Variational Calculus - Week 3 - The Newtonian Universe and Galilean Relativity

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Contents

Based on the notes by Jelle Hartong, Chapter 4

1 The Newtonian Universe

1.1 Definition: Affine Spaces

An ***affine space*** *for the* ***vector space*** $\mathbb{R}^n$ *is a space* A^n *on which* $\mathbb{R}^n$ *acts* ***freely*** *and* ***transitively****.*
That is, there is a map $\mathbb{R}^n \times A^n \to A^n$ *on* A^n *such that:*

$$(\underline{v}, a) \mapsto \underline{v} + a, \qquad \underline{v} \in \mathbb{R}^n, \qquad a, \underline{v} + a \in A^n$$

and satisfying the following properties:

1. ***Identity***
$$\forall a \in A^n, \quad \underline{0} + a = a$$
2. ***Associativity***
$$\forall \underline{v}, \underline{w} \in \mathbb{R}^n, \quad \forall a \in A^n \qquad \underline{v} + (\underline{w} + a) = (\underline{v} + \underline{w}) + a$$
3. ***Free Action***
$$\underline{v} + a = a \iff \underline{v} = 0$$
4. ***Transitivity***
$$\forall a, b \in A^n, \ \exists! \underline{v} \in \mathbb{R}^n \ : \ b = a + \underline{v}$$

- **How does an affine space differ from a normal vector space?**
  - **addition** is **not** defined in A^n (our operation only includes actions of $\mathbb{R}^n$ on A^n)
  - however, **differences** are defined, since:
  $$b - a = \underline{v} \in \mathbb{R}^3$$
  where $\underline{v}$ is the unique vector satisfying **transitivity**:
  $$b = a + \underline{v}$$
  - an **affine space** has **no origin** (this is the reason for why addition is not defined: unless there is an origin, we don't have ways of "comparing" and manipulating elements in A^n)
- **How can a vector space be recovered from an affine space?**
  - once we define $a \in A^n$ as an **origin**, then any other point $b \in A^n$ can be defined by $\underline{v} \in \mathbb{R}^n$ uniquely
  - thus, we obtain $\mathbb{R}^n$ back, but with origin a
- **How are subspaces defined in affine spaces?**
  - an **affine subspace** is a set of points $a, b \in A^n$ such that $b - a$ defines a subspace of $\mathbb{R}^n$

1.2 The Newtonian Universe

- **What is a Newtonian Universe?**
  - an **affine space** A^4 over the vector space $\mathbb{R} \times \mathbb{R}^3$, but with additional structure
  - $\mathbb{R}$ corresponds to a **time** dimension
  - $\mathbb{R}^3$ corresponds to a **space** dimension
  - we require it to be an **affine space**, since space itself doesn't have an "origin"
  - an element $a \in A^4$ is an **event**
- **What additional structure defines a Newtonian Universe?**
  - **Time Interval**: there is a linear map:

$$\tau : \mathbb{R}^4 \to \mathbb{R}$$

    defining a **time interval** between 2 events $a = (t, \underline{a}), b = (t', \underline{b}) \in A^4$ via:

$$\tau(b - a) = |t' - t|$$

    2 events $a, b \in A^4$ are **simulatenous** if:

$$\tau(b - a) = 0$$

  - **Distance**: we can define the distance between 2 **simultaneous events** by the Euclidean norm:

$$\rho(b - a) = \|\underline{b} - \underline{a}\|$$

    Notice, the notion of **distance** for non-simultenous events isn't defined (can't compare between past and present events)
  - a **Newtonian Universe** can then be regarded as the triple:

$$(A^4, \tau, \rho)$$

- **What is the kernel of the time interval mapping τ?**
  - this will be the set of all **simultaneous events**:

$$ker\tau = \{\underline{b} - \underline{a} \mid \tau(b - a)\}$$

  - but $\underline{b} - \underline{a} \in \mathbb{R}^3$, so $ker\tau$ is spanned by vectors in $\mathbb{R}^3$, so it will be a **subspace** of $\mathbb{R}^4$, **isomorphic** to $\mathbb{R}^3$:

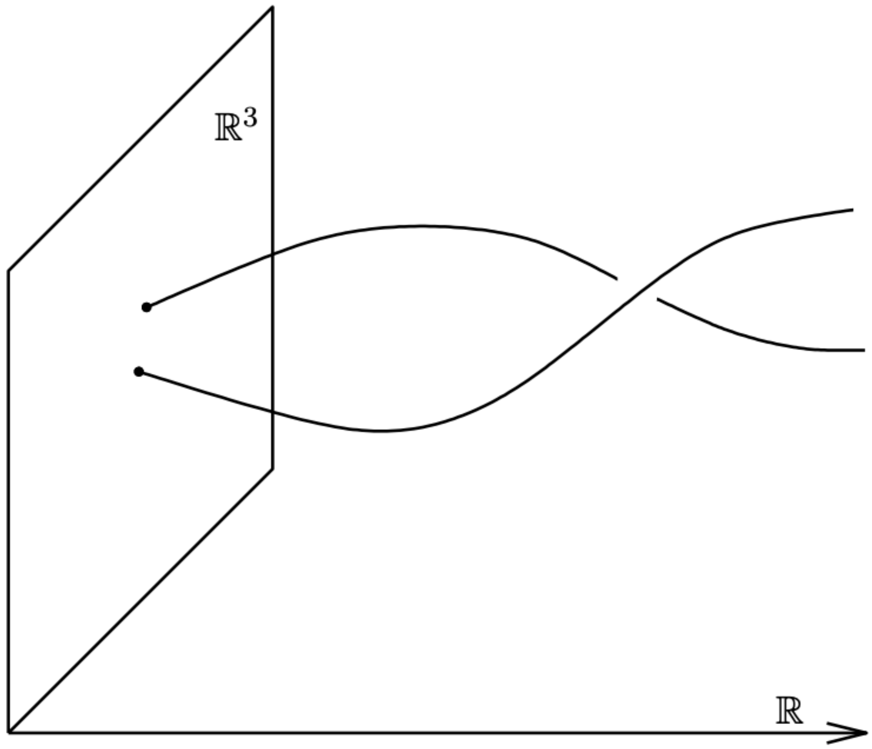
$$ker\tau \cong \mathbb{R}^3$$

  - it is this property that allows us to define ρ as an Euclidean distance, since it is just a standard property of $\mathbb{R}^3$
- **What is a worldline?**
  - consider a particle in the **Newtonian Universe**
  - its **wordline** is the path it follows over time
  - this is curve, a subset of the universe, defined by:

$$\{(t, \underline{x}(t) \mid t \in \mathbb{R}\}$$

  - we assume that $\underline{x}$ is **twice-differentiable**, such that:

* $\underline{x}$ is the **position** of the particle
* $\underline{\dot{x}}$ is the **velocity** of the particle
* $\underline{\ddot{x}}$ is the **acceleration** of the particle



2 Galilean Relativity

2.1 Definition: Affine Homomorphisms

*Consider a **linear transformation** between vector spaces V, W:*

$$T : V \to W$$

*A mapping f is an **affine homomorphism**, if for 2 affine spaces A^n, B^n (defined over V, W respectively), we have that:*

$$T(b - a) = f(b) - f(a), \qquad a \in A^n, b \in B^n$$

*is a **well-defined linear map**.*
Here notice that $b - a \in V$ and $f(b) - f(a) \in W$.
Since A^n is affine, $\exists! \underline{v} \in V$ such that:

$$b = a + \underline{v}$$

so:

$$T(b - a) = f(b) - f(a) \implies f(a + \underline{v}) = f(a) + T(\underline{v})$$

2.2 Definition: Affine Automorphisms

*An **affine automorphism** (or **affine transformation**) is an **affine homomorphism** mapping points in A^n to itself:*

$$f : A^n \to A^n$$

such that the following is a well-defined map:

$$T(b - a) = f(b) - f(a), \qquad a, b \in A^n$$

Now, since $f(a) \in A^n$, it follows that $\exists! \underline{\tilde{v}} \in V$ such that:

$$f(a) = a + \underline{\tilde{v}}$$

Similarly, we know that $b = a + v$.
Thus we have:

$$f(a + \underline{v}) = a + \underline{\tilde{v}} + T(\underline{v})$$

But if we take a to be the origin we get that $a + \underline{v} = v \in A^n$ and $a + \underline{\tilde{v}} = \tilde{v}$ and:

$$f(v) = \tilde{v} + T(\underline{v})$$

*In other words, **affine automorphisms**, are defined by a **linear transformation** T and a **translation** $\tilde{v}$.*

2.3 The Galilean Group

- **What is a relativity group?**
  - a **group** of transformation on the **universe**
  - they preserve the structure endowed to the universe
- **What is a Galilean group?**
  - it is the **relativity group** of the **Newtonian Universe**
  - it is the subgroup of **affined transformations** of $\mathbb{R} \times \mathbb{R}^3$ which preserves the structure of the Newtonian Universe. In other words, if f is an **affine transformation**, then:
    * **time intervals** between events are invariant:

$$\tau(b-a) = \tau(f(b) - f(a))$$

    * **distances** between simulateneous events are invariant:

$$\rho(b-a) = \rho(f(b) - f(a))$$

- **What elements are part of the Galilean group?**
  - let $p, \tilde{v} \in \mathbb{R}^4$ and $T \in \mathbb{R}^{4\times4}$:

$$p = \begin{pmatrix} t \\ \underline{x} \end{pmatrix}, \qquad t \in \mathbb{R}, \underline{x} \in \mathbb{R}^3$$

$$\tilde{v} = \begin{pmatrix} s \\ \underline{a} \end{pmatrix}, \qquad s \in \mathbb{R}, \underline{a} \in \mathbb{R}^3$$

$$T = \begin{pmatrix} \alpha & \underline{w}^T \\ \underline{v} & L \end{pmatrix}, \qquad \alpha \in \mathbb{R}, \underline{w}, \underline{v} \in \mathbb{R}^3$$

    then our affine transformation:

$$f(v) = \tilde{v} + T(\underline{v})$$

    can be interpreted as:

$$v \mapsto \hat{v}$$

    where:

$$\hat{v} = \begin{pmatrix} \alpha & \underline{w}^T \\ \underline{v} & L \end{pmatrix} \begin{pmatrix} t \\ \underline{x} \end{pmatrix} + \begin{pmatrix} s \\ \underline{a} \end{pmatrix} = \begin{pmatrix} \alpha t + \underline{w}^T \underline{x} + s \\ \underline{v} t + L\underline{x} + \underline{a} \end{pmatrix}$$

  - now consider the affine transformation of a second point $q \in \mathbb{R}^4$:

$$\begin{pmatrix} t' \\ \underline{x}' \end{pmatrix} \mapsto \begin{pmatrix} \hat{t}' \\ \hat{\underline{x}}' \end{pmatrix} = \begin{pmatrix} \alpha t' + \underline{w}^T \underline{x}' + s \\ \underline{v} t' + L\underline{x}' + \underline{a} \end{pmatrix}$$

  - if we want to preserve **time intervals**, let $|t' - t| = \gamma$, and consider:

$$\begin{aligned} |\hat{t}' - \hat{t}| &= |(\alpha t' + \underline{w}^T \underline{x}' + s) - (\alpha t + \underline{w}^T \underline{x} + s)| \\ &= |\alpha(t' - t) + \underline{w}^T(\underline{x}' - \underline{x})| \\ \implies |\hat{t}' - \hat{t}| = \gamma &\iff \alpha = \pm 1, \underline{w} = \underline{0} \end{aligned}$$

- if we want to preserve **distance**, let p, q be simultaneous events, so that $|t' - t| =$, and let $\|\underline{x}' - \underline{x}\| = \eta$. Consider:

$$\begin{aligned}
\|\hat{\underline{x}}' - \hat{\underline{x}}\|^2 &= (\hat{\underline{x}}' - \hat{\underline{x}})^T(\hat{\underline{x}}' - \hat{\underline{x}}) \\
&= ((\underline{v}t' + L\underline{x}' + \underline{a}) - (\underline{v}t + L\underline{x} + \underline{a}))^T((\underline{v}t' + L\underline{x}' + \underline{a}) - (\underline{v}t + L\underline{x} + \underline{a})) \\
&= (\underline{v}(t' - t) + L(\underline{x}' - \underline{x}))^T(\underline{v}(t' - t) + L(\underline{x}' - \underline{x})) \\
&= (L(\underline{x}' - \underline{x}))^T(L(\underline{x}' - \underline{x})) \\
&= (\underline{x}' - \underline{x})L^T L(\underline{x}' - \underline{x}) \\
\implies \|\hat{\underline{x}}' - \hat{\underline{x}}\|^2 = \eta^2 &\iff L^T L = \mathbb{I}
\end{aligned}$$

  In other words, for distance invariance, we require that L be an **orthogonal matrix**, and $det(L) = det(L)det(L^T) = det(L)^2 = 1 \implies det(L) = \pm 1$.

- **How does the sign of α and $det(L)$ change the transformations in the Galilean group?**
  - whilst we showed that the Galilean group is composed of transformations with:

$$\alpha = \pm 1 \qquad det(L) = \pm 1$$

    in practice we only use:

$$\alpha = 1 \qquad det(L) = 1$$

  - if $\alpha = -1$, we allow for the transformation to reverse the direction of time
  - if $det(L) = -1$, then L becomes a **reflection** about the origin
  - if $det(L) = 1$, then L becomes a rotation
    * L will have a real eigenvalue 1, whose eigenvector will be the axis of ratoation, and 2 imaginary eigenvalues $\pm\lambda$, which determine the angle of rotation
    * we can thus write $L = R$
    * $R \in SO(3)$, the group of 3×3 orthogonal matrices with unit determinant

- **What are the elementary Galilean transformations?**
  - the **Galilean Group** is a **Lie Group** defining transformations of the form:

$$\begin{pmatrix} t \\ \underline{x} \end{pmatrix} \mapsto \begin{pmatrix} \hat{t} \\ \hat{\underline{x}} \end{pmatrix} = \begin{pmatrix} 1 & \underline{0}^T \\ \underline{v} & R \end{pmatrix} \begin{pmatrix} t \\ \underline{x} \end{pmatrix} + \begin{pmatrix} s \\ \underline{a} \end{pmatrix} = \begin{pmatrix} t + s \\ \underline{v}t + R\underline{x} + \underline{a} \end{pmatrix}$$

  - every such Galilean transformation can be written **uniquely** as the composition of 3 **Elementary Galilean Transformations**:
    1. **Translation in Space and Time**

$$\begin{pmatrix} t \\ \underline{x} \end{pmatrix} = \begin{pmatrix} t + s \\ \underline{x} + \underline{a} \end{pmatrix}$$

    2. **Rotation in Space**

$$\begin{pmatrix} t \\ \underline{x} \end{pmatrix} = \begin{pmatrix} t \\ R\underline{x} \end{pmatrix}$$

    3. **Galilean Boost**

$$\begin{pmatrix} t \\ \underline{x} \end{pmatrix} = \begin{pmatrix} t \\ \underline{x} + t\underline{v} \end{pmatrix}$$

- **Why is time absolute in a Newtonian universe?**
  - because the time difference between events is **invariant** under a Galilean transformation

3 Frames of Reference

- **What is a free particle?**
  - a particle which moves without the influence of **external forces**

> *This is a bad definition, since we don't define these "external forces", but delving into it requires differential geometry and is beyond the scope of the course.*
> *More accurately, a **free particle** is a particle which follows a **geodesic** with respect to an appropriate "**connection**" on a Newtonian universe, where a "**connection**" is a concept from differential geometry.*

- **What is a frame of reference?**
  - a **coordinate system** defined for an **observer**
  - the observer can then make measurements (or observations) of its environment **relative** to its **frame of reference**
- **What is an inertial frame of reference?**
  - a **frame of reference** in which **Newton's Second Law** holds true: if no **external force** is applied on a body, it will **not** accelerate (so either it remains at **rest** or continues in **uniform motion**)
- **How does a free particle behave in an inertial frame of reference?**
  - a **free particle** has no external force applied; in an **inertial frame of reference** this means that its equation of motion satisfies **Newton's Second Law**, so:

$$\ddot{\underline{x}} = \underline{0}$$

  - if we integrate twice, and let $\underline{v}$ denote the velocity of the free particle, then:

$$\underline{x}(t) = \underline{x}_0 + t\underline{v} \qquad \underline{x}_0 = \underline{x}(0)$$

  - hence, in an **inertial frame of reference**, a **free particle** moves along a straight line with constant velocity
- **How can we map between inertial frames of reference?**
  - if we apply a **Galilean Transformation** to an **inertial frame of reference**, then we will obtain another **inertial frame of reference**
- **When is a frame of reference non-inertial?**
  - when its **observer** undergoes **acceleration**
  - for example, if the observer is in a **rotating** frame of reference, or if it is undergoing constant **acceleration**

Consider an ***inertial frame of reference*** *defined by the standard Cartesian coordinates:*

$$(x, y, z, t)$$

Moreover, consider a ***rotating*** *frame of reference defined by:*

$$(x', y', z', t)$$

These 2 coordinate systems are related via:

$$\begin{aligned} t' &= t \\ x' &= x \cos \omega t + y \sin \omega t \\ y' &= -x \sin \omega t + y \cos \omega t \\ z' &= z \end{aligned}$$

The rotating coordinate system rotates about the common z-axis, with ***constant angular velocity*** ω.
After some manipulation (see Exercises), we find that whilst a ***free particle*** *in the* ***inertial frame of reference*** *is defined by:*

$$\ddot{x} = \underline{0}$$

in the ***rotating*** *frame of reference we have:*

$$\underline{\ddot{x}}' = -\underline{w} \times (\underline{w} \times \underline{x}') - 2\underline{w} \times \underline{\dot{x}}'$$

where $\underline{w} = (0, 0, \omega)$.
Since $\underline{\ddot{x}}' \neq \underline{0}$, *the free particle doesn't follow a straight line from the point of view of the rotating FOR: it will look as if there are* ***fictitious forces*** *acting on the particle.*
In fact, these are the ***centrifugal*** *and* ***coriolis*** *forces:*

$$\underline{F}_{cen} = -m\underline{w} \times (\underline{w} \times \underline{x}')$$

$$\underline{F}_{cor} = -2m\underline{w} \times \underline{\dot{x}}'$$

where m *will be the mass of the particle.*

4 Newton's Equation

4.1 Configuration Space

- **What is a configuration space?**
  - consider a set of n particles, such that the Ith particle has a path defined by:

    $$x_I : \mathbb{R} \to \mathbb{R}^3$$

  - the **configuration space** of this system is the **n-fold Cartesian product**:

    $$\mathbb{R}^3 \times \underset{n}{\ldots} \times \mathbb{R}^3 = R^{3n}$$

- the **worldlines** of each particle are represented as a **single curve** in the configuration space
- the **configuration space** is a **manifold**: a surface which locally ressembles Euclidean space

- **What are generalised coordinates?**
  - the set of coordinates which define our **configuration space**
  - the **dimension** of the configuration space is thus given by the number of **generalised coordinates** (that is, the **degrees of freedom**)

- **What is the configuration space for particles constrained to move on the surface of a sphere, S^3?**
  - we can parametrise such trajectories via spherical coordinates:

$$t \mapsto (\theta(t), \varphi(t))$$

  - thus, the configuration space is a 2 dimensional manifold, where the generalised coordinates are θ, φ

4.2 Defining Newtonian Mechanics: Newton's Equation

- **What is the determinacy assumption of Newtonian mechanics?**
  - the **initial state** of a mechanical system **uniquely** determines how the system evolves
  - thus, the **dynamics** of the system are solely determined by knowing:
    - all the generalised positions q
    - all the generalised velocities $\dot{q}$

    at a given instance in time.

- **What is Newton's Equation?**

The second order ODE:

$$\ddot{\underline{q}}(t) = \Phi(\underline{q}(t), \dot{\underline{q}}(t), t)$$

*is **Newton's Equation**, where:*

$$\Phi : U \to \mathbb{R}^N, \qquad U \subset \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$$

*The objective of **Newtonian mechanics** is determining Φ, as this allows us to understand the dynamics of the system. However, we'll have to solve a system of N, second-order ODEs.*

*For example, in an **inertial frame of reference**, if we have a **force field** (i.e gravity, electromagnetism) which doesn't depend explicitly on time, if we have a particle of **inertial mass** m, its motion is defined by:*

$$m\ddot{x} = F(x, \dot{x})$$

- **How can Newton's Equation be reformulated in terms of first order ODEs?**
  - introduce a new function:
  $$v : \mathbb{R} \to \mathbb{R}^N$$
  via:
  $$\underline{\dot{q}} = \underline{v}$$
  - then Newton's equation becomes a system of $2N$ first order ODEs:
  $$\begin{cases} \underline{\dot{q}} = \underline{v} \\ \underline{\dot{v}} = \Phi(\underline{q}, \underline{v}, t) \end{cases}$$
- **Why is it useful to convert Newton's equation into a system of $2N$ ODEs?**
  - it implies that if Φ is sufficiently differentiable, there is a **unique solution**, given initial conditions $q(0), v(0)$, in some time interval
- **What is a physical trajectory?**
  - the **curve** in **state space** of $(q(t), v(t))$ (assuming that q, v satisfy Newton's equation)

4.3 Galilean Gravity

4.3.1 Definition: Galilean Gravity

> ***Galilean gravity*** *is a* ***uniform gravitational force field****, defined by:*
>
> $$F(q, \dot{q}, t) = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$$
>
> *(uniform since it is the same everywhere)*

If $z(t)$ denotes the height of a body at time t, Newton's equation tells us that:

$$m\ddot{z} = -mg$$

*(the m in the LHS is the **inertial mass**, whilst the m in the RHS is the **gravitational mass**. These are shown to be equivalent by the **equivalence principle**).*
Integrating twice:

$$v(t) = v_0 - gt$$

$$z(t) = z_0 + v_0 t - \frac{1}{2}gt^2$$

*Thus, the **state space** will be:*

$$\{(z, v) \mid z \geq 0\} \subseteq \mathbb{R}^2$$

(since we can't pierce through the floor)
*The **physical trajectories** of this system will be parabolae in **state space**:*

$$(z(t), v(t)) = \left(z_0 + v_0 t - \frac{1}{2}gt^2, v_0 - gt\right)$$

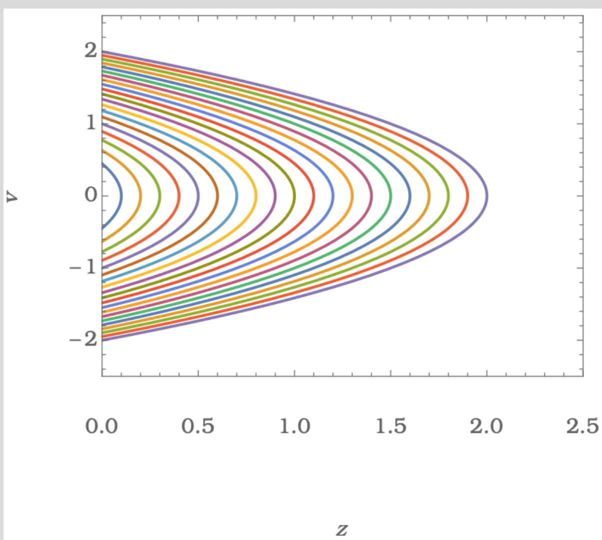
Explicitly:

$$v(t) = v_0 - gt \implies t = \frac{v - v_0}{g}$$

so:

$$z = z_0 + v_0\frac{v - v_0}{g} - \frac{1}{2}g\frac{(v - v_0)^2}{g^2} \geq 0$$

(z is a parabola in terms of v)



- **How does Galilean gravity differ from newton's gravity?**
  - it only applies locally
  - Newton's **universal law of gravitation** showed that gravitational force as distance from Earth increases
  - under Galilean gravity, no matter the initial condition, an object always ends on the floor
  - however, this isn't what we observe empirically (i.e rockets are capable of breaking away from the gravitational pull)

4.3.2 Einstein's Thought Experiment

Einstein reasoned that a sufficiently locally, a gravitational field can't be distinguished from a fictitious force.

To this regard, we have that Galilean gravity can't actually be distinguished from a uniformly accelerating frame of reference.

Consider an inertial frame of reference with coordinates $(txty, z)$ and a non-inertial frame of reference which accelerates uniformly (t, x', y', z'). These are related by:

$$\begin{aligned} t' &= t \\ x' &= x \\ y' &= y \\ z' &= z - \frac{1}{2}gt^2 \end{aligned}$$

Now, a free particle in the inertial frame of reference has motion:

$$m\ddot{z} = 0$$

However, in the accelerating frame of reference:

$$m\ddot{z}' = -mg$$

which precisely defines the motion of the particle under Galilean gravity in the inertial frame of reference.

5 Exercises

1. **Show that every Galilean transformation can be written as the composition of translations in space and time, rotations and Galilean boosts.**

2. **Show that the force-free Newtonian equation $\ddot{x} = 0$ is invariant under the group of Galilean transformations. Furthermore, show that the Galilean transformations transform solutions of $\ddot{x} = 0$ into one another.**

3. **Show that in the rotating frame of reference defined by:**

$$\begin{aligned} t' &= t \\ x' &= x\cos\omega t + y\sin\omega t \\ y' &= -x\sin\omega t + y\cos\omega t \\ z' &= z \end{aligned}$$

The motion of a free particle is defined by:

$$\ddot{\underline{x}}' = -\underline{w} \times (\underline{w} \times \underline{x}') - 2\underline{w} \times \dot{\underline{x}}'$$

where $\underline{w} = (0, 0, \omega)$

4. **Assume that the force field** F **is independnet of the velocity of the praticle. Show taht in this case, Newton's equation is invariant under time reversel; that is, show that if** $x(t)$ **solves the equation, then so does:**

$$\bar{x}(t) = x(-t)$$

5. **A particle of mass** m **is observed, from the point of view of an inertial frame, to be moving in a circular trajectory:**

$$x(t) = (R\cos\omega t, R\sin\omega t, 0)$$

where R, ω **are positive constants. What is the force acting on the particle? You should find that the force is equal in magnitude, and opposite in direction to the centrifugal force discussed above. The force calculated here is called the centripetal force, and it points inwards.**