Variational Calculus - Week 4 - Conservative Forces & Hamiltons' Principle of Least Action

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Contents

Based on the notes by Jelle Hartong, Chapters 5 & 6

Last week we explored mechanical systems, and how Newton's Equation can be used to deterministically understand the behaviour of a system. This week we explore how mechanical systems evolve when Newton's Equation is defined by ***conservative forces****, and how this gives rise to* ***conservation principles****, which any physical trajectory must obey.*

1 Conservative Forces and Potentials

1.1 Conserved Quantities

1.1.1 Definition: Integrals of Motion

> *An* ***integral of motion*** *is a* ***function*** *in* ***state space****:*
>
> $$\mathbb{R}^{2N} \to \mathbb{R}$$
>
> *which remains* ***constant*** *along* ***physical trajectories*** *(that is, curves in state space obeying Newton's Equation).*

- **Why are integrals of motion important in mechanics?**
  - integrals of motion relate to **symmetries** of the system via **Noether's Theorem**
  - as such, integrals of motion lead to **conservation laws**, which define the behaviour of the mechanical system

1.1.2 Definition: Conserved Quantity

> *A* ***conserved quantity*** *is an* ***integral of motion*** *which arises from:*
>
> - ***homogeneity of space and time*** *(there is no origin)*
> - ***isotropy of space*** *(there is no direction)*
>
> *In particular, these* ***integrals of motion*** *are* ***additive****: the value of a quantity in the system is the sum of the values of each of its non-ineracting parts.*

- **What is an example of a conserved quantity for a free particle in an inertial frame of reference?**

- the **momentum** of **free particles** in an **inertial frame of reference**:

$$\underline{p}(\underline{x}, \underline{v}) = m\underline{v}$$

- in particular, given n free-particles, the **total momentum** of the system is:

$$\underline{p}_{system} = \sum_{i=1}^{n} m_i \underline{v}_i$$

- **What is *not* an example of a conserved quantity for a free particle in an inertial frame of reference?**
  - the **velocities** of **free particles** in an **inertial frame of reference**
  - the **velocity** of the system is **not** additive, since it is defined in terms of the total momentum of the system:

$$\underline{v}_{system} = \frac{1}{\sum_{i=1}^{n} m_i} p_{system} = \frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} m_i \underline{v}_i \neq \sum_{i=1}^{n} \underline{v}_i$$

- **What is an example of a conserved quantity for a free-falling apple under Galilean gravity?**
  - recall, under Galilean gravity the equations of motion are:

$$m\ddot{z} = -mg \qquad v = \dot{z}$$

  which has solution:

$$z(t) = z_0 + tv_0 - \frac{1}{2}gt^2 \qquad v(t) = v_0 - gt$$

  - then, define the **energy** as:

$$E(z, v) = \frac{1}{2}mv^2 + mgz$$

  - then:

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\left(\frac{1}{2}mv^2 + mgz\right) \\ &= mv\dot{v} + mg\dot{z} \\ &= mv\ddot{z} + mgv \\ &= v(m\ddot{z} + mg) \\ &= 0 \end{aligned}$$

  - hence, **energy** remains constant across time

- **Why does energy conservation define the mechanics of the falling apple?**
  - since energy remains constant, we have:

$$E(z, v) = E(z(0), v(0)) = \frac{1}{2}mv_0^2 + mgz_0$$

  - if we rearrange our expression for E to find z, we get:

$$\begin{aligned} E(z, v) = \frac{1}{2}mv^2 + mgz \implies z &= -\frac{v^2}{2g} + \frac{E}{gm} \\ &= -\frac{v^2}{2g} + \frac{\frac{1}{2}mv_0^2 + mgz_0}{gm} \\ &= z_0 + \frac{v_0^2 - v^2}{2g} \end{aligned}$$

– and if we rearrange z to make it a function of v, using $t = -\frac{v-v_0}{g}$:

$$\begin{aligned} z &= z_0 - \frac{v - v_0}{g} v_0 - \frac{1}{2} g \left(-\frac{v - v_0}{g} \right)^2 \\ &= z_0 + \frac{2v_0^2 - 2vv_0 - (v^2 - 2vv_0 + v_0^2)}{2g} \\ &= z_0 + \frac{v_0^2 - v^2}{2g} \end{aligned}$$

– hence, simply knowing that energy is conserved gives us the full motion z of the apple - that is, we solved Newton's equation, without even looking at it!

1.2 Conservative Forces

The Galilean gravity was caused by a constant gravitational force, which lead to conservation of energy, which in turn "defined" the mechanics of the system. Such forces are what are known as ***conservative force fields****, and they lead to nice conservation laws, as we will see.*

1.2.1 Definition: Conservative Force Field

A ***conservative force field*** *is a* ***vector field****:*

$$F : \mathbb{R}^3 \to \mathbb{R}^3$$

if it can be defined as the ***negative*** *of the* ***gradient*** *of a* ***scalar field****:*

$$V : \mathbb{R}^3 \to \mathbb{R}$$

That is:

$$F = -\nabla V$$

1.2.2 Definition: Potential

The ***potential*** *is the scalar field:*

$$V : \mathbb{R}^3 \to \mathbb{R}$$

defining a ***conservative force field****.*
More generally, if we have an n-particle system with configuration space $\mathbb{R}^N$*, a* ***potential*** *is a* ***scalar field****:*

$$U : \mathbb{R}^N \to \mathbb{R}$$

such that we can write Newton's equation as:

$$\ddot{\underline{x}} = -\nabla U$$

- **Why are potentials defined up to a constant?**
  - because 2 potentials V, V' differing by a constant lead to the **same** conservative force field:
$$V' = V + k \implies \nabla V' = \nabla V$$
- **What is the gravitational potential in Galilean gravity?**
  - the gravitational force in Galilean gravity was defined by:
$$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial y} = 0 \qquad \frac{\partial F}{\partial z} = -mg$$
  - hence, the potential must've been:
$$V(x, y, z) = mgz$$
  - it is no coincidence that the energy contained the term mgz, as we will see below
- **Why are potentials in conservative force fields described as instantaneous?**
  - the **potential** is defined as a **scalar field** from the **configuration space**
  - it **only** depends on the position of the particles
  - thus, any **sudden** change to the particle positions will be **propagated instantly**, and all other particles will "feel" it (since ∇U defines the position of all particles)

1.3 Lemma: Energy is Conserved in Conservative Force Fields

*Define the **energy** of a system:*
$$E(\underline{x}, \underline{v}) = \frac{1}{2}m\|\underline{v}\|^2 + V(\underline{x})$$
*Then, if F is a **conservative force field**, the **energy** $E(\underline{x}, \underline{v})$ is **conserved** along **physical trajectories**.*
*In other words, **physical trajectories** lie on constant energy surfaces in state space:*
$$\{(\underline{x}, \underline{v}) \mid E(\underline{x}, \underline{v}) = E_0\}$$

Proof. By Newton's Equation (the version in which there is time-independent force field in an inertial frame of reference), any physical trajectory satisfies:
$$m\ddot{\underline{x}} = -\nabla V(\underline{x})$$
Moreover:
$$\begin{aligned}\frac{d}{dt}\|\underline{v}\|^2 &= \frac{d}{dt}\left(\sum_{i=1}^{n} v_i^2\right) \\ &= \sum_{i=1}^{n} 2v_i\dot{v}_i \\ &= 2\langle \underline{v}, \dot{\underline{v}}\rangle\end{aligned}$$

$$\frac{d}{dt}V(\underline{x}) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i$$
$$= \langle \dot{\underline{x}}, \nabla V(\underline{x}) \rangle$$

So if we differentiate the energy:

$$\frac{dE}{dt} = m \langle \underline{v}, \dot{\underline{v}} \rangle + \langle \dot{\underline{x}}, \nabla V(\underline{x}), \rangle$$
$$= \langle \underline{v}, m\dot{\underline{v}} \rangle + \langle \underline{v}, \nabla V(\underline{x}) \rangle$$
$$= \langle \underline{v}, m\ddot{\underline{x}} + \nabla V(\underline{x}) \rangle$$
$$= \langle \underline{v}, \underline{0} \rangle$$
$$= 0$$

Hence, in a conservative force field the energy remains constant, as required.

□

2 Deriving Simple Harmonic Motion: Points of Equilibrium

2.1 Definition: Points of Equilibrium

A ***point of equilibrium*** *is a point at which particles feel no force:*

$$F(\underline{x}_0) = 0, \qquad \underline{x}_0 \in \mathbb{R}^3$$

In ***conservative force fields***, ***points of equilibrium*** *are the* ***critical points*** *of the potential, since* $\nabla V(\underline{x}_0) = \underline{0}$.

2.2 Hooke's Law and Simple Harmonic Motion

- **How does the potential behave close to a critical point?**
  - consider a 1-dimensional system defined by a conservative force field:

$$m\ddot{x} = -\frac{dV}{dx}$$

  - without loss of generality:
    * $x_0 = 0$ is a critical point (we can just define a suitable change of basis)
    * $V(x_0) = 0$ (since $\frac{dV(0)}{dx} = 0$, then $V(0) = k$ for some k, but potentials are defined up to a constant, so we can just set $V(0) = 0$)
  - to understand the behaviour of $V(x)$ close to $x_0 = 0$, we can consider its Taylor Expansion:

$$V(x) = V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2 + \mathcal{O}(x^3) \approx \frac{1}{2}V''(x_0)x^2$$

  - hence, for small displacements about x_0:

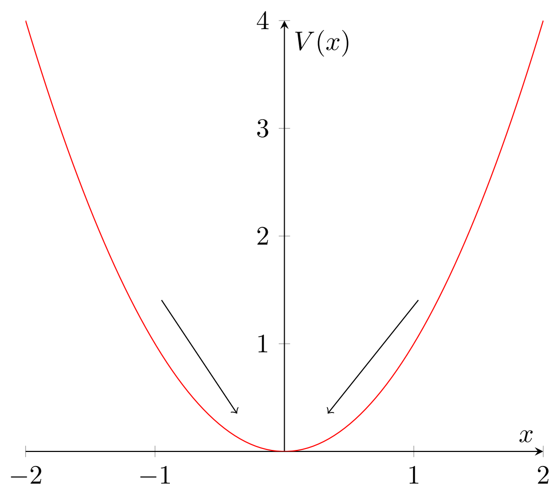
$$V(x) = \frac{1}{2}kx^2, \qquad k = V(x_0)$$

- **What is Hooke's Law?**

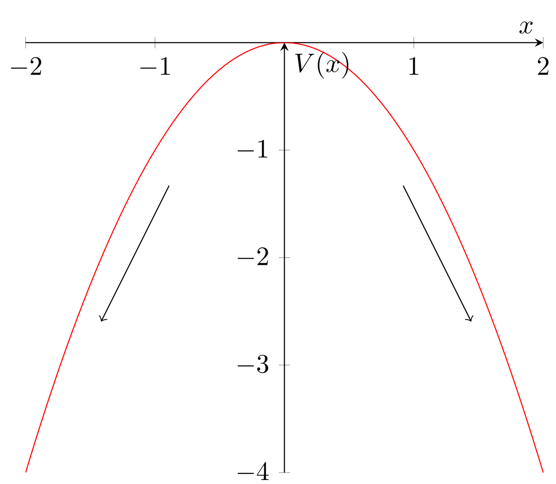
- the equation derived by applying Newton's Equation to the above system:

$$m\ddot{x} = -kx$$

- **What is a degenerate critical point?**
  - if $k = 0$ above, $x_0 = 0$ is a **degenerate critical point**
  - this is because we can't establish any result about stability at x_0 (since $F(x) = -kx = 0$ for any x)
- **What is a stable critical point?**
  - x_0 is a **stable critical point** if $k > 0$ (that is, x_0 is a **local minimum** of the potential $V(x)$)
  - when $k > 0$, solutions become oscillatory and stable (since as x increases, $\ddot{x}$ decreases, pulling the body back towards the origin)



- **What is an unstable critical point?**
  - x_0 is an **unstable critical point** if $k < 0$ (that is, x_0 is a local maximum of the potential $V(x)$)
  - when $k < 0$, moving towards larger values of $|x|$ leads to ever increasing accelerations (i.e the particle will move faster and faster away from x_0)



- **What is simple harmonic motion?**
  - the oscillatory motion arising from stable critical points for the conservative force field:

$$F(x) = \frac{1}{2}kx^2$$

2.2.1 Deriving the Equation for SHM

(1) **Solving the ODE**

We have to solve the second order, homogeneous ODE:

$$m\ddot{x} = -kx$$

The characteristic equation is:

$$mr^2 + k = 0 \implies r = \pm\sqrt{-\frac{k}{m}}$$

Since $k > 0$ (stable critical point), the roots will be imaginary. Let $\omega^2 = \frac{k}{m}$. Then the general solution becomes:

$$x(t) = A\cos\omega t + B\sin\omega t$$

If $x(0) = x_0$ and $\dot{x}(0) = v_0$, then we differentiate:

$$\dot{x}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

Hence:

$$x_0 = A \qquad v_0 = B\omega \implies V = \frac{v_0}{\omega}$$

Hence, SHM is defined by:

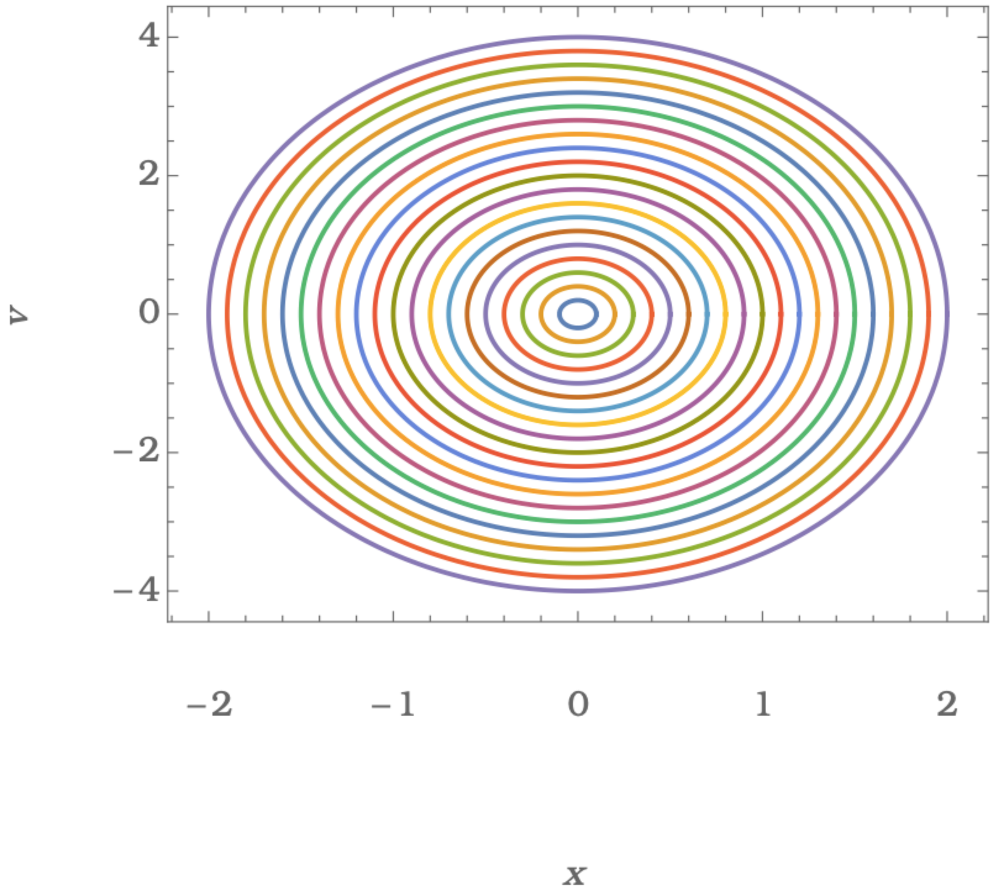
$$x(t) = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

(2) **Geometric Reasoning**

Since $F = -kx$ is a conservative force field, energy is conserved:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Notice, the the physical trajectories in state space will be **ellipses**, each corresponding to a constant energy:



This tells us that x, v should be periodic. If we divide the above by E:

$$1 = \frac{m}{2E}v^2 + \frac{k}{2E}x^2$$

But then, for some function $\alpha(t)$ we can write:

$$x(t) = \sqrt{\frac{2E}{k}}\sin(\alpha(t))$$

$$v(t) = \sqrt{\frac{2E}{m}}\cos(\alpha(t))$$

since $\sin^2(\alpha(t)) + \cos^2(\alpha(t)) = 1$. Finally, we just need to enforce that $\dot{x} = v$:

$$\dot{\alpha}\sqrt{\frac{2E}{k}}\cos(\alpha(t)) = \sqrt{\frac{2E}{m}}\cos(\alpha(t)) \iff \dot{\alpha}\sqrt{\frac{2E}{k}} = \sqrt{\frac{2E}{m}}$$

which imposes:

$$\dot{\alpha}^2 = \frac{k}{m} \implies \alpha(t) = \sqrt{\frac{k}{m}}t + \varphi$$

As above, setting $\omega = \sqrt{\frac{k}{m}}$ we get that:

$$x(t) = \sqrt{\frac{2E}{k}}\sin\left(\omega t + \varphi\right)$$

If in ① we use the formula:

$$a\cos\theta + b\sin\theta = c\sin(\theta + \phi)$$

where:

$$c = sgn(a)\sqrt{a^2 + b^2} \qquad \phi = \arctan\left(-\frac{b}{a}\right)$$

we get the result for ②.

3 Energy Conservation in a 1-Dimensional Conservative System

Throughout this section we consider 1 dimensional systems, to simplify explanations.

3.1 Applying Energy Conservation: Potential Barriers

- **What is a potential barrier?**
  - the **energy** in a **conservative** system is defined by:

$$E = \frac{1}{2}mv^2 + V(x)$$

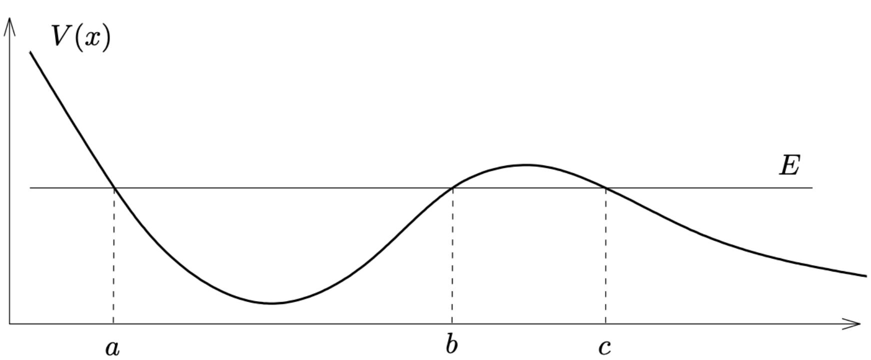
  - notice, since $v^2 \geq 0$, we get a restriction on the **potential**:

$$E \geq V(x), \qquad \forall x$$

  - this means that, since E is conserved in physical trajectories, trajectories for which we would have $V(x) > E$ are not feasible
  - that is, there is a **potential barrier**, which can only be trespassed if E is large enough
- **What is a turning point?**
  - a point at which:

$$E = V(x)$$

  - these occurs at those points in which $v = 0$
- **How does a particle behave at a turning point?**
  - consider the following potential:



- there are 3 turning points, and 2 intervals on which physical trajectories are defined: $[a, b]$ and $[c, \infty)$
- if $x \in [a, b]$, then the particle will **oscillate**:
  * say a particle moves from left to right $a \to b$
  * upon reaching b, its velocity becomes 0
  * it can't continue moving past b, since there is a potential barrier
  * it can't stay in place, since $\nabla V \neq \underline{0}$ (as b isn't a critical point), so it must keep moving
  * the only possibility is for the particle to turn back and start moving towards a (this is why when $v = 0$ we have **turning points**: particles will turn back at these points)
  * the same happens if the particle moves from right to left
  * hence, the particle must oscillate between its 2 turning points
- if $x \in [c, \infty)$ the particle will undergo **unbounded motion** ($x \to \infty$):
  * if already moving from left to right, then the particle will keep moving further right
  * if moving from right to left, the particle eventually reaches the turning point c, so it will turn back and go to infinity

3.1.1 Lemma: Oscillatory Motion at Turning Points

If a particle is stuck between 2 turning points, it will undergo ***oscillatory, periodic motion****.*

Proof. Let a, b be 2 turning points, such that $\forall x \in [a, b]$:

$$V(x) \leq E$$

with equality only at $x = a$ and $x = b$.

We first show that $x(t)$ behaves periodically. Without loss of generality, assume that $x(0) = a$. Then, since a is a turning point, $\dot{x}(0) = 0$. Furthermore, assume that at some point later, $t = T$, we again have $x(T) = a$. Then, $\dot{x}(T) = 0$. Notice, the function:

$$x_T(t) = x(t * T)$$

will solve any ODE that x solves. Moreover, $x_T(0) = a, \dot{x}_T(0) = 0$, so x_T satisfies the same initial conditions as x. Thus, by the uniqueness theorem, $x_T = x$, so:

$$x(t) = x(t + T)$$

and x is periodic.

We can even find the period T. The energy is defined as:

$$E = \frac{1}{2} m \|\dot{x}\|^2 + V(x)$$

We can rearrange for the velocity:

$$\frac{dx}{dt} = \pm \sqrt{\frac{2(E - V(x))}{m}}$$

So:

$$dt = \sqrt{\frac{m}{2}} \times \frac{dx}{\sqrt{E - V(x)}}$$

Hence, the time taken for a particle to move from a to b (which will be positive) must be given by:

$$T(a \to b) = \int_{t=0}^{T/2} dt = \sqrt{\frac{m}{2}} \int_a^b \frac{dx}{\sqrt{E - V(x)}}$$

But Newton's Equation in an inertial frame of reference, under a conservative force, is time invariant, so:

$$T(a \to b) = T(b \to a)$$

so the period will be:

$$T = 2T(a \to b) = \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E - V(x)}}$$

□

3.1.2 Verifying Period for SHM

Above, we found that the frequency of oscillation in SHM was given by:

$$\omega = \sqrt{\frac{k}{m}}$$

From physics, we know that period and frequency are related by:

$$T = \frac{2\pi}{\omega}$$

we verify this now.

As a potential we use $V(x) = \frac{1}{2}kx^2$. Then:

$$\begin{aligned} T &= \sqrt{2m} \int_a^b \frac{dx}{\sqrt{E - \frac{1}{2}kx^2}} \\ &= 2\sqrt{\frac{m}{k}} \int_a^b \frac{dx}{\sqrt{\frac{2E}{k} - x^2}} \end{aligned}$$

Now, let:

$$x = \sqrt{\frac{2E}{k}} \sin\theta$$

Then:

$$\frac{dx}{d\theta} = \sqrt{\frac{2E}{k}} \cos\theta$$

so:

$$T = 2\sqrt{\frac{m}{k}} \int_a^b \frac{dx}{\sqrt{\frac{2E}{k} - x^2}}$$
$$= 2\sqrt{\frac{m}{k}} \int \frac{\sqrt{\frac{2E}{k}} \cos\theta}{\sqrt{\frac{2E}{k} - \frac{2E}{k}\sin^2\theta}} d\theta$$
$$= 2\sqrt{\frac{m}{k}} \int \frac{\cos\theta}{1 - \sin^2\theta} d\theta$$
$$= 2\sqrt{\frac{m}{k}} \int d\theta$$
$$= 2\sqrt{\frac{m}{k}}\theta$$

By the above substitution:

$$\theta = \arcsin\left(\sqrt{\frac{k}{2E}}x\right)$$

a and b are turning points, so $E = V(a) = V(b)$. Since a, b are distinct:

$$ka^2 = kb^2 \iff a = -b$$

Thus:

$$T = 2\sqrt{\frac{m}{k}} = \arcsin\left(\sqrt{\frac{k}{2V(b)}}b\right) - \arcsin\left(\sqrt{\frac{k}{2V(a)}}a\right)$$
$$= \arcsin\left(\sqrt{\frac{k}{kb^2}}b\right) - \arcsin\left(-\sqrt{\frac{k}{kb^2}}a\right)$$
$$= 2\arcsin(1)$$
$$= \pi$$

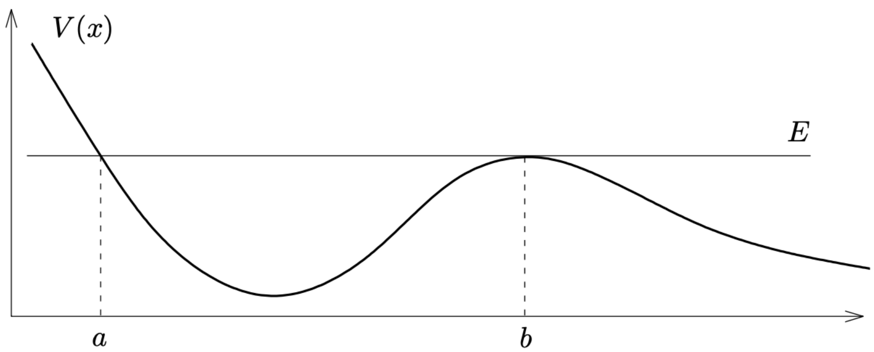
Hence:

$$T = \frac{2\pi}{\sqrt{\frac{k}{m}}} = \frac{2\pi}{\omega}$$

as expected.

3.2 Potential Barriers at Points of Equilibrium

- **How does a particle behave when the turning point is a critical point?**
  - lets consider a potential $V(x)$ and an energy E, such that there is a **turning point** at a **critical point** of V:



- notice, at a critical point, $v = 0$, and $F = 0$, so any particle which reaches a critical point must necessarily remain there forever
- however, in practice, the particle will **never** reach this turning point - or rather, it would take infinite time for this to happen

3.2.1 Lemma: Period of Oscillation at Critical Points

Consider a particle moving between turning points a, b. If b is also a critical point, such that $V'(b) = 0$, then it takes "infinite time" for the particle to reach b. In other words, the following integral diverges:

$$T(a \to b) = \sqrt{\frac{m}{2}} \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} \frac{dx}{\sqrt{E - V(x)}}$$

Proof. Since we are interested at how $V(x)$ looks like close to b (since we integrate to $b - \varepsilon$ as $\varepsilon \to 0^+$), we can consider its Taylor expansion at b:

$$V(x) = V(b) + V'(b)(x-b) + \frac{1}{2}V''(b)(x-b)^2 + \mathcal{O}(x^3)$$

We can thus approximate:

$$V(x) = E + \frac{1}{2}V''(b)(x-b)^2$$

If this approximation is "valid" in some neighbourhood of b, $[b^*, b - \varepsilon$ (Gods of analysis please forgive me for such heretic language and hand-wavy explanations), we can write:

$$
\begin{aligned}
T(a \to b) = \sqrt{\frac{m}{2}} \left(\int_a^{b^*} \frac{dx}{\sqrt{E - V(x)}} + \lim_{\varepsilon \to 0^+} \int_{b^*}^{b-\varepsilon} \frac{dx}{\sqrt{E - V(x)}} \right) \\
&\geq \sqrt{\frac{m}{2}} \lim_{\varepsilon \to 0^+} \int_{b^*}^{b-\varepsilon} \frac{dx}{\sqrt{E - V(x)}} \\
&= \sqrt{\frac{m}{2}} \lim_{\varepsilon \to 0^+} \int_{b^*}^{b-\varepsilon} \frac{dx}{\sqrt{\frac{1}{2}V''(b)(x-b)^2}} \\
&= \sqrt{\frac{m}{V''(b)}} \lim_{\varepsilon \to 0^+} \int_{b^*}^{b-\varepsilon} \frac{dx}{x-b} \\
&= \sqrt{\frac{m}{V''(b)}} \lim_{\varepsilon \to 0^+} [ln|x-b|]_{b^*}^{b-\varepsilon} \\
&= \sqrt{\frac{m}{V''(b)}} \lim_{\varepsilon \to 0^+} [ln\varepsilon + ln|b^* - b|]
\end{aligned}
$$

which diverges, since:

$$\lim_{\varepsilon \to 0^+} ln\varepsilon = -\infty$$

□

4 Hamilton's Principle of Least Action

The ***inverse problem in the calculus of variations*** *concerns determining whether a differential equation is an Euler–Lagrange equation of a functional and, if so, determining the functional. Hamilt'ns Principle of Least Action tells us that Newton's Equation can be thought of as the Euler lagarange equation of the action functional, which gives us a variational way of solving mechanical problems.*

4.1 Lemma: Hamilton's Principle of Least Action

Consider a particle of mass m moving through a ***conservative force field****:*

$$F : \mathbb{R}^3 \to \mathbb{R}^3$$

with potential V.
Solving Newton's equation:

$$m\ddot{\underline{x}} = -\nabla V(\underline{x})$$

is equivalent to ***extremising*** *the* ***action functional****, with Lagrangian*

$$L(\underline{x}, \dot{\underline{x}}, t) = T - V = \frac{1}{2}m\|\dot{\underline{x}}\|^2 - V(\underline{x})$$

Proof. We consider the Euler-Lagrange Equations for the Lagrangian. Since it doesn't depend explicitly on t, we can use Beltrami's Identity:

$$E' = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L$$

which tells us that if $\underline{x}$ satisfies the Euler-Lagrange equation, E' is conserved. We thus compute:

$$\begin{aligned}
E' &= \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial \dot{x}^i} \left(\frac{1}{2} m \|\underline{\dot{x}}\|^2 - V(\underline{x}) \right) \dot{x}^i \frac{1}{2} m \|\underline{\dot{x}}\|^2 + V(\underline{x}) \\
&= \sum_{i=1}^{n} \frac{1}{2} (2m\dot{x}^i) \dot{x}^i - \frac{1}{2} m \|\underline{\dot{x}}\|^2 + V(\underline{x}) \\
&= m\|\underline{\dot{x}}\|^2 - \frac{1}{2} m \|\underline{\dot{x}}\|^2 + V(\underline{x}) \\
&= \frac{1}{2} m \|\underline{\dot{x}}\|^2 + V(\underline{x})
\end{aligned}$$

But notice, this is precisely the expression for energy E. If E' is conserved along paths satisfying the Euler-Lagrange Equation, then the energy E in the conservative force field is also conserved, which implies that the solution to the Euler-Lagrange equations will also be solutions to Euler's Equation. Thus, physical trajectories in state space correspond to solutions to the Euler-Lagrange equations, as required.

□

- **How can we use the Lagrangian to construct physical quantities?**
  - the **force** is:
    $$\frac{\partial L}{\partial x^i}$$
  - the **momentum** is:
    $$\frac{\partial L}{\partial \dot{x}^i}$$
  - the **energy** is:
    $$\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L$$
- **Why is the Lagrangian approach to mechanics more beneficial than Newton's?**
  1. all the physical quantities can be derived directly from the Lagrangian
  2. the Lagrangian is a scalar field (since it uses energies), whilst Newton's equations are vector-valued (since it uses forces)
  3. Newton's equation requires using an inertial frame of reference; Euler-Lagrange are valid in any coordinate system

4.2 Worked Example: Motion of a Planar Pendulum

4.3 Worked Example: Motion of a Planar Pendulum with Moving Pivot

5 Workshop

Consider a bead sliding along a curve on a vertical plane from point A to point B. We work in the $(,xy9)$ plane with gravity acting down along the y-direction. We are free to chose the origin of the coordinate systeam at a point A, so that:

$$A = (0, 0)$$

The total time it takes the bead to slide down a curve is:

$$T = \int_0^L \frac{ds}{v(s)}$$

where L denotes the arclength of the curve γ, s is the arclength parameter, and v is the speed of the bead s units down the curve from A. Since we don't know the value of L we need to rephrase the problem. Let us assume that the curve γ is the graph of y as a function of x; that is, we parametrise the curve by x, with a typical point on the curve being $(x, y(y(x))$.

1. **Use conservation of energy to express the speed v as a function of y, and in this way arrive at the time functional:**

$$T[y] = \frac{1}{\sqrt{2g}} \int_0^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{-y}} dx$$

where the constant g is the acceleration due to gravity and the final point is $B = (x_1, y(x_1))$.

Under Galilean gravity, the energy (up to a constant) is:

$$E = mgy + \frac{mv^2}{2}$$

The particle starts at rest ($v = 0$) from the origin ($y = 0$); in other words, since energy is conserved:

$$E = 0$$

It thus follows that:

$$v = \sqrt{-2gy}$$

Now, we know that:

$$ds = \sqrt{dx^2 + dy^2} \implies ds = \sqrt{1 + (y')^2} dx$$

Thus:

$$T = \int_0^L \frac{ds}{v(s)} \implies T = \frac{1}{\sqrt{2g}} \int_0^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{-y}} dx$$

2. **The first person to ever think about the problem was Galileo, and he already noticed that the straight line was not the fastest. Compute the time it takes for the bead to slide along the straight line segment from $A = (0, 0)$ to $B = (1, -1)$. Express your answer as a function of g.**

A general straight line is defined by:

$$y = mx + c$$

Since it passes through $(0, 0)$:

$$c = 0$$

and since it passes through $(1, -1)$:

$$-1 = m$$

Hence:

$$y = -x$$

Thus:

$$T = \frac{1}{\sqrt{2g}} \int_0^1 \frac{\sqrt{1 + (-1)^2}}{\sqrt{x}} dx = \frac{1}{\sqrt{g}} \int_0^1 \frac{1}{\sqrt{x}} dx = \frac{2}{\sqrt{g}} \left[\sqrt{x}\right]_0^1 = \frac{2}{\sqrt{g}}$$

3. **Now, consider the circular segment:**

$$y = -\sqrt{1 - (x-1)^2}$$

from $A = (0, 0)$ to $B = (1, -1)$ Show that the time taken by the bead along this curve is less than that taken along the straight line.

Again, raw computation:

$$y' = -\frac{1}{2\sqrt{1 - (x-1)^2}} \times -2(x-1) = \frac{x-1}{\sqrt{1 - (x-1)^2}}$$

so:

$$(y')^2 = \frac{(x-1)^2}{1 - (x-1)^2} \implies 1 + (y')^2 = \frac{1}{1 - (x-1)^2}$$

Thus:

$$\begin{aligned}
T &= \frac{1}{\sqrt{2g}} \int_0^1 \frac{\sqrt{\frac{1}{1-(x-1)^2}}}{\sqrt{\sqrt{1 - (x-1)^2}}} dx \\
&= \frac{1}{\sqrt{2g}} \int_0^1 \frac{(1 - (x-1)^2)^{-1/2}}{(1 - (x-1)^2)^{1/4}} dx \\
&= \frac{1}{\sqrt{2g}} \int_0^1 (1 - (x-1)^2)^{-3/4} dx
\end{aligned}$$

Now, let:

$$x - 1 = -\cos\theta \implies \frac{dx}{d\theta} = \sin\theta$$

and the integration bounds become:

$$(0) - 1 = -\cos\theta \implies \theta = 0$$

$$(1) - 1 = -\cos\theta \implies \theta = \frac{\pi}{2}$$

and so:

$$\begin{aligned}
T &= \frac{1}{\sqrt{2g}} \int_0^{\frac{\pi}{2}} (1 - (-\cos\theta)^2)^{-3/4} \sin\theta d\theta \\
&= \frac{1}{\sqrt{2g}} \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{-3/4} \sin\theta d\theta \\
&= \frac{1}{\sqrt{2g}} \int_0^{\frac{\pi}{2}} (\sin\theta)^{-1/2} d\theta
\end{aligned}$$

This has solution:

$$T \approx \frac{1.85407}{\sqrt{g}} \approx 2.62206 < \frac{2}{\sqrt{g}}$$

(requires elliptic integrals)

A more general question would ask for the circle which leads to the shortest time. The general equation of a point going through $(0,0)$ and $(1,-1)$ is parametrised by t:

$$(x-(t+1))^2 + (y-t)^2 = (t+1)^2 + t^2$$

or:

$$(y-t)^2 = t^2 - x^2 + 2x(t+1)$$

If we optimise for t (numerically) we find that:

$$t \approx 0.28866955254$$

such that:

$$T \approx \frac{1.8317}{\sqrt{g}}$$

4. **Galileo had assumed (incorrectaly, as we will see) that the circular arc was the vbrachistochrone curve. Of course, he did not have the calculus of variations at his disposal. Write down the Euler-Lagrange equation for the time functional:**

$$T[y] = \int_0^{x_1} \frac{\sqrt{1+(y')^2}}{\sqrt{-2gy}}\,dx$$

and show that:

$$c = \sqrt{1+(y')^2}\sqrt{-y}$$

is constant along the curve. Solve for y', remembering that $y' < 0$, since the bead is falling.

It is equivalent to extremise the functional with lagrangian:

$$L = \frac{\sqrt{1+(y')^2}}{\sqrt{-y}}$$

Thus:

$$\frac{\partial L}{\partial y} = -\frac{1}{2} \times \frac{\sqrt{1+(y')^2}}{(-y)^{3/2}} \times -1 = \frac{\sqrt{1+(y')^2}}{2(-y)^{3/2}}$$

$$\frac{\partial L}{\partial y'} = \frac{1}{2} \times \frac{1}{\sqrt{-y}\sqrt{1+(y')^2}} \times 2y' = \frac{y'}{\sqrt{-y}\sqrt{1+(y')^2}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial y'} = \frac{y''\sqrt{-y}\sqrt{1+(y')^2} - \left(-\sqrt{1+(y')^2}\frac{1}{2\sqrt{-y}} + \sqrt{-y}\frac{y'y''}{\sqrt{1+(y')^2}}\right)}{(-y)(1+(y')^2)}$$

Overall, this gives:

$$2yy'' + (y')^2 + 1 = 0$$

Sicne the Lagrangian doesn't involve x, the energy is constant along extremal paths, so by Beltrami's Identity:

$$y'\frac{\partial L}{\partial y'} - L = c_0 \implies \frac{(y')^2}{\sqrt{-y}\sqrt{1+(y')^2}} - \frac{\sqrt{1+(y')^2}}{\sqrt{-y}} = -\frac{1}{\sqrt{1+(y')^2}}\sqrt{-y} = c_0$$

Hence, as required:

$$\sqrt{1+(y')^2}\sqrt{-y} = c = -\frac{1}{c_0}$$

must be constant.

We can now solve for y':

$$\begin{aligned}
&\sqrt{1+(y')^2}\sqrt{-y} = c \\
\implies &(1+(y')^2)(-y) = c^2 \\
\implies &(y')^2 = -\frac{c^2}{y} - 1 \\
\implies &y' = -\sqrt{-\frac{c^2+y}{y}}
\end{aligned}$$

where we take the negative root, since $y' < 0$.

5. **Introduce ϕ via:**

$$\tan\phi = \sqrt{-\frac{y}{c^2+y}}$$

and show that the brachistochrone is given parametrically by:

$$x(\phi) = c^2(\phi - \sin\phi\cos\phi)$$

and

$$y(\phi) = -c^2\sin^2\phi$$

where c is determined in such a way for some $\phi_1, (x(\phi_1), y(\phi_1)) = (1, -1)$. Show that the time taken by teh bead along this curve is:

$$T = \frac{\sqrt{2}c\phi_1}{\sqrt{g}}$$

How much faster is this compared to the circular arc? Or the straight line?

We now try solving for y:

$$\frac{dy}{dx} = -\sqrt{-\frac{c^2+y}{y}} \implies -\int\sqrt{-\frac{y}{c^2+y}}dy = x + d$$

Defining

$$\tan\phi = \sqrt{-\frac{y}{c^2+y}}$$

it follows that:

$$\begin{aligned}
&\tan^2\phi = -\frac{y}{c^2+y} \\
\implies &\tan^2\phi(c^2+y) + y = 0 \\
\implies &c^2\tan^2\phi + y(1+\tan^2\phi) = 0 \\
\implies &c^2\tan^2\phi + y\sec^2\phi = 0 \\
\implies &y = -c^2\frac{\tan^2}{\sec^2\phi} \\
\implies &y = -c^2\sin^2\phi
\end{aligned}$$

Personally, it seems more natural to start with this substitution, and then deriving the expression for $\tan\phi$*, but this is what the question asks.*

This tells us that:

$$\frac{dy}{d\phi} = -2c^2\sin\phi\cos\phi$$

Hence:

$$\begin{aligned}
x + d &= -\int\sqrt{-\frac{y}{c^2+y}}dy \\
&= 2c^2\int\tan\phi\sin\phi\cos\phi d\phi \\
&= 2c^2\int\sin^2\phi d\phi \\
&= c^2\int 1-\cos 2\phi d\phi \\
&= c^2\left(\phi - \frac{\sin 2\phi}{2}\right) \\
&= c^2\left(\phi - \sin\phi\cos\phi\right)
\end{aligned}$$

Now, the curve passes through the origin, which implies that:

$$0 = -c^2\sin^2\phi \implies \phi = 0$$

and so:

$$0 + d = 0 \implies d = 0$$

Thus, we get that:

$$\begin{aligned}
x(\phi) &= c^2\left(\phi - \sin\phi\cos\phi\right) \\
y(\phi) &= -c^2\sin^2\phi
\end{aligned}$$

as required.

To compute the time it takes, there are 2 ways:

① **Smart Way (from Solutions)**

Notice, we know that:

$$y' = -\sqrt{-\frac{c^2+y}{y}}$$

so:

$$(y')^2 = -\frac{c^2+y}{y}$$

and:

$$1+(y')^2 = -\frac{c^2}{y}$$

Hence:

$$\frac{\sqrt{1+(y')^2}}{\sqrt{-y}} = -\frac{c}{y}$$

Finally, using the parametrisation for x:

$$\frac{dx}{d\phi} = c^2(1+\sin^2\phi - \cos^2\phi) = 2c^2\sin^2\phi$$

Thus, we have:

$$\begin{aligned} T &= \frac{1}{\sqrt{2g}}\int_0^1 \frac{\sqrt{1+(y')^2}}{\sqrt{-y}}dx \\ &= -\frac{1}{\sqrt{2g}}\int_0^1 -\frac{c^2}{y}dx \\ &= \frac{1}{\sqrt{2g}}\int_0^{\phi_1}\frac{1}{\sin^2\phi}(2c^2\sin^2\phi)d\phi \\ &= \frac{\sqrt{2}c^2}{\sqrt{g}}\int_0^{\phi_1} 1 d\phi \\ &= \frac{\sqrt{2}c\phi_1}{\sqrt{g}} \end{aligned}$$

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② **Dumb Way (Mine)**

We can rewrite:

$$\sqrt{1+(y')^2}dx = \sqrt{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2}d\phi$$

We know that:

$$\frac{dx}{d\phi} = 2c^2\sin^2\phi$$

$$\frac{dy}{d\phi} = -2c^2\sin\phi\cos\phi$$

so:

$$\sqrt{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2} = \sqrt{4c^4\sin^4\phi + 4c^4\sin^2\phi\cos^2\phi} = 2c^2\sqrt{\sin^2\phi(\sin^2\phi+\cos^2\phi)} = 2c^2\sin\phi$$

Similarly:

$$\sqrt{-y} = \sqrt{c^2 \sin^2 \phi} = c \sin \phi$$

so:

$$T = \frac{1}{\sqrt{2g}} \int_0^{\phi_1} \frac{2c^2 \sin \phi}{c \sin \phi} d\phi = \frac{\sqrt{2} c \phi_1}{\sqrt{g}}$$

Numerically, we find that:

$$\phi_1 \approx 1.260601$$

so:

$$T \approx \frac{1.82568}{\sqrt{g}}$$

6. **Consider now a slight variant of the Brachistochrone problem:**

   (a) (3 marks) **Find the curve along which the bead, starting at rest from the origin, will reach the line $x = 1$ in the shortest time.**

   We know that the brachistochrone is a curve parametrised by:

   $$x(\theta) = a(\theta - \sin(\theta)) \qquad y(\theta) = a(\cos(\theta) - 1)$$

   We seek to find a value of a, such that the resulting brachistochrone results in the bead reaching the line $x = 1$ the fastest. Recall from lectures, this is a variable endpoint problem, which imposes that the optimal brachistochrone traces a curve which incides *orthogonally* on the line $x = 1$. In other words, we require that when $x = 1$:

   $$y'(\theta) = 0$$

   since $x = 1$ is a vertical line parallel to the y-axis (the value of x' at the line is irrelevant). Hence, we compute:

   $$y'(\theta) = 0 \implies a(-\sin(\theta)) = 0 \implies \sin(\theta) = 0$$

   The smallest, positive $\theta \in (0, 2\pi]$ satisfying this is:

   $$\theta = \pi$$

   (if $\theta = 0$, then $(x, y) = (0, 0)$ so we can't have $x = 1$).

   Thus, the brachistochrone is perpendicular to $x = 1$ when $\theta = \pi$. At this point, we must have that $x(\pi) = 1$, so:

   $$x(\pi) = 1 \implies a(\pi - \sin(\pi)) = 1 \implies a = \frac{1}{\pi}$$

   Hence, the curve along which the bead reaches $x = 1$ in the shortest time is parametrised by:

   $$x(\theta) = \frac{1}{\pi}(\theta - \sin(\theta)) \qquad y(\theta) = \frac{1}{\pi}(\cos(\theta) - 1)$$

   (b) (1 mark) **At what point $(1, y)$ does the curve hit the line?**

   We saw above that the brachistochrone incides on $x = 1$ when $\theta = \pi$. The value of y at this point is:

   $$y(\pi) = \frac{1}{\pi}(\cos(\pi) - 1) = -\frac{2}{\pi}$$

   so the curve hits the line at:

   $$\left(1, -\frac{2}{\pi}\right)$$

(c) (3 marks) **How long does it take to get there?**

The time functional for a curve y from 0 to x_1 is:

$$T[y] = \int_0^{x_1} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}} dx$$

Notice, we can write:

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} dx = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

Moreover, $x_1 = 1 = x(\theta = \pi) = 1$. Thus, we can change the time functional to be an integral over θ:

$$T[y] = \int_0^{x_1} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}} dx = \int_0^{\pi} \frac{\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}}{\sqrt{-2gy(\theta)}} d\theta$$

Using:

$$x(\theta) = \frac{1}{\pi}(\theta - \sin(\theta)) \qquad y(\theta) = \frac{1}{\pi}(\cos(\theta) - 1)$$

we first compute the derivatives:

$$\frac{dx}{d\theta} = \frac{1}{\pi}(1-\cos(\theta)) \qquad \frac{dy}{d\theta} = -\frac{1}{\pi}\sin(\theta)$$

Thus:

$$\begin{aligned}
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \frac{1}{\pi^2}\left[(1-\cos(\theta))^2 + \sin^2\theta\right] \\
&= \frac{1}{\pi^2}\left[1 - 2\cos(\theta) + \cos^2(\theta) + \sin^2\theta\right] \\
&= \frac{2}{\pi^2}(1-\cos(\theta))
\end{aligned}$$

We can thus compute the integral:

$$\begin{aligned}
T[y] &= \int_0^{\pi} \frac{\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}}{\sqrt{-2gy(\theta)}} d\theta \\
&= \int_0^{\pi} \frac{\sqrt{\frac{2}{\pi^2}(1-\cos(\theta))}}{\sqrt{-2g\frac{1}{\pi}(\cos(\theta)-1)}} d\theta \\
&= \frac{1}{\sqrt{\pi g}} \int_0^{\pi} d\theta \\
&= \frac{\pi}{\sqrt{\pi g}} \\
&= \sqrt{\frac{\pi}{g}}
\end{aligned}$$

So it takes $\sqrt{\frac{\pi}{g}}$ time units.