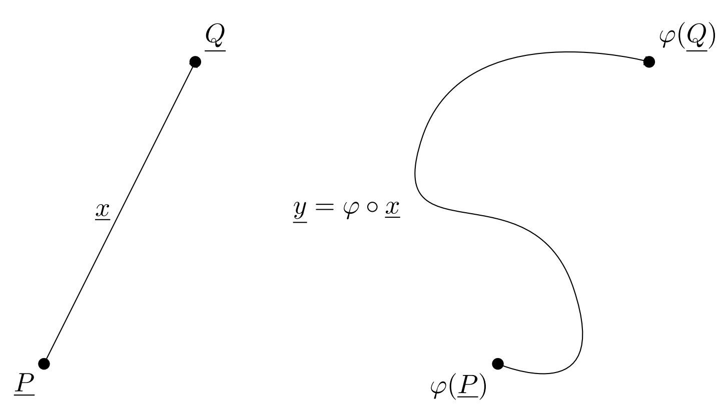
*A C^2 **diffeomorphism** is a mapping:*

$$\varphi : \mathbb{R}^n \to \mathbb{R}^n$$

satisfying:

1. $\varphi \in C^2$
2. $\exists \varphi^{-1}$
3. $\varphi^{-1} \in C^2$

- **How does applying diffeomorphismsm affect the extremising functions for the action?**
  - consider the action:
  $$I[\underline{x}] = \int_0^1 L(\underline{x}, \dot{\underline{x}}, t)dt$$
  for a **regular** C^2 curve:
  $$\underline{x} : [0,1] \to \mathbb{R}^n \qquad \underline{x}(0) = \underline{P} \qquad \underline{x}(1) = \underline{Q}$$
  - if $\underline{x}$ extremises I, consider the new path:
  $$\underline{y} = \varphi \circ \underline{x} = \varphi(\underline{x}(t))$$
  between points $\varphi(\underline{P})$ and $\varphi(\underline{Q})$ (since $\varphi \in C^2$ so it's continuous)
  - if L had been the **arclength**, then $\underline{x}$ would be a straight line
  - however, for most choices of φ, $\underline{y}$ would almost certainly not be a straight line, so $\underline{y}$ will **not** extremise I



– however, **diffeomorphisms** become extremely useful if, after being applied, they **still** extremise the functional - this is the basis of **Noether's Theorem**

1.2 Lemma: Extremising Actions Under Diffeomorphisms

Let:

$$I[\underline{x}] = \int_0^1 K(\underline{x}, \dot{\underline{x}}, t)dt \qquad J[\underline{y}] = \int_0^1 L(\underline{y}, \dot{\underline{y}}, t)dt$$

such that:

$$K(\underline{x}, \dot{\underline{x}}, t) = L(\underline{y}, \dot{\underline{y}}, t)$$

where:

$$\underline{y} = \varphi \circ \underline{x}$$

*and φ is a C^2 **diffeomorphism**.*
*Then, $\underline{x}$ **extremises** I **if and only if** $\underline{y} = \varphi \circ \underline{x}$ **extremises** J.*
In other words, φ sets up a bijective correspondence between extremals of I and extremals of J.
(Lemma 7.1)

Proof. We have that:

$$K(\underline{x}, \dot{\underline{x}}, t) = L(\underline{y}, \dot{\underline{y}}, t) = L\left(\varphi(\underline{x}(t)), \frac{d}{dt}\varphi(\underline{x}(t)), t\right)$$

We first need to compute an expression for $\dot{\underline{y}}$ in terms of $\underline{x}$. This will be the **total derivative**

$$\dot{\underline{y}}(t) = \frac{d}{dt}\varphi(\underline{x}(t)) = D_{\varphi(\underline{x}(t))}\dot{\underline{x}}(t)$$

where the jth component is given by:

$$\dot{y}^j = \sum_{k=1}^{n} \frac{\partial \varphi^j}{\partial x^i}\frac{d}{dt}x^i = \sum_{k=1}^{n} \frac{\partial \varphi^j}{\partial x^k}\dot{x}^k$$

(this follows by the fact that φ^j depends on t through each of its n variables x^k)

We know that the ***total derivative*** *is a* ***linear map****. In fact, in a given coordinate system, the* ***total derivative*** *can be shown to be the* ***Jacobian Matrix****:*

$$D_{\varphi(\underline{x}(t))} = \begin{pmatrix} (\nabla\varphi^1)^T \\ (\nabla\varphi^2)^T \\ \vdots \\ (\nabla\varphi^n)^T \end{pmatrix} = \begin{pmatrix} \frac{\partial\varphi^1}{\partial x^1} & \frac{\partial\varphi^1}{\partial x^2} & \cdots & \frac{\partial\varphi^1}{\partial x^n} \\ \frac{\partial\varphi^2}{\partial x^1} & \frac{\partial\varphi^2}{\partial x^2} & \cdots & \frac{\partial\varphi^2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\varphi^n}{\partial x^1} & \frac{\partial\varphi^n}{\partial x^2} & \cdots & \frac{\partial\varphi^n}{\partial x^n} \end{pmatrix}$$

We can indeed see that $\underline{\dot{y}} = D_{\varphi(\underline{x}(t))}\underline{\dot{x}}$*, since:*

$$\dot{y}^j = \sum_{k=1}^{n} \frac{\partial\varphi^j}{\partial x^k}\dot{x}^k$$

Most importantly, $D_{\phi(\underline{x}(t))}$ *will be* ***invertible****.*

Hence, we can compute the Euler-Lagrange Equation for K by using the RHS. For a given variable x^i:

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial K}{\partial \dot{x}^i} - \frac{\partial K}{\partial x^i} \\ &= \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} \end{aligned}$$

Now, since φ is a vector field, each of its components is a function φ^j which itself can depend on x^i. Moreover, $\dot{y}^j$ also depend on x^i. Hence, we get that:

$$\begin{aligned} \frac{\partial L}{\partial x^i} &= \sum_{j=1}^{n} \left(\frac{\partial L}{\partial y^j}\frac{\partial y^j}{\partial x^i} + \frac{\partial L}{\partial \dot{y}^j}\frac{\partial \dot{y}^j}{\partial x^i} \right) \\ &= \sum_{j=1}^{n} \left[\frac{\partial L}{\partial y^j}\frac{\partial \varphi^j}{\partial x^i} + \frac{\partial L}{\partial \dot{y}^j}\frac{\partial}{\partial x^i} \left(\sum_{k=1}^{n} \frac{\partial\varphi^j}{\partial x^k}\dot{x}^k \right) \right] \\ &= \sum_{j=1}^{n} \left[\frac{\partial L}{\partial y^j}\frac{\partial \varphi^j}{\partial x^i} + \frac{\partial L}{\partial \dot{y}^j} \left(\sum_{k=1}^{n} \frac{\partial^2\varphi^j}{\partial x^i \partial x^k}\dot{x}^k \right) \right] \end{aligned}$$

*This can get messy, so an alternative is to use **Einstein's Summation Notation**, whereby if a "dummy variable" appears twice, we can infer that there is summation over said variable. In other words, we can write:*

$$\sum_{j=1}^{n}\left[\frac{\partial L}{\partial y^j}\frac{\partial \varphi^j}{\partial x^i}+\frac{\partial L}{\partial \dot{y}^j}\left(\sum_{k=1}^{n}\frac{\partial^2 \varphi^j}{\partial x^i \partial x^k}\dot{x}^k\right)\right]$$
$$=\sum_{j=1}^{n}\left[\frac{\partial L}{\partial y^j}\frac{\partial \varphi^j}{\partial x^i}\right]+\sum_{j,k=1}^{n}\left[\frac{\partial L}{\partial \dot{y}^j}\frac{\partial^2 \varphi^j}{\partial x^i \partial x^k}\dot{x}^k\right]$$

as:

$$\frac{\partial L}{\partial y^j}\frac{\partial \varphi^j}{\partial x^i}+\frac{\partial L}{\partial \dot{y}^j}\frac{\partial^2 \varphi^j}{\partial x^i \partial x^k}\dot{x}^k$$

However, I personally find this more confusing, so I'll continue using the full notation, alongside brackets.

Moreover, L only depends on $\dot{x}^i$ through $\underline{\dot{y}}$ so:

$$\frac{\partial L}{\partial \dot{x}^i}=\sum_{j=1}^{n}\frac{\partial L}{\partial \dot{y}^j}\frac{\partial \dot{y}^j}{\partial \dot{x}^i}$$
$$=\sum_{j=1}^{n}\frac{\partial L}{\partial \dot{y}^j}\frac{\partial}{\partial \dot{x}^i}\left(\sum_{k=1}^{n}\frac{\partial \varphi^j}{\partial x^k}\dot{x}^k\right)$$
$$=\sum_{j=1}^{n}\frac{\partial L}{\partial \dot{y}^j}\frac{\partial \varphi^j}{\partial x^i}$$

Thus, the Euler-Lagrange Equation for K becomes:

$$\begin{aligned}
0 &= \frac{d}{dt}\frac{\partial K}{\partial \dot{x}^i} - \frac{\partial K}{\partial x^i} \\
&= \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} \\
&= \frac{d}{dt}\left(\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{y}^j}\frac{\partial \varphi^j}{\partial x^i}\right) - \sum_{j=1}^{n}\left[\frac{\partial L}{\partial y^j}\frac{\partial \varphi^j}{\partial x^i} + \frac{\partial L}{\partial \dot{y}^j}\left(\sum_{k=1}^{n}\frac{\partial^2 \varphi^j}{\partial x^i \partial x^k}\dot{x}^k\right)\right] \\
&= \sum_{j=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^j}\frac{\partial \varphi^j}{\partial x^i}\right) - \frac{\partial L}{\partial y^j}\frac{\partial \varphi^j}{\partial x^i} - \frac{\partial L}{\partial \dot{y}^j}\left(\sum_{k=1}^{n}\frac{\partial^2 \varphi^j}{\partial x^i \partial x^k}\dot{x}^k\right)\right] \\
&= \sum_{j=1}^{n}\left[\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^j}\right)\frac{\partial \varphi^j}{\partial x^i} + \frac{\partial L}{\partial \dot{y}^j}\frac{d}{dt}\left(\frac{\partial \varphi^j}{\partial x^i}\right)\right) - \frac{\partial L}{\partial y^j}\frac{\partial \varphi^j}{\partial x^i} - \frac{\partial L}{\partial \dot{y}^j}\left(\sum_{k=1}^{n}\frac{\partial^2 \varphi^j}{\partial x^i \partial x^k}\dot{x}^k\right)\right] \\
&= \sum_{j=1}^{n}\left[\frac{\partial \varphi^j}{\partial x^i}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^j}\right) - \frac{\partial L}{\partial y^j}\right) + \left(\frac{\partial L}{\partial \dot{y}^j}\frac{d}{dt}\left(\frac{\partial \varphi^j}{\partial x^i}\right)\right) - \frac{\partial L}{\partial \dot{y}^j}\left(\sum_{k=1}^{n}\frac{\partial^2 \varphi^j}{\partial x^i \partial x^k}\dot{x}^k\right)\right] \\
&= \sum_{j=1}^{n}\left[\frac{\partial \varphi^j}{\partial x^i}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^j}\right) - \frac{\partial L}{\partial y^j}\right) + \frac{\partial L}{\partial \dot{y}^j}\left(\frac{d}{dt}\left(\frac{\partial \varphi^j}{\partial x^i}\right) - \sum_{k=1}^{n}\frac{\partial^2 \varphi^j}{\partial x^i \partial x^k}\dot{x}^k\right)\right] \\
&= \sum_{j=1}^{n}\left[\frac{\partial \varphi^j}{\partial x^i}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^j}\right) - \frac{\partial L}{\partial y^j}\right) + \frac{\partial L}{\partial \dot{y}^j}\left(\sum_{k=1}^{n}\frac{\partial^2 \varphi^j}{\partial x^k \partial x^i}\dot{x}^k - \sum_{k=1}^{n}\frac{\partial^2 \varphi^j}{\partial x^i \partial x^k}\dot{x}^k\right)\right] \\
&= \sum_{j=1}^{n}\left[\frac{\partial \varphi^j}{\partial x^i}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^j}\right) - \frac{\partial L}{\partial y^j}\right)\right], \qquad \textit{since } \varphi \in C^2, \textit{ so by continuity } \frac{\partial^2 \varphi^j}{\partial x^i \partial x^k} = \frac{\partial^2 \varphi^j}{\partial x^k \partial x^i}
\end{aligned}$$

But we saw,

$$\sum_{j=1}^{n}\frac{\partial \varphi^j}{\partial x^i}\Box$$

represents a transformation of a vector by applying the Jacobian to it. This is saying that the Jacobian times the vector with components $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^j}\right) - \frac{\partial L}{\partial y^j}$ gives the 0 vector. Since the Jacobian is invertible, this implies that:

$$\sum_{j=1}^{n}\left[\frac{\partial \varphi^j}{\partial x^i}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^j}\right) - \frac{\partial L}{\partial y^j}\right)\right] = 0 \iff \forall j \in [1, n], \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^j}\right) - \frac{\partial L}{\partial y^j} = 0$$

In other words, $\underline{x}$ satisfies the Euler-Lagrange Equations for the Lagrangian K **if and only if** $\underline{y}$ satisfies the Euler-Lgrange Equations for the Lagrangian L, as required.

$\Box$

1.3 Symmetries of the Lagrangian

- **When is a diffeomorphism a symmetry of a Lagrangian?**
  - when φ is such that:
  $$\underline{y} = \varphi(\underline{x}(t)) \implies L(\underline{x}, \dot{x}, t) = L(\underline{y}, \underline{\dot{y}}, t)$$
  - we say that L is **invariant under** φ
  - if φ is a **symmetry**, then it maps extrema to extrema

- **Why are the Euler-Lagrange equations true in any coordinate system?**
  - we can think of **diffeomorphisms** as mapping $\underline{x}$ into another coordinate system $\underline{y}$
  - but if L is invariant under φ, changing the coordinate system doesn't affect the Lagrangian, or the Euler-Lagrange Equations
  - thus, the Euler-Lagrange Equations will apply in any coordinate system

It is important to note the logical direction of this Lemma: ***if*** *the Lagrangians are equal under diffeomorphism* ***then*** *the extremals of the Lagrangian will agree under the transformation φ.*
This does ***not*** *mean that if φ maps between extrema of 2 Lagrangians, then the φ will be a symmetry.*

2 Noether's Theorem (Version 1)

2.1 Definition: One-Parameter Subgroup of Diffeomorphisms

Consider a ***one-parameter family****:*

$$\varphi_s : \mathbb{R}^n \to \mathbb{R}^n, \qquad \forall s \in \mathbb{R}$$

of C^2 ***diffeomorphisms****. These depend* ***differentiably*** *on s.*
These form a ***subgroup*** *of the* ***diffeomorphism group****, via* ***function composition****, where:*

1. $$\varphi_0(\underline{x}) = \underline{x}, \qquad \forall \underline{x} \in \mathbb{R}^n$$

2. $$\varphi_s \circ \varphi_t = \varphi_{s+t}, \qquad \forall s, t \in \mathbb{R}$$

The properties of the diffeomorphism immediately give us the group structure, where the identity if:

$$\varphi_0 \circ \varphi_s = \varphi_s = \varphi_s \circ \varphi_s$$

the inverse is in the subgroup:

$$(\phi_s)^{-1} = \phi_{-s}$$

and the elements are associative (since function composition is associative).

2.2 Theorem: Noether's Theorem (I)

Let:

$$I[\underline{x}] = \int_0^1 L(\underline{x}, \dot{\underline{x}}, t) dt$$

be an ***action*** *for* ***regular curves****:*

$$\underline{x} : [0,1] \to \mathbb{R}^n$$

and let L be ***invariant*** *under a* ***one-parameter*** *group of* ***diffeomorphisms*** *$\{\varphi_s\}$ (the family is known as a* ***continuous symmetry****).*
Then, the ***Noether charge****:*

$$N(\underline{x}, \dot{\underline{x}}, t) = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}^i} \frac{\partial \varphi_s^i(\underline{x})}{\partial s}\bigg|_{s=0}$$

is ***conserved****; that is, along* ***extrema*** *of I:*

$$\frac{dN}{dt} = 0$$

(Theorem 7.2)

Proof. Let $\underline{x}(t)$ be a solution to the Euler-Lagrange Equation for L. Then, by the above Lemma, $\underline{y}(s,t) = \varphi_s \circ \underline{x}(t)$ also satisfies the Euler-Lagrange Equations:

$$\frac{\partial L}{\partial y^i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{y}^i}$$

Moreover, by assumption, L is invariant under φ_s **for all** $s \in \mathbb{R}$, so it doesn't depend on s. Thus:

$$0 = \frac{dL}{ds} = \sum_{i=1}^{n} \left(\frac{\partial L}{\partial y^i} \frac{\partial y^i}{\partial s} + \frac{\partial L}{\partial \dot{y}^i} \frac{\partial \dot{y}^i}{\partial s} \right)$$

Using the Euler-Lagrange Equation thus implies that:

$$0 = \sum_{i=1}^{n} \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^i}\right) \frac{\partial y^i}{\partial s} + \frac{\partial L}{\partial \dot{y}^i} \frac{\partial \dot{y}^i}{\partial s} \right)$$

But now notice that:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^i}\frac{\partial y^i}{\partial s}\right) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^i}\right)\frac{\partial y^i}{\partial s} + \frac{\partial L}{\partial \dot{y}^i}\frac{d}{dt}\left(\frac{\partial y^i}{\partial s}\right) \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^i}\right)\frac{\partial y^i}{\partial s} + \frac{\partial L}{\partial \dot{y}^i}\frac{\partial}{\partial s}\left(\dot{y}^i\right) \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}^i}\right)\frac{\partial y^i}{\partial s} + \frac{\partial L}{\partial \dot{y}^i}\frac{\partial \dot{y}^i}{\partial s}
\end{aligned}$$

where we have used the fact that $\underline{y}$ is twice continuously differentiable, so we can exchange the order of differentiation.

Hence we have that:

$$0 = \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{y}^i}\frac{\partial y^i}{\partial s}\right)$$

Now, if we evaluate the above expression at $s = 0$, we get that $\underline{y} = \varphi_0(\underline{x}(t)) = \underline{x}(t)$, so:

$$\begin{aligned}
0 &= \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{y}^i}\frac{\partial y^i}{\partial s}\right)\Bigg|_{s=0} \\
&= \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}^i}\frac{\partial \varphi_s^i(\underline{x})}{\partial s}\right)\Bigg|_{s=0} \\
&= \frac{dN}{dt}
\end{aligned}$$

as required

□

3 Exercises

1. **Show that the family $\{\varphi_s\}$ defines a group of transformation isomorphic to $(\mathbb{R}, +)$.**
2. **Consider the Lagrangian:**

$$L = \frac{1}{2}m\|\underline{\dot{x}}\|^2 - V(\underline{x})$$

for plane curves:

$$\underline{x} : [0, 1] \to \mathbb{R}^2$$

Assume that the potential V only depends on $\|\underline{x}\|$. Show that L is invariant under the one-parameter symmetry group:

$$\varphi_s : \mathbb{R}^2 \to \mathbb{R}^2$$

defined by:

$$\varphi_s(\underline{x}) = \begin{pmatrix} x^1\cos(s) - x^2\sin(s) \\ x^1\sin(x) + x^2\cos(s) \end{pmatrix}$$

Find the expression for the Noether charge associtated to this symmetry.

We begin by showing that L is invariant under φ_s. We have that:

$$\dot{y}^i(s,t) = \frac{\partial \varphi_s^i}{\partial x^1}\dot{x}^1 + \frac{\partial \varphi_s^i}{\partial x^2}\dot{x}^2$$

so:

$$\begin{aligned}
\dot{y}^1(s,t) &= \dot{x}^1\cos(s) - \dot{x}^2\sin(s) \\
\dot{y}^2(s,t) &= \dot{x}^1\sin(s) + \dot{x}^2\cos(s)
\end{aligned}$$

Hence:

$$\begin{aligned}
\|\dot{y}\|^2 &= (\dot{x}^1\cos(s) - \dot{x}^2\sin(s))^2 + (\dot{x}^1\sin(s) + \dot{x}^2\cos(s))^2 \\
&= (\dot{x}^1)^2\cos^2(s) - 2\sin(s)\cos(s)\dot{x}^1\dot{x}^2 + (\dot{x}^2)^2\sin^2(s) + (\dot{x}^2)^2\cos^2(s) + 2\sin(s)\cos(s)\dot{x}^1\dot{x}^2 + (\dot{x}^1)^2\sin^2(s) \\
&= (\dot{x}^1)^2 + (\dot{x}^2)^2 \\
&= \|\dot{x}\|^2
\end{aligned}$$

Moreover:

$$
\begin{aligned}
\|\underline{y}\|^2 &= (x^1\cos(s) - x^2\sin(s))^2 + (x^1\sin(x) + x^2\cos(s))^2 \\
&= (x^1\cos(s) - x^2\sin(s))^2 + (x^1\sin(s) + x^2\cos(s))^2 \\
&= (x^1)^2\cos^2(s) - 2\sin(s)\cos(s)x^1x^2 + (x^2)^2\sin^2(s) + (x^2)^2\cos^2(s) + 2\sin(s)\cos(s)x^1x^2 + (x^1)^2\sin^2(s) \\
&= (x^1)^2 + (x^2)^2 \\
&= \|x\|^2
\end{aligned}
$$

In other words:

$$
\begin{aligned}
L(\underline{y}, \dot{\underline{y}}, t) &= \frac{1}{2}m\|\dot{\underline{y}}\|^2 - V(\|\underline{y}\|) \\
&= \frac{1}{2}m\|\dot{\underline{x}}\|^2 - V(\|\underline{x}\|) \\
&= L(\underline{x}, \dot{\underline{x}}, t)
\end{aligned}
$$

Hence, L is invariant under φ_s. In particular, φ_s represent a series of rotations, which means that L is invariant under rotations; that is, paths extremising I will be **rotationally symmetric**.

We now seek to find Noether's charge. We have:

$$
\begin{aligned}
\varphi_s^1 &= x^1\cos(s) - x^2\sin(s) \\
\varphi_s^2 &= x^1\sin(s) + x^2\cos(s)
\end{aligned}
$$

so:

$$
\begin{aligned}
\frac{\partial \varphi_s^1}{\partial s} &= -x^1\sin(x) - x^2\cos(s) \\
\frac{\partial \varphi_s^2}{\partial s} &= x^1\cos(x) - x^2\sin(s)
\end{aligned}
$$

Moreover, if we take the partial derivative of L with respect to $\dot{x}^i$, we don't need to consider V, since it only depends on $\|\underline{x}\|$. Hence:

$$\frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i$$

Thus, by Noether's Theorem:

$$
\begin{aligned}
N &= \frac{\partial L}{\partial \dot{x}^1}\frac{\partial \varphi_s^1}{\partial s} + \frac{\partial L}{\partial \dot{x}^2}\frac{\partial \varphi_s^2}{\partial s}\bigg|_{s=0} \\
&= m\dot{x}^1(-x^1\sin(x) - x^2\cos(s)) + m\dot{x}^2(x^1\cos(x) - x^2\sin(s))\big|_{s=0} \\
&= m(x^1\dot{x}^2 - \dot{x}^1x^2)
\end{aligned}
$$

The **angular momentum** of an object with mass m, position $\underline{x}$ and velocity $\underline{v}$ is defined by:

$$L = m(\underline{x} \times \underline{v})$$

where $\times$ denotes the vector cross product. If we compute the angular momentum for this particle:

$$
\begin{aligned}
L &= m\begin{pmatrix} x^1 \\ x^2 \\ 0 \end{pmatrix} \times \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ x^1\dot{x}^2 - x^2\dot{x}^1 \end{pmatrix}
\end{aligned}
$$

In other words, Noether's Theorem tells us:

$$\textit{rotational symmetry} \iff \textit{conservation of angular momentum}$$