Variational Calculus - Week 10 - Variational Calculus in Higher Dimensions

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Contents

Based on the notes by Jelle Hartong, Section 11

Until now we have only considered variational problems which involved curves in $\mathbb{R}^n$. We now focus on how we can generalise variational problems to apply to hypersurfaces in space.

1 Multidimensional Euler-Lagrange Equations

1.1 Useful Theorems

1.1.1 Theorem: The Divergence Theorem

*Let $D \subset \mathbb{R}^m$ be a **bounded** and **open** set. Let ∂D be its **piecewise smooth** boundary.*
Let:

$$\underline{X} = (X^1, \ldots, X^m)$$

*be a **smooth vector field**, defined on $D \cup \partial D$.*
*If $\underline{N}$ is the **unit, outward-pointing** normal of ∂D, then:*

$$\int_D \nabla \cdot \underline{X} dV = \int_{\partial D} \langle \underline{X}, \underline{N} \rangle \, dA$$

*where $\nabla \cdot$ is the **divergence** operator:*

$$\nabla \cdot \underline{X} = \sum_{\mu=1}^{m} \frac{\partial X^\mu}{\partial x^\mu}$$

(Theorem 11.1)

1.1.2 Theorem: Multidimensional Fundamental Lemma of Variational Calculus

*Let $D \subset \mathbb{R}^m$ be a **bounded**, **open set** with **piecewise smooth** boundary ∂D.*
Let:

$$f : D \to \mathbb{R}^n$$

*be a **continuous** function satisfying:*

$$\int_D \langle f(\underline{x}), h(\underline{x}) \rangle \, d^m x = 0$$

*for **all** C^∞ functions:*

$$h : D \to \mathbb{R}^n$$

*which **vanish** on the boundary ∂D.*
Then:

$$f \equiv 0$$

(Theorem 11.2)

1.2 Theorem: The Multi-Dimensional Euler-Lagrange Equations

*Let $D \subset \mathbb{R}^m$ be a **bounded** region, with a **piecewise smooth** boundary.*
Let $\underline{x} = (x^1, \ldots, x^m)$ be coordinates for D.
*Consider **vector fields**:*

$$\underline{y} : D \to \mathbb{R}^n$$

*and define ∇y to be the collection of **mn** partial derivatives of y, corresponding to each of the partial derivatives associated with each component function y^i and some variable x^μ in D:*

$$y^i_\mu = \frac{\partial y^i}{\partial x^\mu}$$

*If we define a **Lagrangian**:*

$$L(\underline{y}, \nabla \underline{y}, \underline{x})$$

*then the **general multi-dimensional Euler-Lagrange equations** are given by:*

$$\frac{\partial L}{\partial y^i} = \sum_{\mu=1}^{m} \frac{\partial}{\partial x^\mu} \left(\frac{\partial L}{\partial y^i_\mu} \right)$$

where we have to think of $\frac{\partial L}{\partial y^i_\mu}$ as a function of $\underline{x}$.
(Equation 11.2)

As a sanity check, we can consider this for the case when D is a one-dimensional space. Then:

$$\underline{y} : [0,1] \to \mathbb{R}^n$$

Moreover, our coordinates for D are defined by a single variable x. Thus:

$$\frac{\partial L}{\partial y^i} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial y^{i\prime}}\right) = \frac{d}{dx}\left(\frac{\partial L}{\partial y^{i\prime}}\right)$$

where:

$$y^{i'} = \frac{\partial y^i}{\partial x} = \frac{dy^i}{dx}$$

Proof. We can consider a simple case, where $D \subset \mathbb{R}^2$ is an open, bounded region, with piecewise smooth boundary ∂D. Our coordinates for D are defined by 2 parameters:

$$\underline{x} = (u, v)$$

Define a C^2 vector field:

$$\underline{y} : D \to \mathbb{R}$$

such that:

$$\forall (u,v) \in D, \qquad \underline{y}(u,v) \in \mathbb{R}^n$$

We can define a **Lagrangian** which depends on $\underline{y}, u, v$, alongside the partial derivatives of $\underline{y}$ with respect to u, v. This has the form:

$$L : \underbrace{\mathbb{R}^n}_{\underline{y}} \times \underbrace{\mathbb{R}^{2n}}_{\underline{y}_u, \underline{y}_v} \times \underbrace{\mathbb{R}^2}_{u,v} \to \mathbb{R}$$

where:

$$y^i_u = \frac{\partial y^i}{\partial u} \qquad y^i_v = \frac{\partial y^i}{\partial v}$$

With this Lagrangian, we thus have a corresponding action:

$$S[\underline{y}] = \int_D L(\underline{y}, \underline{y}_u, \underline{y}_v, u, v)\,du dv$$

where the **boundary conditions** are now defined by a **function**:

$$\underline{y}(\underline{x}) = \underline{\phi}(\underline{x}), \qquad \underline{x} \in \partial D$$

and:

$$\underline{\phi} : \partial D \to \mathbb{R}^n$$

(analogosuly, when $D = [0,1]$ then $\partial D = \{0\} \cup \{1\}$, which is where we defined our boundary terms)

We now consider **variations**, which will be C^1 functions:

$$\underline{\varepsilon} : D \to \mathbb{R}^n$$

where:

$$\forall \underline{x} \in \partial D, \qquad \underline{\varepsilon}(\underline{x}) = 0$$

If $\underline{y}$ is a critical point of $S[\underline{y}]$, then:

$$\begin{aligned}
0 &= \frac{d}{ds} S[\underline{y} + s\underline{\varepsilon}]\Big|_{s=0} \\
&= \int_D \frac{d}{ds} \left(L(\underline{y} + s\underline{\varepsilon}, L(\underline{y}_u + s\underline{\varepsilon}_u, L(\underline{y}_v + s\underline{\varepsilon}_v, u, v \right) dudv \Big|_{s=0} \\
&= \int_D \left(\sum_{i=1}^{n} \left[\frac{\partial L}{\partial y^i} \varepsilon^i + \frac{\partial L}{\partial y^i_u} \varepsilon^i_u + \frac{\partial L}{\partial y^i_v} \varepsilon^i_v \right] \right) dudv \\
&= \int_D \left(\sum_{i=1}^{n} \left[\frac{\partial L}{\partial y^i} - \frac{\partial}{\partial u} \frac{\partial L}{\partial y^i_u} - \frac{\partial}{\partial v} \frac{\partial L}{\partial y^i_v} \right] \varepsilon^i \right) dudv \\
&+ \int_D \left(\sum_{i=1}^{n} \left[\frac{\partial}{\partial u} \left(\frac{\partial L}{\partial y^i_u} \varepsilon^i \right) + \frac{\partial}{\partial v} \left(\frac{\partial L}{\partial y^i_v} \varepsilon^i \right) \right] \right) dudv
\end{aligned}$$

The second integral becomes 0 by applying the Divergence Theorem, since it becomes an integral over ∂D, and $\varepsilon^i = 0$ on the surface. Moreover, by the Fundamental Lemma:

$$\int_D \left(\sum_{i=1}^{n} \left[\frac{\partial L}{\partial y^i} - \frac{\partial}{\partial u} \frac{\partial L}{\partial y^i_u} - \frac{\partial}{\partial v} \frac{\partial L}{\partial y^i_v} \right] \varepsilon^i \right) dudv = 0 \iff \sum_{i=1}^{n} \left[\frac{\partial L}{\partial y^i} - \frac{\partial}{\partial u} \frac{\partial L}{\partial y^i_u} - \frac{\partial}{\partial v} \frac{\partial L}{\partial y^i_v} \right] = 0$$

which are our multi-dimensional EL equations

□

2 Solutions to Laplace's Equation as Variational Extrema

2.1 Definition: Laplace's Equation and Harmonic Functions

*Let $D \subset \mathbb{R}^2$ be the **unit disc**. **Laplace's Equation** is the PDE:*

$$\Delta\phi = \phi_{xx} + \phi_{yy} = 0$$

where:

$$\phi : D \to \mathbb{R}$$

*and ϕ satisfies some boundary condition (we will consider **Dirichlet**, **Neumann** and **Robin** boundary conditions).*

---

*If ϕ satisfies **Laplace's Equation**, then ϕ is called **harmonic**. (Equation 11.3)*

2.2 Proposition: Harmonic Functions Minimise an Action

*Define the **energy functional**:*

$$E[\phi] = \int_D \frac{1}{2}(\phi_x^2 + \phi_y^2) dx\ dy$$

*Then, the **extremals** of E are **harmonic functions**; that is, solutions to $\Delta\phi = 0$. Moreover, such extremals **minimise** the **energy**. (Equation 11.4)*

Proof. We could directly compute the Euler-Lagrange equations, but since we'll want to discuss boundary conditions, we apply variations to the functional.

Indeed, consider variations $\varepsilon : D \to \mathbb{R}$. Then, if ϕ is an extremal of E:

$$\frac{d}{ds} E[\phi + s\varepsilon]\bigg|_{s=0} = 0$$

Hence:

$$\begin{aligned} 0 &= \frac{d}{ds} E[\phi + s\varepsilon]\bigg|_{s=0} \\ &= \int_D \frac{d}{ds}\left(\frac{1}{2}((\phi_x + s\varepsilon_x)^2 + (\phi_y + s\varepsilon_y)^2)\right) dx\ dy\bigg|_{s=0} \\ &= \int_D (\phi_x + s\varepsilon_x)\varepsilon_x + (\phi_y + s\varepsilon_y)\varepsilon_y dx\ dy\bigg|_{s=0} \\ &= \int_D \phi_x\varepsilon_x + \phi_y\varepsilon_y dx\ dy \end{aligned}$$

Now, notice we have that:

$$\begin{aligned} \frac{\partial}{\partial x}(\phi_x\varepsilon) &= \phi_{xx}\varepsilon + \phi_x\varepsilon_x \\ \frac{\partial}{\partial y}(\phi_y\varepsilon) &= \phi_{yy}\varepsilon + \phi_y\varepsilon_y \end{aligned}$$

Hence:

$$0 = -\int_D (\phi_{xx} + \phi_{yy})\varepsilon dx\ dy + \int_D \frac{\partial}{\partial x}(\phi_x\varepsilon) + \frac{\partial}{\partial y}(\phi_y\varepsilon) dx\ dy$$

Now, recalling the Divergence Theorem (or Green's Theorem, which is equivalent in $\mathbb{R}^2$):

$$\int_D \frac{\partial}{\partial x}(\phi_x\varepsilon) + \frac{\partial}{\partial y}(\phi_y\varepsilon) dx\ dy = \int_{\partial D} \langle \underline{N}, \nabla\phi\rangle\, \varepsilon ds$$

where ds denotes an infinitesimal arclength of our surface ∂D, and $\underline{N}$ is a **unit, outward** normal vector to ∂D. The quantity $\langle \underline{N}, \nabla\phi\rangle$ is known as the **normal derivative** of ϕ.

*Using **Green's Theorem** we arrive at the same conclusion. If we parametrise ∂D using $(x(t), y(x(t))$:*

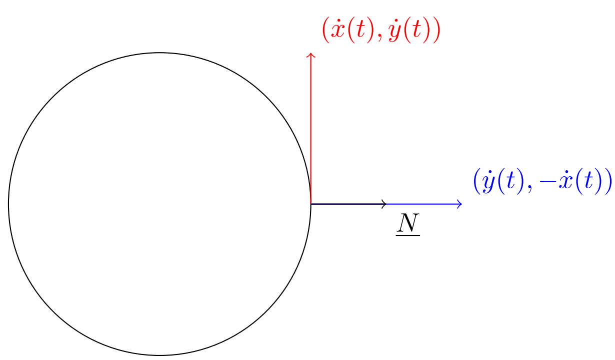
$$\begin{aligned}\int_D \frac{\partial}{\partial x}(\phi_x \varepsilon) + \frac{\partial}{\partial y}(\phi_y \varepsilon) dx\, dy &= \int_{\partial D} \varepsilon(-\phi_y dx + \phi_x dy) \\ &= \int_{\partial D} \varepsilon(-\phi_y \dot{x} + \phi_x \dot{y}) dt \\ &= \int_{\partial D} \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \varepsilon(-\phi_y \dot{x} + \phi_x \dot{y}) ds \\ &= \int_{\partial D} \langle \underline{N}, \nabla \phi \rangle \, \varepsilon ds\end{aligned}$$

where we have used the fact that:

$$ds = \sqrt{\dot{x}^2 + \dot{y}} dt$$

*and that the **unit** outward normal vector $\underline{N}$ to ∂D is precisely given by:*

$$\underline{N} = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}}} (\dot{y}, -\dot{x})$$



Thus, overall we get that if ϕ extremises $E[\phi]$ then by the Fundamental Lemma (and assuming endpoint fixed variations $\varepsilon|_{\partial D} = 0$) then:

$$0 = \phi_{xx} + \phi_{yy}$$

so ϕ will be harmonic, as required.

□

2.3 Types of Boundary Conditions for Laplace's Equation

2.3.1 Dirichlet Boundary Condition

- **What are Dirichlet boundary conditions?**
  - **Dirichlet** boundary conditions impose conditions on the value of ϕ on ∂D:

$$\phi(\underline{x})|_{\underline{x} \in \partial D} = f$$

where:

$$f : \partial D \to \mathbb{R}$$

- **How do Dirichlet boundary conditions affect the solution to Laplace's Equation?**
  - under **Dirichlet** conditions,

$$\int_{\partial D} \langle \underline{N}, \nabla \phi \rangle \, \varepsilon ds = 0$$

    if we have **endpoint fixed variations**:

$$\varepsilon|_{\partial D} = 0$$

2.3.2 Neumann Boundary Condition

- **What are Neumann boundary conditions?**
  - **Neumann** boundary conditions impose conditions on the value of $\nabla \phi$ on the boundary:

$$\langle \underline{N}, \nabla \phi \rangle = 0$$

- **How do Neumann boundary conditions affect the solution to Laplace's Equation?**
  - under **Neumann** conditions,

$$\int_{\partial D} \langle \underline{N}, \nabla \phi \rangle \, \varepsilon ds = 0$$

    for **any** free variation ε
  - this states that **harmonic solutions** ϕ will have a vanishing **normal derivative** at the boundary

2.3.3 Robin Boundary Condition

- **What are Robin boundary conditions?**
  - **Robin** boundary conditions impose conditions on the values of ϕ and $\nabla \phi$ on the boundary:

$$\langle \underline{N}, \nabla \phi \rangle + \alpha \phi = g, \quad \alpha \in \mathbb{R}$$

    where:

$$g : \partial D \to \mathbb{R}$$

- **How do Robin boundary conditions affect the solution to Laplace's Equation?**
  - **Robin** conditions arise when we use **free** variations at the boundary for the functional:

$$E[\phi] = \int_D \frac{1}{2}(\phi_x^2 + \phi_y^2) dx \, dy + \int_{\partial D} \left(\frac{\alpha}{2} \phi^2 - g\phi \right) ds$$

  - working as above, we'd get that:

$$\begin{aligned} 0 &= \frac{d}{ds} E[\phi + s\varepsilon] \bigg|_{s=0} \\ &= -\int_D (\phi_{xx} + \phi_{yy}) \varepsilon dx \, dy + \int_{\partial D} (\langle \underline{N}, \nabla \phi \rangle + \alpha \phi - g) \varepsilon ds \end{aligned}$$

  - for arbitrary, free variations ε we'd get that by the Fundamental Lemma:

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{on } D$$

$$\langle \underline{N}, \nabla \phi \rangle + \alpha \phi - g = 0 \quad \text{on } \partial D$$

3 Minimal Surfaces

3.1 Lemma: Surface Area of a Parametric Surface

Let $\Sigma \subset \mathbb{R}^3$ be a ***surface****, parametrised by u, v:*

$$\underline{r}(u,v) = x(u,v)\underline{e}_1 + y(u,v)\underline{e}_2 + z(u,v)\underline{e}_3$$

Then, the ***surface area*** *of Σ is given by:*

$$\int_D \|\underline{r}_u \times \underline{r}_v\| dA$$

In particular, if Σ can be described by a function f:

$$x = u \qquad y = v \qquad z = f(x,y)$$

then the ***surface area*** *of Σ is given by:*

$$\int_D \sqrt{1 + f_x^2 + f_y^2} dxdy$$

Proof. The idea is to notice that, at any point on the surface, $\underline{r}_u$ and $\underline{r}_v$ will be tangents, and together, they define a linear approximation to the surface (as a plane). The area of the parallelogram spanned by the tangent vectors is precisely:

$$\|\underline{r}_u \times \underline{r}_v\|$$

Doing this over each point gives us the desired result.

If the surface is defined by a function f, then notice that:

$$\underline{r}(u,v) = \underline{r}(x,y) = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$$

so:

$$\begin{aligned}\|\underline{r}_x \times \underline{r}_y\| &= \left\| \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} 1 \\ -f_y \\ -f_x \end{pmatrix} \right\| \\ &= \sqrt{1 + f_x^2 + f_y^2}\end{aligned}$$

as required.

□

3.2 Proposition: The Plateau Problem and Minimal Surfaces

*The **Plateau Problem** is the problem of finding a **minimal surface** (i.e surfaces with minimal surface area), subject to **boundary conditions**.*
*If f defines a surface, then f will be a **minimal surface** provided it satisfies the PDE:*

$$(1 + f_y^2)f_{xx} + (1 + f_x^2)f_{yy} - 2f_x f_y f_{xy} = 0$$

(Equation 11.5)

Proof. We have a surface area functional:

$$S[f] = \int_d \sqrt{1 + f_x^2 + f_y^2} dxdy$$

Extremals of $S[f]$ will be minimal surfaces, so we seek to find the corresponding Euler-Lagrange equations. f depends on $x.y$ so we need to use the multidimensional EL equations:

$$\frac{\partial L}{\partial f} = \frac{\partial}{\partial x}\frac{\partial L}{\partial f_x} + \frac{\partial}{\partial y}\frac{\partial L}{\partial f_y}$$

Since $L(f, f_x, f_y, x, y) = \sqrt{1 + f_x^2 + f_y^2}$ doesn't depend (explicitly) on f:

$$\frac{\partial}{\partial x}\frac{\partial L}{\partial f_x} + \frac{\partial}{\partial y}\frac{\partial L}{\partial f_y} = 0$$

We can compute these partial derivatives:

$$\frac{\partial L}{\partial f_x} = \frac{f_x}{\sqrt{1 + f_x^2 + f_y^2}}$$

$$\frac{\partial L}{\partial f_y} = \frac{f_y}{\sqrt{1+f_x^2+f_y^2}}$$

Moreover:

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial f_x}\right) &= \frac{\partial}{\partial x}\left(\frac{f_x}{\sqrt{1+f_x^2+f_y^2}}\right) \\
&= \frac{f_{xx}\sqrt{1+f_x^2+f_y^2} - \frac{\partial}{\partial x}(\sqrt{1+f_x^2+f_y^2})f_x}{1+f_x^2+f_y^2} \\
&= \frac{f_{xx}\sqrt{1+f_x^2+f_y^2} - (\frac{1}{2}(1+f_x^2+f_y^2)^{-1/2}(2f_xf_{xx}+2f_yf_{yx}))f_x}{1+f_x^2+f_y^2} \\
&= \frac{f_{xx}(1+f_x^2+f_y^2) - (f_xf_{xx}+f_yf_{yx})f_x}{(1+f_x^2+f_y^2)^{3/2}} \\
&= \frac{f_{xx}+f_x^2f_{xx}+f_y^2f_{xx}-f_x^2f_{xx}-f_yf_xf_{yx}}{(1+f_x^2+f_y^2)^{3/2}} \\
&= \frac{f_{xx}(1+f_y^2)-f_yf_xf_{xy}}{(1+f_x^2+f_y^2)^{3/2}}
\end{aligned}$$

Identical working gives us:

$$\frac{\partial}{\partial y}\left(\frac{\partial L}{\partial f_y}\right) = \frac{f_{yy}(1+f_x^2)-f_yf_xf_{xy}}{(1+f_x^2+f_y^2)^{3/2}}$$

Thus putting it all together:

$$\begin{aligned}
0 &= \frac{\partial}{\partial x}\frac{\partial L}{\partial f_x} + \frac{\partial}{\partial y}\frac{\partial L}{\partial f_y} \\
&= \frac{f_{xx}(1+f_y^2)+f_{yy}(1+f_x^2)-2f_yf_xf_{xy}}{(1+f_x^2+f_y^2)^{3/2}}
\end{aligned}$$

In particular, the denominator is non-zero, so the Euler-Lagrange Equations are:

$$f_{xx}(1+f_y^2)+f_{yy}(1+f_x^2)-2f_yf_xf_{xy} = 0$$

as required.

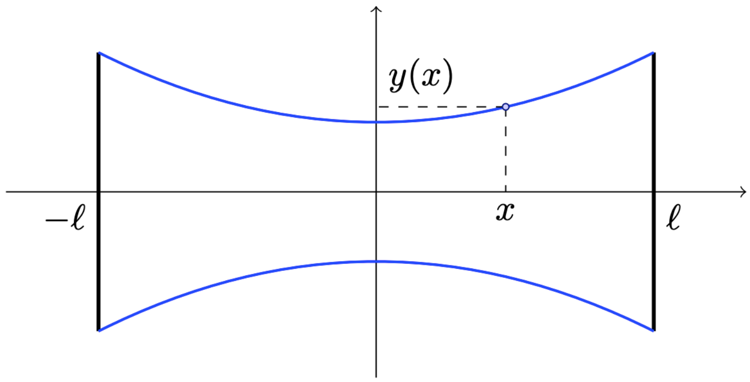
□

3.3 Worked Example: Soap Films

Soap films take shapes according to what minimises the ***surface tension****. This corresponds to minimising the* ***surface area*** *of the film itself. We now consider a particular (simple) example of how a soap film is created, and use variational methods to derive the shape of the resulting film.*

Consider 2 rings of radius r, and places a distance 2ℓ apart, along the same axis (i.e the centers of the rings are aligned). Suppose a soap film has formed between the 2 rings. Determine the shape of the film.

Notice, such a soap film should have rotational symmetry. In particular, it can be thought of as a **surface of revolution**:



That is, we have some curve in 1 dimension $y(x)$ where $x \in [-\ell, \ell]$ and $y(\pm\ell) = r$. The soap film is obtained by rotating y 2π degrees, using the x axis as an axis of rotation. The area of the surface of revolution is:

$$J[y] = \int_{-\ell}^{\ell} 2\pi y(x)\sqrt{1 + y'(x)}dx$$

To derive this area, we can think of cutting y into strips of length:

$$ds = \sqrt{1 + y'(x)}dx$$

If ds is small, then it can be thought of as the height of a cylinder, whose radius will be $y(x)$. In particular, this small cylinder contributes an area of:

$$\underbrace{2\pi y}_{\text{circumference of circle of radius } y} \times \underbrace{\sqrt{1 + y'(x)}dx}_{\text{height of cylinder}}$$

to the surface of revolution.

But recall, we have already since a functional very similar to this, when discussing the **catenary**:

*The **catenary** is an extremal of the functional:*

$$\int_{-\ell_0}^{\ell_0} y(x)\sqrt{1+y'(x)^2}dx$$

subject to a length constraint.
Finding the EL equations leads to the ODE:

$$\left(\frac{y-\lambda}{c}\right)^2 = 1 + (y')^2$$

*(here λ is a **Lagrange Multiplier**)*
Solving the ODE gives:

$$y = c\cosh\frac{x}{c} + h - c\cosh\frac{\ell_0}{c}$$

Notice, since 2π is a constant, minimising J is equivalent to minimising the catenary action, using $\lambda = 0$. In particular, we get the ODE:

$$(y')^2 = \left(\frac{y}{c}\right)^2 - 1$$

which from the working of the catenary (see W8 notes) implies that:

$$y(x) = c\cosh\left(\frac{x}{c}\right)$$

y is known as a **catenoid**.

We need to satisfy the boundary conditions:

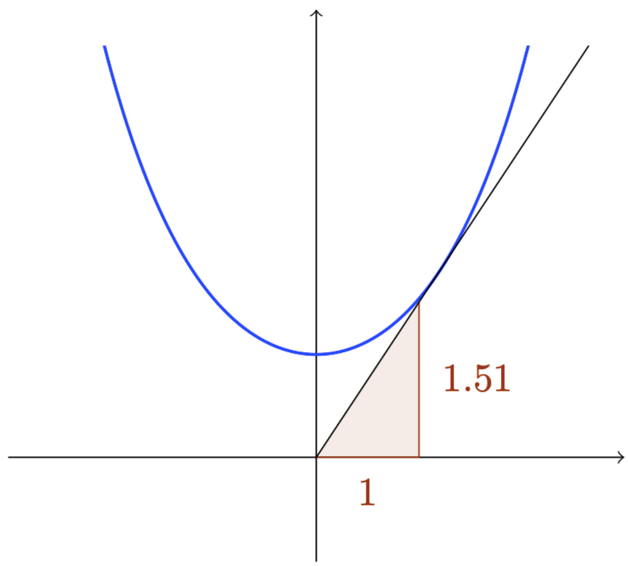
$$y(\ell) = r \implies c\cosh\left(\frac{\ell}{c}\right) = r \;\therefore\; \cosh\left(\frac{\ell}{c}\right) = \frac{r}{c}$$

(since cosh is even, $\cosh\left(\frac{\ell}{c}\right) = \cosh\left(-\frac{\ell}{c}\right)$, so we only need to satisfy one of the conditions to satisfy both)

To see whether this has any solutions, we can define:

$$\zeta = \frac{\ell}{c} \implies \cosh(\zeta) = \frac{r}{\ell}\zeta$$

In other words, the boundary conditions are satisfied if cosh intersects witha line through the origin with gradient $\frac{r}{\ell}$. Since $\cosh(\zeta) \geq 1$, clearly there are some gradient settings for which this intersection won't happen (in fact, an intersection only happens if $\frac{r}{\ell} \geq \rho_c \approx 1.51$).

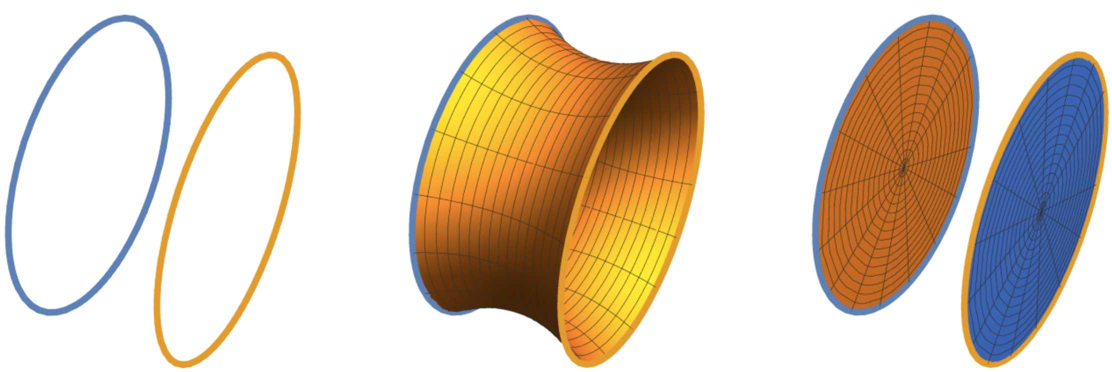


If $\frac{r}{\ell} > \rho_c$, we are guaranteed at least one solution to the boundary constraints. The line intersects $\cosh(\zeta)$ twice, then we'd need to use the action to compute the area of the corresponding surface, and then pick the surface with smallest area (the method has found 2 extremals, but only one will be a minimum).

If $\frac{r}{\ell} = \rho_c$, then the line intersects $\cosh(\zeta)$ precisely once, and there is a unique solution.

If $\frac{r}{\ell} < \rho_c$, then this method doesn't tell us anything. J might still be minimisible, but by a class of functions which isn't a surface of revolution. For instance, if $y(x)$ is a discontinuous function, such that the soap film has "broken", and it extends between the 2 rings individually:

(a) Boundary (b) Catenoid (c) Disks



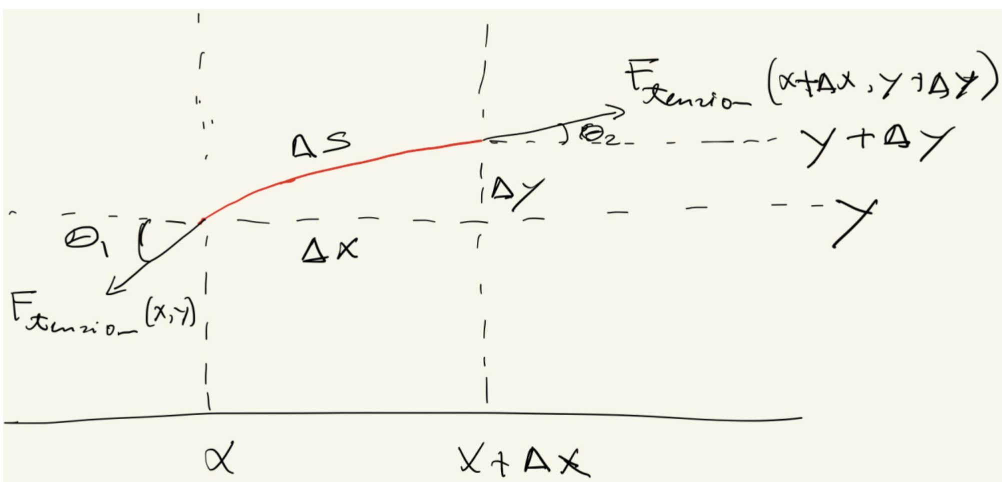
4 The One-Dimensional Wave Equation

4.1 Proposition: The One-Dimensional Wave Equation

> *Let $y(t, x)$ denote the position of an* ***oscillating string*** *at time t and position $x \in [0, \ell]$. Moreover, assume that the string is* ***fixed*** *at the* ***endpoints****:*
>
> $$\forall t > 0, \quad y(t, 0) = y(t, \ell) = 0$$
>
> *and that it has* ***constant mass density*** *ρ.*
> *Then, assuming that the* ***string*** *vibrates with a* ***small amplitude****, y satisfies the* ***one-dimensional wave equation****:*
>
> $$\rho \frac{\partial^2 y}{\partial t^2} = \tau \frac{\partial^2 y}{\partial x^2}$$
>
> *where τ is a* ***tension*** *term.*

4.1.1 From Physics

Proof. We shall use the following diagram to aid the derivation:



- we consider a small segment of horizontal position, from x to $x + \Delta x$
- similarly, we consider the vertical displacement on this interval, from y to $y + \Delta y$
- the length of the string on these intervals is Δs
- at the endpoints of the intervals, there are forces due to **tension**:

$$F_{tension}(x, y) \qquad F_{tension}(x + \Delta x, y + \Delta y)$$

- tension forces have the **same magnitude** τ, and act **tangentially** to the string in opposite directions:

$$F_{tension}(x, y) = -\tau(\cos(\theta_1), \sin(\theta_1))$$

$$F_{tension}(x, y) = \tau(\cos(\theta_2), \sin(\theta_2))$$

- moreover, recall that for some function $f(x)$, its **Taylor Expansion** about the point a is:

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

In particular, if we expand $f(x + \Delta x)$ about the point x:

$$f(x+\Delta x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(x)}{n!}(x+\Delta x - x)^n = \sum_{n=1}^{\infty} \frac{f^{(n)}(x)}{n!}\Delta x^n$$

We can define:

$$\Delta\theta = \theta_2 - \theta_1$$

The tangent is the quotient of opposite and adjacent, so:

$$\tan(\theta_1) \approx \frac{\partial y}{\partial x}(x)$$

$$\tan(\theta_2) \approx \frac{\partial y}{\partial x}(x+\Delta x)$$

We now consider the forces along the x and y directions. Along the x direction, the total force is:

$$\tau(\cos(\theta_2) - \cos(\theta_1))$$

But we can write:

$$\cos(\theta_2) = \cos(\theta_1 + \Delta\theta)$$

so Taylor Expanding about θ_1:

$$\cos(\theta_2) = \cos(\theta_1 + \Delta\theta) = \cos(\theta_1) - \Delta\theta\sin(\theta_1) + \mathcal{O}(\Delta\theta^2)$$

Hence:

$$\tau(\cos(\theta_2) - \cos(\theta_1)) \approx \tau(-\Delta\theta\sin(\theta_1) + \mathcal{O}(\Delta\theta^2))$$

But now Taylor expanding $\sin(\theta_1)$ about 0:

$$\sin(\theta_1) = \theta_1 + \mathcal{O}(\theta_1^3)$$

and since θ_1 is small:

$$\tau(\cos(\theta_2) - \cos(\theta_1)) \approx 0$$

Hence, the horizontal force is 0.

We now consider the vertical force:

$$\begin{aligned}
\tau(\sin(\theta_2) - \sin(\theta_1)) &= \tau(\cos(\theta_2)\tan(\theta_2) - \cos(\theta_1)\tan(\theta_2)) \\
&\approx \tau\left(\cos(\theta_2)\frac{\partial y}{\partial x}(x+\Delta x) - \cos(\theta_1)\frac{\partial y}{\partial x}(x)\right) \\
&= \tau\left(\cos(\theta_1+\Delta\theta)\frac{\partial y}{\partial x}(x+\Delta x) - \cos(\theta_1)\frac{\partial y}{\partial x}(x)\right) \\
&\approx \tau\left(\left[\cos(\theta_1) - \Delta\theta\sin(\theta_1) + \mathcal{O}(\Delta\theta^2)\right]\left[\frac{\partial y}{\partial x}(x) + \Delta x\frac{\partial^2 y}{\partial x^2} + \mathcal{O}(\Delta x^2)\right] - \cos(\theta_1)\frac{\partial y}{\partial x}(x)\right) \\
&\approx \tau\left(\cos(\theta_1)\Delta x\frac{\partial^2 y}{\partial x^2} - \Delta\theta\sin(\theta_1)\frac{\partial y}{\partial x}(x)\right) \\
&\approx \tau\Delta x\frac{\partial^2 y}{\partial x^2}
\end{aligned}$$

where the terms involving $\Delta\theta\Delta x$ are very small and thus have been dropped; similarly, since θ_1 is small, $\cos(\theta_1) \approx 1$ and $\sin(\theta_1) \approx 0$.

Hence, if we apply Newton's Second Law on the segment of string of length Δs, the only force comes from the vertical tension so:

$$\underbrace{\tau \Delta x \frac{\partial^2 y}{\partial x^2}}_{force} = \underbrace{\rho \Delta s}_{mass} \underbrace{\frac{\partial^2 y}{\partial t^2}}_{acceleration}$$

But now, we know that, since we have a small amplitude:

$$\Delta s = \int_x^{x+\Delta x} \sqrt{1 + \left(\frac{dy}{dz}\right)^2} dz \approx \int_x^{x+\Delta x} \sqrt{1+0} dz = \Delta x$$

so:

$$\tau \Delta x \frac{\partial^2 y}{\partial x^2} = \rho \Delta x \frac{\partial^2 y}{\partial t^2} \implies \tau \frac{\partial^2 y}{\partial x^2} = \rho \frac{\partial^2 y}{\partial t^2}$$

as required.

□

4.1.2 From Euler-Lagrange Equations

Recall, for functions of 2 variables (in our case $y = y(x,t)$), the Euler-Lagrange Equations are:

$$\frac{\partial L}{\partial y} = \frac{\partial}{\partial t} \frac{\partial L}{\partial y_t} + \frac{\partial}{\partial x} \frac{\partial L}{\partial y_x}$$

Again, assuming a small amplitude, the kinetic energy of the string will be $\frac{1}{2}\rho y_t^2$, whilst its potential energy will be $\frac{1}{2}\tau y_x^2$. Hence, we define a Lagrangian:

$$L(y_t, y_x) = \frac{1}{2}\rho y_t^2 - \frac{1}{2}\tau y_x^2$$

The Euler-Lagrange Equation is thus:

$$\frac{\partial L}{\partial y_t} = \rho y_t$$

$$\frac{\partial L}{\partial y_x} = -\tau y_x$$

so:

$$0 = \rho y_{tt} - \tau y_{xx} \implies \rho y_{tt} = \tau y_{xx}$$

as required.

4.2 Theorem: Solutions to the One-Dimensional Wave Equation

The general solution for the wave equation:

$$y_{xx} - \frac{1}{v^2} y_{tt} = 0$$

subject to the initial condition:

$$y(t, 0) = y(t, \ell) = 0$$

where:

$$v^2 = \frac{\tau}{\rho} \qquad x \in [0, \ell] \qquad t > 0$$

is:

$$y(t, x) = f(x + vt) + g(x - vt)$$

where f, g are 2ℓ periodic functions, and:

$$f(x + vt) = -g(-x - vt)$$

*In other words, solutions to the wave equation are 2 identical waves, moving in opposite directions, thus forming a **standing wave**.*

Proof. We introduce the change of variables:

$$U = x + vt \qquad V = x - vt$$

Thus, if we define:

$$y(t, x) = \Psi(U, V)$$

we have that:

$$y_t = \Psi_U U_t + \Psi_V V_t = \Psi_U v - \Psi_V v$$

$$\begin{aligned} y_{tt} &= \Psi_{UU} v U_t + \Psi_{UV} v V_t - \Psi_{VV} v V_t - \Psi_{VU} v U_t \\ &= \Psi_{UU} v^2 - \Psi_{UV} v^2 + \Psi_{VV} v^2 - \Psi_{VU} v^2 \\ &= v^2 (\Psi_{UU} + \Psi_{VV} - 2\Psi_{UV}) \end{aligned}$$

$$y_x = \Psi_U U_x + \Psi_V V_x = \Psi_U + \Psi_V$$

$$\begin{aligned} y_x &= \Psi_{UU} U_x + \Psi_{UV} V_x + \Psi_{VV} V_x + \Psi_{VU} U_x \\ &= \Psi_{UU} + \Psi_{UV} - \Psi_{VV} + \Psi_{VU} \\ &= \Psi_{UU} + \Psi_{VV} + 2\Psi_{UV} \end{aligned}$$

So the wave equation becomes:

$$y_{xx} - \frac{1}{v^2} y_{tt} = 0 \implies 4\Psi_{UV} = 0 \implies \Psi_{UV} = 0$$

Explicitly, this says that:

$$\begin{aligned}
&\frac{\partial}{\partial V}\left(\frac{\partial \Psi}{\partial U}\right) = 0 \\
\implies &\frac{\partial \Psi}{\partial U} = f(U) \\
\implies &\Psi(U, V) = f(U) + g(V) \\
\implies &y(t, x) = f(x + vt) + g(x - vt)
\end{aligned}$$

(here we are being a bit careless, by ignoring rewriting some integrals in terms of the function being integrated)

We need to make sure that y satisfies the boundary conditions:

$$\forall t > 0, \quad y(t, 0) = y(t, \ell) = 0$$

If $x = 0$:

$$y(t, 0) = 0 \implies f(vt) + g(-vt) = 0$$

But this relation must be true for **any** t, so in particular it must be true for $t := x + vt$ for any fixed x:

$$f(x + vt) = -g(-x - vt)$$

If $x = \ell$:

$$y(t, \ell) = 0 \implies f(\ell + vt) + g(\ell - vt) = 0$$

but using the relation for the condition $x = 0$:

$$f(\ell + vt) = -g(-\ell - vt)$$

so we have that:

$$-g(-\ell - vt) = -g(\ell - vt)$$

In other words, g must be 2ℓ periodic, since adding 2ℓ to its argument leaves the value of g unchanged.

We can thus write:

$$y(t, x) = g(x - vt) - g(-x - vt)$$

To fully specify y on $[0, \ell]$, we need to provide the initial conditions for y at $t = 0$, noting that:

$$y(0, x) = g(x) - g(-x)$$

$$y_t(0, x) = -v(g_x(x) + g_x(-x))$$

□