Introduction to Representation Theory - Week 6 & 7 - Induced Representations

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Contents

Based on the notes by Konstantin Ardakov, Section 6

1 Induced Modules

1.1 Recap: Invariant Modules

Let V be a $\mathbb{C}G$-module. The ***invariant submodule*** *of V is:*

$$V^G = \{v \in V \mid \forall g \in G,\ g \cdot v = v\}$$

V^G is the ***largest subspace*** *of V which is fixed by G.*
(Definition 5.19)

1.2 Coinvariants

1.2.1 Definition: Space of H-Coinvariants

Let H be a ***finite group****, and let V be a* ***kH-module****. The* ***vector space of H-coinvariants*** *of V is:*

$$V_H = V/(H-1)V$$

where:

$$(H-1)V = k\{h \cdot v - v \mid h \in H, v \in V\}$$

V_H is the ***largest quotient*** *kH-module of V which is* ***isomorphic*** *to the* ***trivial module****.*
(Definition 6.1)

1.2.2 Lemma: Factoring Through Coinvariants

Let G be a ***finite group*** *with* ***subgroup*** *$H \leq G$.*

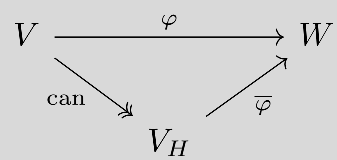
1. *Let W be a kH*-***module***, *defined by H acting* ***trivially***. *If:*

   $$\varphi : V \to W$$

   is a kH-***module homomorphism***, *then:*

   $$\exists! \overline{\varphi} : V_H \to W$$

   such that:

   commutes

2. *If*

   $$\psi : V \to W$$

   is a kH-***linear morphism***, *then it* ***induces*** *a morphism:*

   $$\psi_H : V_H \to W_H$$

3. *Given an* ***external direct sum*** *of kH*-***modules*** *$V \oplus W$, the H*-***coinvariant*** *distributes:*

   $$(V \oplus W)_H \cong V_H \oplus W_H$$

1.2.3 Remark: Isomorphism Between Invariants and Coinvariants

*We always have **mappings** from **invariants** to **coinvariants**:*

$$V^H \hookrightarrow V \twoheadrightarrow V_H$$

*where $V^H \hookrightarrow V$ is simply an **inclusion** (as $V^H \subseteq V$), and $V \twoheadrightarrow V_H$ is the **surjective canonical map** defined by taking a quotient.*

---

*If **Maschke's Theorem** applies (so that $\text{char}(k)$ doesn't divide $|G|$), then the **composition***

$$V^H \to V_H$$

*is an **isomorphism**. This is because by **Maschke's Theorem** and **complete reducibility** either:*

- *V^H, V_H are **one-dimensional***
- *V^H, V_H are both **trivial** ($\{0\}$)*

*In particular, the **invariants** of a non-trivial **irreducible** representation are 0 (otherwise, we'd have a non-trivial map onto the non-trivial representation, and so the kernel would be non-zero, which contradicts irreducibility, as the kernel is always a G-stable subspace).*

1.2.4 Example: Invariants and Coinvariants Differ

- let

$$H = \mathbb{Z} = \langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$$

- then, we can view kH as a **ring of polynomials**, since:

$$f \in kH \implies f = \sum_{z \in H} k_z \cdot z = \sum_{n \in \mathbb{Z}} k_z \cdot g^n$$

  so f is a polynomial with coefficients in k, and whose "variable" is g (and g^{-1}):

$$kG \cong k[g, g^{-1}] \left(= \bigoplus_{n \in \mathbb{Z}} kg^n \right)$$

- let V be a free kH module of **rank** 1 (i.e V is a 1-dimensional module over kH)
- we have that:

$$\begin{aligned} V^H &= \{f \in kH \mid \forall n \in \mathbb{Z},\ g^n \cdot f = f\} \\ &\subseteq \{f \in kH \mid \forall n \in \mathbb{Z},\ g \cdot f = f\} \\ &= \{f \in kH \mid \forall n \in \mathbb{Z},\ (g - e) \cdot f = 0\} \end{aligned}$$

- but since $\mathbb{Z}$ and k are integral domains, then the polynomial ring $k[g, g^{-1}]$ is itself an integral domain. Moreover, $g - e \in k[g, g^{-1}]$ clearly, so:

$$\forall f \in kH, (g-e) \cdot f = 0 \iff f = 0$$

so:

$$V^H \subseteq \{0\} \implies V^H = \{0\}$$

- now, define:

$$\varphi : V \to k$$

via:

$$\varphi(g^n) = 1_k$$

and φ is trivial on elements of k. Then, there exists some non-trivial morphism

$$\overline{\varphi} : V_H \to k$$

by properties of the coinvariant, so in particular:

$$V_H \neq \{0\}$$

- hence, we see than when **Mashcke's Theorem** doesn't apply, we don't always have an isomorphism between **invariants** and **coinvariants**

1.3 Lemma: Group Actions on $kG \otimes W$

*Let G be a **finite group**, with **subgroup** H. Let W be a kH**-module**. Then:*

1. *there is a **left** G**-action** on $kG \otimes W$, defined on **elementary tensors** via:*

$$\forall g, x \in G, w \in W, \qquad g \cdot (x \otimes w) = (gx) \otimes w$$

2. *there is a **right** H**-action** on $kG \otimes W$, defined on **elementary tensors** via:*

$$\forall h \in H, x \in G, w \in W, \qquad h * (x \otimes w) = (xh^{-1}) \otimes (h \cdot w)$$

3. *the **left** G**-action** and the **right** H**-action commute pointwise**:*

$$\forall g \in Gm, h \in H, u \in kG \otimes W, \qquad g \cdot (h * u) = h * (g \cdot u)$$

(Lemma 6.2)

Proof. These G-actions are defined on the elementary tensors. The crux of the proof is showing that these G-actions are indeed well-defined. To do this, the principal strategy is to use the Universal Property of the Tensor Product:

> *Let V, W, U be* ***vector spaces****. Then, for every* ***bilinear map***
>
> $$b : V \times W \to U$$
>
> *there is a* ***unique linear map***
>
> $$\tilde{b} : V \otimes W \to U$$
>
> *such that:*
>
> $$b = \tilde{b} \cdot \otimes$$
>
> *In other words:*
>
> $$\forall v, w \in V, \qquad b(v, w) = \tilde{b}(v \otimes w)$$
>
> *(Lemma 4.9)*

For example, for (1). Fix $g \in G$, and define:

$$\varphi_g : kG \times W \to kG \otimes W$$

via:

$$(x, w) \mapsto gx \otimes w$$

It is straightforward (albeit tedious) to show that φ_g is bilinear. Then, by the Universal Property, we get a map:

$$\rho_g : kG \otimes W \to kG \otimes W$$

defined precisely by:

$$\rho_G(x \otimes w) = \varphi_g(x, w) = gx \otimes w$$

which is our G-action. Now that we know that the G-action can be seen as a unique linear morphism, it is sufficient to check that it satisfies the properties of a G-action on the elementary tensors. Indeed:

$$\rho_e(x \otimes w) = (ex) \otimes w = x \otimes w$$

and

$$\rho_g(\rho_h(x \otimes w)) = \rho_g(hx \otimes w) = (gh)x \otimes w = \rho_{gh}(x \otimes w)$$

so the right G-action is well-defined.

(2) can be proven in a similar way.

For (3), we just apply the definition. If $u = x \otimes w$:

$$\begin{aligned} g \cdot (h * (x \otimes w)) &= g \cdot (xh^{-1} \otimes h \cdot w) \\ &= gxh^{1} \otimes h \cdot w \\ &= h * (gx \otimes w) \\ &= h * (g \cdot (x \otimes w)) \end{aligned}$$

□

1.4 Corollary: Space of Coinvariants of $kG \otimes W$ is kG-Module

Let:

- *G be a* ***finite group***
- *H be a* ***subgroup*** *$H \leq G$*
- *W be a kH*-***module***

Then, the space of H ***coinvariants*** *of $kG \otimes W$:*

$$(kG \otimes W)_H$$

with respect to the ***right*** *H*-***action*** *is a kG*-***module****.*
(Corollary 6.3)

Proof. When we say "$(kG \otimes W)_H$ with respect to the right H-action", we mean that:

$$(kG \otimes W)_H = (kG \otimes W) / \left[(H-1) * (kG \otimes W)\right]$$

To show that this is a kG-module, it is sufficient to see that both $kG \otimes W$ and $[(H-1) * (kG \otimes W)]$ are preserved under the right G-action (i.e they are G-stable).

That $kG \otimes W$ is G-stable is immediate from the definition of G-action on $kG \otimes W$.

That $[(H-1) * (kG \otimes W)]$ is G-stable follows from the pointwise commutativity of the left H-action and the right G-action, since if $u \in kG \otimes W$ and:

$$v = h * u - u \in (H-1) * (kg \otimes W)$$

then:

$$g \cdot v = g \cdot (h * u - u) = h * (g \cdot u) - (g \cdot u) = (h-1) * (g \cdot u) \in (H-1) * (kG \otimes W)$$

□

1.5 Induced Modules

1.5.1 Definition: The Induced kG-Module

Let:

- *G be a **finite group***
- *H be a **subgroup** $H \leq G$*
- *W be a kH**-module***

Then:

1. *the **induced** kG**-module** is:*

$$\mathrm{Ind}_H^G W = (kG \otimes W)_H$$

*the space of H**-coinvariants** of $kG \otimes W$ under the **right** H**-action***

2. *we write $g \overline{\otimes} w$ as the **image** of $g \otimes w \in kG \otimes W$ in the **quotient space**:*

$$\mathrm{Ind}_H^G W = (kG \otimes W)_H$$

(Definition 6.4)

1.5.2 Definition: The Induced Representation

Let:

- G *be a* ***finite group***
- H *be a* ***subgroup*** $H \leq G$
- W *be a* kH***-module***

Let:

$$\sigma : H \to GL(W)$$

be a representation of H *afforded by* W*.*
Then, the ***induced representation*** *is the representation of* G *afforded by* $\operatorname{Ind}_H^G W$*:*

$$\operatorname{Ind}_H^G \sigma : G \to GL(\operatorname{Ind}_H^G W)$$

(Definition 6.4)

1.5.3 Lemma: Actions in the Induced kG-Module

Let:

- G *be a* ***finite group***
- H *be a* ***subgroup*** $H \leq G$
- W *be a* kH***-module***

If $g \in G, w \in W$*, then:*

1. $$\forall h \in H, \qquad gh \mathbin{\overline{\otimes}} w = g \mathbin{\overline{\otimes}} h \cdot w$$

2. $$\forall x \in G, \qquad g \cdot (x \mathbin{\overline{\otimes}} w) = gx \mathbin{\overline{\otimes}} w$$

(Lemma 6.5)

Proof. To show that $g_1 \overline{\otimes} w_1 = g_2 \overline{\otimes} w_2$, we need to show (by definition) that:

$$g_1 \otimes w_1 - g_2 \otimes w_2 \in (H-1) * (kg \otimes W)$$

(1)

$$\begin{aligned} g \otimes h \cdot w - gh \otimes w &= h * (gh \otimes w) - gh \otimes w \\ &= (h-1) * (gh \otimes w) \in (H-1) * (kg \otimes W) \end{aligned}$$

so as required:

$$gh \overline{\otimes} w = g \overline{\otimes} h \cdot w$$

(2)

We have that:

$$g \cdot (x \otimes w) = gx \otimes w$$

so immediately:

$$g \cdot (x \otimes w) - gx \otimes w = 0 \in (H-1) * (kg \otimes W)$$

□

1.5.4 Lemma: H-Stable Subspaces of $kG \otimes W$

Let:

- *G be a **finite group***
- *H be a **subgroup** $H \leq G$*
- *W be a kH**-module***

Let $x \in G$. Then:

$$xkH \otimes W$$

*is an H**-stable subspace** of $kG \otimes W$ under the **right** H**-action**, and there is a **linear isomorphism**:*

$$\alpha : W \to (xkH \otimes W)_H$$

defined by:

$$w \mapsto x \overline{\otimes} w$$

such that:

$$(xkH \otimes W)_H = x \overline{\otimes} W$$

(Lemma 6.6)

Proof. For the first part, suppose that we have any element:

$$(kxh) \otimes w$$

where $h \in H, g \in G, w \in W$. Then, for any $z \in H$:

$$z * ((kxh) \otimes w) = kx(hz^{-1}) \otimes z \cdot w$$

but:

- since H is a subgroup, and $h, z \in H$:
$$hz^{-1} \in H$$
- $w \in W$, and W is a kH submodule, so $z \cdot w \in W$

Thus:

$$\forall z \in H, \qquad z * ((kxh) \otimes w) \in xkH \otimes W$$

so $z * ((kxh) \otimes w)$ is a H-stable subspace (that it is a subspace is clear, as we are just identifying a subset kxH of kG which satisfies the subspace structure).

For the second assertion, define a (bilinear) map

$$\tilde{\beta} : xkH \times W \to W$$

via:

$$\tilde{\beta}\left(\sum_{h \in H} \lambda_h xh, w\right) = \sum_{h \in H} \lambda_h (h \cdot w)$$

In particular:

$$(xh, w) \mapsto h \cdot w$$

Now, recalling the Universal Property of the Tensor Product:

> *Let V, W, U be* ***vector spaces****. Then, for every* ***bilinear map***
>
> $$b : V \times W \to U$$
>
> *there is a* ***unique linear map***
>
> $$\tilde{b} : V \otimes W \to U$$
>
> *such that:*
>
> $$b = \tilde{b} \cdot \otimes$$
>
> *In other words:*
>
> $$\forall v, w \in V, \qquad b(v, w) = \tilde{b}(v \otimes w)$$
>
> *(Lemma 4.9)*

this implies that we have a unique linear map:

$$\beta : xkH \otimes W \to W$$

which is defined on elementary tensors via:

$$\forall h \in H, w \in W, \qquad \beta(xh \otimes w) = h \cdot w$$

In particular, for any $y \in H$:

$$\begin{aligned}\beta(y * (xh \otimes w)) &= \beta(xhy^{-1} \otimes y \cdot w) \\ &= (hy^{-1}) \cdot (y \cdot w) \\ &= h \cdot w \\ &= \beta(xh \otimes w)\end{aligned}$$

This implies that:

$$\beta((y-1) * (xh \otimes w))$$

so in particular, β is 0 on $(H-1) * (xkH \otimes W)$.

Hence, using

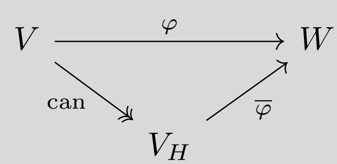
*Let W be a kH**-module**, defined by H acting **trivially**. If:*

$$\varphi : V \to W$$

*is a kH**-module homomorphism**, then:*

$$\exists! \overline{\varphi} : V_H \to W$$

such that:



commutes

implies there exists a well-defined lienar map:

$$\overline{\beta} : (xkH \otimes W)_H \to W$$

defined by:

$$xh \,\overline{\otimes}\, w \mapsto h \cdot w$$

We claim that $\overline{\beta}$ defines an inverse to:

$$\alpha : W \to (xkH \otimes W)_H$$

defined by:

$$w \mapsto x \,\overline{\otimes}\, w$$

Indeed, for any $h \in H, w \in W$:

$$\alpha(\overline{\beta}(xh \,\overline{\otimes}\, w)) = \alpha(h \cdot w) = x \,\overline{\otimes}\, h \cdot w = xh \,\overline{\otimes}\, w$$

where we have used:

Let:

- *G be a **finite group***
- *H be a **subgroup** $H \leq G$*
- *W be a kH**-module***

If $g \in G, w \in W$, then:

1. $$\forall h \in H, \qquad gh \overline{\otimes} w = g \overline{\otimes} h \cdot w$$
2. $$\forall x \in G, \qquad g \cdot (x \overline{\otimes} w) = gx \overline{\otimes} w$$

(Lemma 6.5)

Thus,

$$\alpha \circ \overline{\beta} = 1_{(xkH \otimes W)_H}$$

Similarly:

$$\overline{\beta}(\alpha(w)) = \overline{\beta}(x \overline{\otimes} w) = 1_H \cdot w = w$$

so:

$$\overline{\beta} \circ \alpha = 1_W$$

Hence, α is an isomorphism, as required.

□

1.5.5 Proposition: Decomposing the Induced kG-Module

Let:

- G *be a **finite group***
- H *be a **subgroup** $H \leq G$*
- W *be a kH**-module***

Let

$$\{x_1, \dots, x_m\}$$

*be a complete set of **left coset representatives** for H in G, such that:*

$$G = x_1 H \sqcup x_2 H \sqcup \dots \sqcup x_m H$$

Then:

1. *there is a **vector space decomposition** for the **induced** kG**-module** defined by the **cosets**:*

$$\operatorname{Ind}_H^G W = \bigoplus_{i=1}^{m} x_i \overline{\otimes} W$$

2. $$\dim\left(\operatorname{Ind}_H^G W\right) = |G/H| \dim(W)$$

(Proposition 6.7)

Proof. By Lemma 6.6 above:

$$\forall i \in [1, m], \qquad xkH \otimes W$$

are H-stable subspaces of $kG \otimes W$ under the right H-action. Since G is a disjoint union of cosets $x_i H$, we can write:

$$kG \otimes W = \left(\bigoplus_{i=1}^{m} x_i kH \right) \otimes W = \bigoplus_{i=1}^{m} (x_i kH \otimes W)$$

Now, using:

> *Given an **external direct sum** of kH**-modules** $V \oplus W$, the H**-coinvariant** distributes:*
>
> $$(V \oplus W)_H \cong V_H \oplus W_H$$

alongside Lemma 6.6:

> *Let:*
>
> - *G be a **finite group***
> - *H be a **subgroup** $H \leq G$*
> - *W be a kH**-module***
>
> ---
>
> *Let $x \in G$. Then:*
>
> $$xkH \otimes W$$
>
> *is an H**-stable subspace** of $kG \otimes W$ under the **right** H**-action**, and there is a **linear isomorphism**:*
>
> $$\alpha : W \to (xkH \otimes W)_H$$
>
> *defined by:*
>
> $$w \mapsto x \overline{\otimes} w$$
>
> *such that:*
>
> $$(xkH \otimes W)_H = x \overline{\otimes} W$$
>
> *(Lemma 6.6)*

we get that:

$$\operatorname{Ind}_H^G W = (kG \otimes W)_H \cong \bigoplus_{i=1}^{m} (x_i kH \otimes W)_H = \bigoplus_{i=1}^{m} (x_i \overline{\otimes} W)$$

Now, since $x_i \in G$, in particular:

$$\dim(x_i \overline{\otimes} W) = \dim(x_i) \dim(W) = \dim(W)$$

Then, computing the dimension over a direct sum yields that since we have $|G/H|$ representatives $x_i \in G$:

$$\dim \left(\operatorname{Ind}_H^G W \right) = |G/H| \dim(W)$$

as required.

$\square$

1.5.5.1 Example: Induced Module for the Trivial Representation

- let G be some finite group with subgroup H

- let $\mathbb{1}$ be a kH-module under the trivial action of H (so that $W = k\{1\}$)
- then, by the above Proposition:

$$\dim(\operatorname{Ind}_H^G \mathbb{1}) = |G/H| = \dim(k(G/H))$$

- moreover, a basis for $\operatorname{Ind}_H^G \mathbb{1}$ is given by:

$$\{x_1 \overline{\otimes} 1, \ldots, x_m \overline{\otimes} 1\}$$

- the action of G on this basis is given by:

$$g \cdot (x_i \overline{\otimes} 1) = gx_i \overline{\otimes} 1$$

- if $\sigma(g)$ is some permutation of $[1, m]$, we can then write:

$$gx_i = x_{\sigma(g)(i)} h$$

  where:

$$h = x_{\sigma(g)(i)}^{-1} g x_i$$

  In particular, we thus have that

$$g \cdot (x_i \overline{\otimes} 1) = x_{\sigma(g)(i)} h \overline{\otimes} 1 = x_{\sigma(g)(i)} \overline{\otimes} h \cdot 1 = x_{\sigma(g)(i)} \overline{\otimes} 1$$

  by using:

> *Let:*
>
> - G *be a* ***finite group***
> - H *be a* ***subgroup*** $H \leq G$
> - W *be a* kH***-module***
>
> ---
>
> *If* $g \in G, w \in W$*, then:*
>
> *1.*
> $$\forall h \in H, \qquad gh \overline{\otimes} w = g \overline{\otimes} h \cdot w$$
> *2.*
> $$\forall x \in G, \qquad g \cdot (x \overline{\otimes} w) = gx \overline{\otimes} w$$
>
> *(Lemma 6.5)*

- this tells us that the action of G is thus determined by its effect on the representatives x_i
- in particular, since $\sigma(g)$ is a bijection, this defines an isomorphism of kG-modules:

$$\rho : k(G/H) \to \operatorname{Ind}_H^G$$

  via:

$$gH \mapsto g \overline{\otimes} 1$$

2 Characters of Induced Representations

2.1 Restriction Module

2.1.1 Definition: Restriction Module

- *G be a **finite group***
- *H be a **subgroup** $H \leq G$*
- *V be a kG**-module***

Then, define

$$\mathrm{Res}_H^G V$$

*to be the kH**-module** resulting from restricting the action of kG on V to the action of the **subring** kH.*
(Definition 6.9)

2.1.2 Definition: Induced and Restricted Characters

*Let $H \leq G$ be a **subgroup** of the **finite group** G. Then:*

- *if ψ is the **character** of G afforded by the $\mathbb{C}G$-module V, then*

  $$\mathrm{Res}_H^G \psi$$

  *is the **restricted character** of the $\mathbb{C}H$-module $\mathrm{Res}_H^G V$*
- *if φ is the **character** of H afforded by the $\mathbb{C}H$-module W, then*

  $$\mathrm{Ind}_H^G \varphi$$

  *is the **induced character** of the $\mathbb{C}G$-module $\mathrm{Ind}_H^G W$*

(Definition 6.11)

2.1.3 Proposition: Induction is the Left Adjoint of Restriction

Let:

- *G be a **finite group***
- *H be a **subgroup** $H \leq G$*
- *U be a kG**-module***
- *W be a kH**-module***

*Then, there exists a **linear isomorphism***

$$\Phi : \operatorname{Hom}_{kG}(\operatorname{Ind}_H^G W, U) \to \operatorname{Hom}_{kH}(W, \operatorname{Res}_H^G U)$$

defined by:

$$\forall \alpha \in \operatorname{Ind}_H^G W, w \in W, \qquad \Phi(\alpha)(w) = \alpha(1 \overline{\otimes} w)$$

(Proposition 6.10)

Proof. First, we need to show that Ψ is well-defined, as it is mapping between homomorphism which go between cosets. To do this, we show that Φ can be realised as a composition of well-defined mappings. In particular, suppose we are given a kG-linear map:

$$\alpha : \operatorname{Ind}_H^G W \to U$$

Then, we obtain a well-defined kH-linear mapping via restriction:

$$\operatorname{Res}_H^G \alpha : \operatorname{Res}_H^G \operatorname{Ind}_H^G W \to \operatorname{Res}_H^G U$$

Now, define a map:

$$\gamma : W \to \operatorname{Res}_H^G \operatorname{Ind}_H^G W$$

via:

$$w \mapsto 1 \overline{\otimes} w$$

This will be kH-linear, since if we use:

Let:

- *G be a* ***finite group***
- *H be a* ***subgroup*** *$H \leq G$*
- *W be a kH**-module***

If $g \in G, w \in W$, then:

1. $$\forall h \in H, \qquad gh \overline{\otimes} w = g \overline{\otimes} h \cdot w$$

2. $$\forall x \in G, \qquad g \cdot (x \overline{\otimes} w) = gx \overline{\otimes} w$$

(Lemma 6.5)

we get that:

$$\gamma(h \cdot w) = 1 \overline{\otimes}(h \cdot w) = h \overline{\otimes} w = h \cdot (1 \overline{\otimes} w) = h \cdot \gamma(w)$$

Then, if we precompose γ and $\operatorname{Res}_H^G \alpha$, we obtain a map:

$$\gamma \circ \operatorname{Res}_H^G \alpha : W \to \operatorname{Res}_H^G U$$

which defines out kH-linear map $\Phi(\alpha)$. Thus, since Φ is defined by well-defined operations (like kH-linear maps), Φ is well-defined.

To show that there is a bijection, we construct a map:

$$\Psi : \operatorname{Hom}_{kH}(W, \operatorname{Res}_H^G U) \to \operatorname{Hom}_{kG}(\operatorname{Ind}_H^G W, U)$$

For this, given some:

$$\beta : W \to \operatorname{Res}_H^G U$$

define:

$$\Psi(\beta)(g \overline{\otimes} w) = g \cdot \beta(w)$$

We again need to show that Ψ is well-defined. To this end, we need to show that

$$\forall g \in G, h \in H, w \in W, \qquad (gh) \cdot \beta(w) = g \cdot \beta(h \cdot w)$$

since this ensures that the action of gh on $\beta(w)$ is compatible with the action on induced representations. However, this is immediate from the fact that β is, by definition, a kH-linear map. Thus, it remains to shwo that $\Psi(\beta)$ indeed defines a kG-linear mapping. Taking any $g, x \in G, w \in W$, we have that:

$$\begin{aligned} \Psi(\beta)(g \cdot (x \overline{\otimes} w)) &= \psi(\beta)(gx \overline{\otimes} w) \\ &= (gx) \cdot \beta(w) \\ &= g \cdot (x \cdot \beta(w)) \\ &= g \cdot \Psi(\beta()(x \overline{\otimes} w) \end{aligned}$$

so it is indeed a kG-linear map. In particular, this shows that indeed Ψ defines the desired linear map:

$$\Psi : \mathrm{Hom}_{kH}(W, \mathrm{Res}_H^G U) \to \mathrm{Hom}_{kG}(\mathrm{Ind}_H^G W, U)$$

Finally, we show that Φ, Ψ are mutual inverses, and thus, define the desired isomorphism. Firstly, suppose that we have a kG-linear map:

$$\alpha : \mathrm{Ind}_H^G W \to U$$

Then, for any $g \in G, w \in W$:

$$\Psi(\Phi(\alpha))(g \overline{\otimes} w) = g \cdot \Phi(\alpha)(w) = g \cdot \alpha(1 \overline{\otimes} w)$$

By kG-linearity of α, we then have that:

$$\Psi(\Phi(\alpha))(g \overline{\otimes} w) = \alpha(g \overline{\otimes} w)$$

which shows that as required:

$$\Psi(\Phi(\alpha)) = \alpha$$

Secondly, suppose that we have a kH-linear map:

$$\beta : W \to \mathrm{Res}_H^G U$$

Then $\forall w \in W$:

$$\Phi(\Psi(\beta))(w) = \Psi(\beta)(1 \overline{\otimes} w) = 1 \cdot \beta(w) = \beta(w)$$

so as required:

$$\Phi(\Psi(\beta)) = \beta$$

□

2.1.4 Corollary: Frobenius Reciprocity

*Let G be a **finite group**, and $H \leq G$ a **subgroup**. Then, if:*

- *φ is a **character** of H*
- *ψ is a **character** of G*

it follows that:

$$\left\langle \mathrm{Ind}_H^G \varphi, \psi \right\rangle_G = \left\langle \varphi, \mathrm{Res}_H^G \psi \right\rangle_H$$

---

*For any group K, $\langle \rangle_K$ denotes the **inner product** on the **class functions** of K:*

$$\langle -, - \rangle_K : \mathcal{C}(K) \times \mathcal{C}(K) \to \mathbb{C}$$

(Corollary 6.12)

Proof. Identifying representations with modules, let:

- ψ correspond to the character associated to a $\mathbb{C}G$-module U
- φ correspond to the character associated to a $\mathbb{C}H$-module W

Then, using:

> *Let V, W be **finite diemnsional** $\mathbb{C}G$-modules. Then:*
>
> *1.*
> $$\mathrm{Hom}_{\mathbb{C}G}(V, W) = \mathrm{Hom}(V, W)^G$$
>
> *2.*
> $$\langle \chi_V, \chi_W \rangle = \dim\left(\mathrm{Hom}_{\mathbb{C}G}(V, W)\right)$$
>
> *(Proposition 5.22)*

it follows that:

$$\left\langle \mathrm{Ind}_H^G \varphi,\ \psi \right\rangle_G = \dim\left(\mathrm{Hom}_{\mathbb{C}G}(\mathrm{Ind}_H^G W,\ U)\right) \qquad \left\langle \varphi,\ \mathrm{Res}_H^G \psi \right\rangle_H = \dim\left(\mathrm{Hom}_{\mathbb{C}H}(W,\ \mathrm{Res}_H^G U)\right)$$

But by Proposition 6.10 above, we have an isomorphism:

$$\mathrm{Hom}_{\mathbb{C}G}(\mathrm{Ind}_H^G W, U) \cong \mathrm{Hom}_{\mathbb{C}H}(W, \mathrm{Res}_H^G U)$$

so it follows that:

$$\left\langle \mathrm{Ind}_H^G \varphi, \psi \right\rangle_G = \left\langle \varphi, \mathrm{Res}_H^G \psi \right\rangle_H$$

as required.

□

2.1.5 Example: Counting Simple Modules

- suppose that U is a **simple $\mathbb{C}G$-module**
- let $\mathbb{1}$ denotes the **trivial character** (which is 1 for each group element)
- finally, let $H = \{e\}$ be the trivial subgroup of some finite group G
- we saw in an example above that:

$$\mathbb{C}(G/H) \cong \mathrm{Ind}_H^G \mathbb{1} \implies \mathbb{C}G \cong \mathrm{Ind}_{\{e\}}^G \mathbb{1}$$

  For this particular example, this is simple to see, since the induced representation will have as basis elements of the form $x_i \overline{\otimes} 1$, whereby x_i are representatives of the left cosets of $\{e\}$, so in fact G forms a basis and thus we must have that $\mathrm{Ind}_{\{e\}}^G \mathbb{1} \cong \mathbb{C}G$

- thus, by **Frobenius Reciprocity**:

$$\langle \mathbb{C}G, \chi_U \rangle_G = \left\langle \mathbb{1}, \mathrm{Res}_H^G \chi_U \right\rangle_{\{} e\} = \chi_U(e) = \dim(U)$$

  where we have used the fact that since H is trivial, the dot product is a sum over the single term $\{e\}$; then we've just applied the fact that characters on trivial group elements yield the dimension (when U is simple)

- but how can we interpret $\langle \mathbb{C}G, \chi_U \rangle_G$?
- if we think of $\mathbb{C}G$ as a direct sum of simple modules:

$$\langle \mathbb{C}G, \chi_U \rangle_G = \left\langle \bigoplus \chi_{V_i}, \chi_U \right\rangle_G = \sum \langle \chi_{V_i}, \chi_U \rangle$$

  That is, $\langle \mathbb{C}G, \chi_U \rangle_G$ will count the number of times that U appears in the decomposition of $\mathbb{C}G$, which by the above work will be $\dim(U)$!

- this is what we'd expect based on

> *Suppose that k is* ***algebraically closed****.*
> *Let G be a* ***finite group*** *such that $|G| \neq 0$ in k, and let*
>
> $$V_1, \ldots, V_r$$
>
> *be a complete list of* ***pairwise nonisomorphic simple kG-modules****.*
> *Then:*
>
> 1. *kG (as a kG-module) is such that:*
>
> $$kG \cong V_1^{\dim(V_1)} \oplus \ldots \oplus V_r^{\dim(V_r)}$$
>
> 2. $$|G| = \sum_{i=1}^{r} \dim(V_i)^2$$
>
> *(Corollary 3.20)*

2.2 From Characters to Induced Characters

2.2.1 Lemma: Computing Induced Characters

*Let $H \leq G$ be a **subgroup** of the **finite group** G. Let*

$$\{x_1, \ldots, x_m\}$$

*be a **complete set** of **left coset representatives** for H, such that:*

$$G = x_1 H \sqcup \ldots \sqcup x_m H$$

Suppose that $g \in G$ acts on $[1, m]$ via a permutation such that

$$\forall i \in [1, m], \qquad g x_i H = x_{g \cdot i} H$$

and define:

$$\mathrm{Fix}(g) = \{i \in [1, m] \mid g \cdot i = i\}$$

*Then, for every **finite dimensional** $\mathbb{C}H$-module W, we have that:*

1. $$\forall i \in [1, m], \qquad g \cdot (x_i \overline{\otimes} W) \subseteq x_{g \cdot i} \overline{\otimes} W$$

2. $$\left(\mathrm{Ind}_H^G \chi_W\right)(g) = \sum_{i \in \mathrm{Fix}(g)} \chi_W(x_i^{-1} g x_i)$$

(Lemma 6.14)

Proof.

(1)

Using

Let:

- *G be a* ***finite group***
- *H be a* ***subgroup*** *$H \leq G$*
- *W be a kH****-module***

If $g \in G, w \in W$, then:

1.
$$\forall h \in H, \qquad gh \,\overline{\otimes}\, w = g \,\overline{\otimes}\, h \cdot w$$

2.
$$\forall x \in G, \qquad g \cdot (x \,\overline{\otimes}\, w) = gx \,\overline{\otimes}\, w$$

(Lemma 6.5)

alongside the fact that:

$$gx_i H = x_{g \cdot i} H \iff x_{g \cdot i}^{-1} g x_i H = H \iff x_{g \cdot i}^{-1} g x_i \in H$$

it follows that:

$$\begin{aligned} g \cdot (x_i \,\overline{\otimes}\, w) &= gx_i \,\overline{\otimes}\, w \\ &= (x_{g \cdot i}(x_{g \cdot i}^{-1}) g x_i \,\overline{\otimes}\, w \\ &= x_{g \cdot i}((x_{g \cdot i}^{-1} g x_i) \,\overline{\otimes}\, w \\ &= x_{g \cdot i} \,\overline{\otimes}((x_{g \cdot i}^{-1} g x_i) \cdot w \\ &\in x_{g \cdot i} \,\overline{\otimes}\, W \end{aligned}$$

W is a $\mathbb{C}H$-module, so $((x_{g \cdot i}^{-1} g x_i) \cdot w \in W$

(2) Suppose that the representation afforded by the $\mathbb{C}G$-module $\operatorname{Ind}_H^G W$ is

$$\rho : G \to \mathrm{GL}(\operatorname{Ind}_H^G W)$$

By

Let:

- *G be a **finite group***
- *H be a **subgroup** $H \leq G$*
- *W be a kH**-module***

Let

$$\{x_1, \ldots, x_m\}$$

*be a complete set of **left coset representatives** for H in G, such that:*

$$G = x_1 H \sqcup x_2 H \sqcup \ldots \sqcup x_m H$$

Then:

1. *there is a **vector space decomposition** for the **induced** kG**-module** defined by the **cosets**:*

$$\operatorname{Ind}_H^G W = \bigoplus_{i=1}^{m} x_i \overline{\otimes} W$$

2. $$\dim\left(\operatorname{Ind}_H^G W\right) = |G/H| \dim(W)$$

(Proposition 6.7)

we can write:

$$\operatorname{Ind}_H^G W = \bigoplus_{i=1}^{m} x_i \overline{\otimes} W$$

for a given choice of representatives. Moreover, by (1), $\rho(g)$ acts on these summands by permutation of the representatives via g.

If we consider the matrix $\rho(g)$, this will be a block matrix, with the block diagonal matrices corresponding to the matrix representation of $\rho(g)$ when restricted to act on each of the $x_i \overline{\otimes} W$. In particular, a block matrix contributes to the trace of $\rho(g)$ if and only if g fixes $x_i \overline{\otimes} W$. The trace of the restriction of $\rho(g)$ to $x_i \overline{\otimes} W$ (when $g \cdot i = i$) is then given by the trace of the H-action given by $x_{g \cdot i}^{-1} g x_i = x_i^{-1} g x_i$ on W, from which the result follows:

$$\left(\operatorname{Ind}_H^G \chi_W\right)(g) = \sum_{i \in \operatorname{Fix}(g)} \chi_W(x_i^{-1} g x_i)$$

□

2.2.2 Theorem: Computing Induced Characters Invariantly

The above definition of the induced character depends on a choice of representative for each of the left coset. We can make the induced character invariant to this by defining an extension by zero of the character.

*Let $H \leq G$ be a **subgroup** of the **finite grouop** G. Given a morphism:*

$$\varphi : H \to \mathbb{C}$$

*we define its **extension by zero to** G as the morphism*

$$\varphi^0 : G \to \mathbb{C}$$

defined by:

$$\varphi^0(x) = \begin{cases} \varphi(x), & x \in H \\ 0, & x \in G \setminus H \end{cases}$$

---

*Then, if W is a **finite dimensional** $\mathbb{C}H$-module, for any $g \in G$ we have that:*

$$\left(\mathrm{Ind}_H^G \chi_W\right)(g) = \frac{1}{|H|} \sum_{x \in G} \chi_W^0(x^{-1}gx)$$

(Theorem 6.16)

Proof. First note that for any $g \in G$:

$$i \in \mathrm{Fix}(g) \iff gx_iH = x_iH \iff x_i^{-1}gx_iH = H \iff x_i^{-1}gx_i \in H$$

In particular, given m representatives $\{x_1, \dots, x_m\}$, this means that we can rewrite:

$$\left(\mathrm{Ind}_H^G \chi_W\right)(g) = \sum_{i \in \mathrm{Fix}(g)} \chi_W(x_i^{-1}gx_i) = \sum_{i=1}^{m} \chi_W^0(x_i^{-1}gx_i)$$

since if $i \notin \mathrm{Fix}(g)$, $x_i^{-1}gx_i \in G \setminus H$, so $\chi_W^0(x_i^{-1}gx_i) = 0$.

Moreover, suppose that $y \in G$ and $h \in H$. Then, $hyh^{-1} \in H$ if $y \in H$, and $hyh^{-1} \in G \setminus H$ if $y \in G \setminus H$ (otherwise we'd have the contradiction that $y \in H$). In particular, and since χ_W^0 will be a class function on H, we have that:

$$\chi_W^0(hyh^{-1}) = \chi_W^0(y)$$

Lastly, notice that since the cosets of H partition G:

$$\forall x \in G, \exists! x_i \in G, \exists! h \in H \; : \; x = x = x_ih$$

Putting all this together:

$$\begin{aligned}
\sum_{x\in G} \chi_W^0(x^{-1}gx) &= \sum_{i=1}^m \sum_{h\in H} \chi_W^0((x_ih)^{-1}g(x_ih)) \\
&= \sum_{i=1}^m \sum_{h\in H} \chi_W^0(h^{-1}(x_igx_i)h)) \\
&= \sum_{i=1}^m \sum_{h\in H} \chi_W^0(x_igx_i)) \\
&= |H| \sum_{i=1}^m \chi_W^0(x_igx_i)) \\
&= |H| \left(\operatorname{Ind}_H^G \chi_W\right)(g)
\end{aligned}$$

□

- **Under what conditions does the above guarantee that $\left(\operatorname{Ind}_H^G \chi_W\right)(g)$?**
  - suppose that
    $$g^G \cap H = \emptyset$$
  - then
    $$\forall x \in G, \qquad xgx^{-1} \notin H$$
    so we'd have that:
    $$\forall x \in G, \qquad \chi_W^0(xgx^{-1}) = 0$$

2.2.3 Corollary: Induction of Class Function is Left Adjoint to Restriction of Class Functions

*Let $H \leq G$ be a **subgroup** of the **finite group** G. For any class function of H $\varphi \in \mathcal{C}(H)$, define:*

$$\left(\operatorname{Ind}_H^G \varphi\right)(g) = \frac{1}{H} \sum_{x\in G} \varphi^0(x^{-1}gx)$$

Then:

$$\operatorname{Ind}_H^G : \mathcal{C}(H) \to \mathcal{C}(G) \dashv \operatorname{Res}_H^G : \mathcal{C}(G) \to \mathcal{C}(\mathcal{H})$$

*where Res_H^G sends $\psi \in \mathcal{C}(G)$ to its **restriction** on H:*

$$\psi_{|H} : H \to \mathbb{C}$$

The adjunction is defined in the sense that:

$$\forall \varphi \in \mathcal{C}(G), \psi \in \mathcal{C}(H) \qquad \left\langle \operatorname{Ind}_H^G \varphi, \psi \right\rangle_G = \left\langle \varphi, \operatorname{Res}_H^G \psi \right\rangle_G$$

(Corollary 6.18)

Proof. If φ, ψ are characters of the representations of H, G, by Theorem 6.16 above, we know that we can write:

$$\left(\operatorname{Ind}_H^G \varphi\right)(g) = \frac{1}{|H|} \sum_{x \in G} \varphi_W^0(x^{-1}gx)$$

Moreover, by Frobenius Reciprocity, we have that:

$$\left\langle \operatorname{Ind}_H^G \varphi, \psi \right\rangle_G = \left\langle \varphi, \operatorname{Res}_H^G \psi \right\rangle_H$$

Lastly, by

> *The* ***irreducible characters*** *of G form an* ***orthonormal basis*** *for $\mathcal{C}(G)$.*
> *(Corollary 5.15)*

we have that by properties of the inner product, the result must follow for any class function $\varphi \in \mathcal{C}(G), \psi \in \mathcal{C}(H)$.

□

2.2.4 Corollary: Reformulation of Induced Character

We can reformulate Theorem 6.16 in a way which makes it more practical for calculations.

> *Let $H \leq G$ be a* ***subgroup*** *of the* ***finite group*** *G. For any $g \in G$, consider the intersection:*
>
> $$g^G \cap H$$
>
> *Define $h_1, \dots, h_\ell \in H$ to be a* ***complete set of representatives*** *of the* ***conjugacy classes*** *of H contained in $g^G \cap H$, such that:*
>
> $$g^G \cap H = \bigsqcup_{i=1}^{\ell} h_i^H$$
>
> *Then, if W is a* ***finite dimensional*** *$\mathbb{C}H$-module:*
>
> $$\chi_{\operatorname{Ind}_H^G W}(g) = \frac{|G|}{|H|} \sum_{i=1}^{\ell} \frac{|h_i^H|}{|g^G|} \chi_W(h_i)$$
>
> *(Corollary 6.19)*

Notice here we are using a slight change in notation for the induced character, but we have that:

$$\chi_{\operatorname{Ind}_H^G W}(g) = \left(\operatorname{Ind}_H^G \chi_W\right)(g)$$

Proof. For any $g \in G$, define the set:

$$\begin{aligned} S &= \{x \in G \mid xgx^{-1} \in g^G \cap H\} \\ &= \{x \in G \mid xgx^{-1} \in H\} \\ &= \bigcup_{y \in g^G \cap H} \{x \in G \mid x^{-1}gx = y\} \end{aligned}$$

Now, if we fix:

$$y = x_0^{-1} g x_0 \in g^G \cap H$$

we can define a mapping:

$$\tau_y : C_G(g) \to \{x \in G \mid x^{-1}gx = y\}$$

via:

$$\tau_y(z) = zx_0$$

since indeed:

$$(zx_0)^{-1} g (zx_0) = x_0 z^{-1} gzx_0 = x_0^{-1} z^{-1} zgx_0 = x_0^{-1} gx_0 = y$$

so clearly $zx_0 \in \{x \in G \mid x^{-1}gx = y\}$. We claim that τ defines a bijection:

$$\begin{aligned} & ax_0 \in \{x \in G \mid x^{-1}gx = y\} \\ \iff & x_0^{-1} a^{-1} gax_0 = y = x_0^{-1} gx_0 \\ \iff & a^{-1}ga = g \\ \iff & ga = ag \\ \iff & a \in C_G(g) \end{aligned}$$

Now, by Theorem 6.16, we can write:

$$|H| \chi_{\mathrm{Ind}_H^G W}(g) = \sum_{x \in G} \chi_W^0(x^{-1}gx)$$

In particular, notice that:

$$x \in S \implies xgx^{-1} \in H \implies \chi_W^0(x^{-1}gx) = \chi_W(x^{-1}gx)$$

Thus, we can rewrite the summand to run over S:

$$|H| \chi_{\mathrm{Ind}_H^G W}(g) = \sum_{x \in S} \chi_W(x^{-1}gx)$$

Now, applying the definition of S:

$$S = \bigcup_{y \in g^G \cap H} \{x \in G \mid x^{-1}gx = y\}$$

since each element of $x \in S$ is defined by some $y \in g^G \cap H$, and $g^G \cap H$ is a disjoint union of h_i^H, we can write:

$$|H| \chi_{\mathrm{Ind}_H^G W}(g) = \sum_{i=1}^{\ell} \sum_{y \in h_i^H} \sum_{x \in \tau_y^{-1}(C_G(g))} \chi_W(x^{-1}gx)$$

But by definition, if $x \in \tau_y^{-1}(C_G(g))$ then $x^{-1}gx = y$, so:

$$|H|\chi_{\mathrm{Ind}_H^G W}(g) = \sum_{i=1}^{\ell} \sum_{y \in h_i^H} |C_G(g)|\chi_W(y)$$

Similarly, since χ_W is a class function, $\chi_W(y)$ is invariant under conjugation, so without loss of generality, we may assume that:

$$\forall y \in h_i^H, \qquad \chi_W(y) = \chi_W(h_i)$$

so:

$$|H|\chi_{\mathrm{Ind}_H^G W}(g) = |C_G(g)| \sum_{i=1}^{\ell} |h_i^H|\chi_W(h_i)$$

Finally, since:

$$\forall g \in G, \qquad |G| = |g^G||C_G(g)|$$

(Lemma 5.17)

we get that:

$$\chi_{\mathrm{Ind}_H^G W}(g) = \frac{|G|}{|H|} \sum_{i=1}^{\ell} \frac{|h_i^H|}{|g^G|} \chi_W(h_i)$$

as required.

□

3 Characters for Normal Subgroups

3.1 x-Twists

3.1.1 Motivating x-Twists

*Suppose that we consider **induced representations** over **normal subgroups**.*
*Recall, when computing the **induced character***

$$\left(\operatorname{Ind}_H^G \chi_W\right)(g)$$

we decomposed $\operatorname{Ind}_H^G \chi_W$ *according to the decomposition of* G *into **left cosets** of* H*:*

$$G = \bigsqcup_{i=1}^{m} x_i H$$

Then, the action of g *on each* $x_i \overline{\otimes} W$ *could be thought of as a **permutation** of the **block matrices** which composed the **block diagonal matrix** representation of* g*.*
Now, if $H \triangleleft G$*, the permutation action of* $g \in H$ *on* G/H *is **trivial**, since:*

$$g(x_i H) = x_i(x_i^{-1} g x_i)H = x_i H$$

where we've used the fact that H *is normal, so* $x_i^{-1} g x_i \in H$*. This implies that the action of* $g \in H$ *on* $\operatorname{Ind}_H^G W$ *will preserve the block diagonals, since:*

$$g \cdot (x_i \overline{\otimes} w) = x_i(\underbrace{x_i^{-1} g x_i}_{\in H}) \overline{\otimes} w = x_i \overline{\otimes} (x_i^{-1} g x_i) w \in x_i \overline{\otimes} W$$

We call the construction $x^{-1} g x$ *an **x-twist**.*

3.1.2 Definition: x-Twist

*Let $N \triangleleft G$ be a **normal subgroup** of the **finite group** G. Consider some morphism:*

$$\varphi : N \to \mathbb{C}$$

*Then, for any $x \in G$, the **x-twist** of φ is the function:*

$$\varphi^x : N \to \mathbb{C}$$

*defined by **conjugation with** x:*

$$\varphi^x(h) = \varphi(x^{-1}hx)$$

(Definition 6.20)

3.1.3 Proposition: Properties of the x-Twist

*Let $N \triangleleft G$ be a **normal subgroup** of the **finite group** G. If φ^x is an **x-twist**, then:*

1. *If $\varphi \in \mathcal{C}(N)$, then $\varphi^x \in \mathcal{C}(N)$*
2. *If $\varphi \in \mathcal{C}(N)$, there is a **permutation action** of G/N on $\mathcal{C}(N)$ via:*

$$xN \cdot \varphi = \varphi^x$$

Proof.

(1)

By definition of a normal subgroup N, conjugation of elements in N by some $x \in G$ defines an automorphism of N. Thus, if φ is a class function:

$$\varphi^x(g^{-1}hg) = \varphi((x^{-1}g^{-1})h(gx)) = \varphi(h)$$

$$\varphi^x(h) = \varphi(x^{-1}hx) = \varphi(h)$$

so indeed:

$$\varphi^x(g^{-1}hg) = \varphi^x(h)$$

and φ^x is a class function on N.

(2)

As we saw above, φ^x depends on the choice of coset $xN \in G/N$, so in particular it defines the permutation action of G/N on $\mathcal{C}(N)$ defined above.

□

3.2 Proposition: x-Twists as Characters

Let $N \triangleleft G$ be a ***normal subgroup*** *of the* ***finite group*** *G. If φ is a* ***character*** *of N (so that for some $\mathbb{C}N$ module W, we have that $\varphi = \chi_W$), then:*

1. *φ^x is a* ***character*** *of N*
2. *if*

$$\{x_1, \ldots, x_m\}$$

defines a ***complete set of left coset representatives*** *for N in G, then:*

$$\operatorname{Res}_N^G \operatorname{Ind}_N^G \varphi = \sum_{i=1}^{m} \varphi^{x_i}$$

(Proposition 6.21)

Proof.

(1)

Consider W as a $\mathbb{C}$ vector space, and define an N-linear action via:

$$n *_x w = (x^{-1}nx)w$$

This action corresponds to the representation W^x, and thus φ^x defines a character of N too (the representation associated to φ just acting directly $n \cdot w = nw$).

Here, suppose that ρ is the representation associated to χ_W. Then, the representation associated to $\chi_W^x = \varphi^x$ is ρ^x, which we can define via:

$$\rho^x(g) = \rho(x^{-1}gx)$$

and since $x^{-1}gx \in N$ (as conjugation is an automorphism), ρ^x gives a well-defined representation $\rho^x : N \to GL(W)$, and thus, $\varphi^x = \chi_W^x$ is a well-defined character.

(2)

As we saw in the motivation, the N-action on $x_i \overline{\otimes} W$ preserves the space, so $x_i \overline{\otimes} W$ defines a $\mathbb{C}N$-module. In particular, we can then decompose $\operatorname{Res}_N^G \operatorname{Ind}_N^G \chi_W$ as a direct sum of $\mathbb{C}N$ modules, based on the representatives x_i (here is is important to apply the restriction, as otherwise we wouldn't be considering N-actions). In particular, this yields that:

$$\operatorname{Res}_N^G \operatorname{Ind}_N^G \varphi = \operatorname{Res}_N^G \operatorname{Ind}_N^G \chi_W = \sum_{i=1}^{m} \chi_{x_i \overline{\otimes} W}$$

But in the motivation we saw that if $h \in N$, then:

$$h \cdot x_i \overline{\otimes} w = x_i \overline{\otimes}(x_i^{-1} h x_i) w$$

which is equivalent to applying ρ^{x_i}, since:

$$\rho^{x_i}(h) \cdot (x_i \overline{\otimes} w) = (x_i^{-1} h x_i) \cdot (x_i \overline{\otimes} w) = x_i \overline{\otimes}(x_i^{-1} h x_i) w$$

so:

$$\chi_{x_i \overline{\otimes} W} = \chi_W^{x_i} = \varphi^{x_i}$$

so as required:

$$\operatorname{Res}_N^G \operatorname{Ind}_N^G \varphi = \sum_{i=1}^{m} \varphi^{x_i}$$

Alternatively, recall that by the proof of Lemma 6.14, we have that $\rho(g)$ preserves $x_i \overline{\otimes} W$, and restriction to the subspace will have trace equal to the trace of the action $x_i^{-1} g x_i \in H$ on W. In other words:

$$\chi_{x_i \overline{\otimes} W} = \varphi^{x_i}$$

□

3.2.1 Corollary: Constructing Irreducible Characters of Normal Subgroups

Let $N \triangleleft G$ be a ***normal subgroup*** *of the finite group G. Suppose that φ is an* ***irreducible character*** *of N, such that:*

$$\forall x \in G \setminus N, \qquad \varphi^x \neq \varphi$$

Then, the ***induced character***

$$\operatorname{Ind}_N^G \varphi$$

is ***irreducible****.*
(Corollary 6.22)

Proof. By Frobenius Reciprocity:

$$\| \operatorname{Ind}_N G\varphi \|^2 = \langle \operatorname{Ind}_N G\varphi \rangle_G = \left\langle \operatorname{Res}_N^G \operatorname{Ind}_N G\varphi, \varphi \right\rangle_N$$

Then, using Proposition 6.21 above, we know that if x_i are representatives of the left cosets of N in G:

$$\operatorname{Res}_N^G \operatorname{Ind}_N G\varphi = \sum_{i=1}^{m} \varphi^{x_i}$$

In particular, since x_i are representatives, we may assume that at least one pf the x_i are elements of N; in particular, WLOG let $x_1 = e_G$. Then, since by assumption

$$\forall x \in G \setminus N, \qquad \varphi^x \neq \varphi$$

we have that:

$$\forall i \geq 2, \qquad \varphi^{x_i} \neq \varphi$$

Using **row orthogonality**:

> *Let φ, ψ be* ***irreducible characters*** *of the* ***finite group*** *G. Then:*
>
> $$\langle \varphi, \psi \rangle = \begin{cases} 1, & \varphi = \psi \\ 0, & \varphi \neq \psi \end{cases}$$
>
> *(Theorem 5.13)*

and since φ, φ^{x_i} are irreducible characters, it follows that:

$$\langle \varphi^{x_i}, \varphi \rangle_N = \begin{cases} 1, & i = 1 \\ 0, & i \geq 2 \end{cases}$$

since when $x_i = e_G$, $\varphi^{e_G}(h) = \varphi(e_G^{-1} h e_G) = \varphi(h)$. In particular, it follows that by the linearity of the dot product:

$$\| \operatorname{Ind}_N^G \varphi \|^2 = 1$$

Now, let $\chi = \operatorname{Ind}_N^G \varphi$. Using

> *The* ***irreducible characters*** *of G form an* ***orthonormal basis*** *for $\mathcal{C}(G)$.*
> *(Corollary 5.15)*

if $\chi_i, i \in [1, r]$ are the irreducible characters of N which span $\mathcal{C}(N)$, then we have that:

$$\chi = \sum_{i=1}^{n} m_i \chi_i$$

(notice here that each χ_i corresponds to the character obtained by restricting the representation to $x_i \overline{\otimes} W$, and m_i is nothing but the multiplicity of $x_i \overline{\otimes} W$ in the decomposition of $\operatorname{Ind}_N^G W$).

Thus, we have that:

$$\|\chi\|^2 = \sum_{i=1}^{n} m_i^2 |\chi_i|^2 = 1$$

In particular, since $m_i \in \mathbb{N}$, this is possible if and only if there is a unique non-zero m_i which is non-zero, and equal to 1. This forces that $\exists i \in [1, r]$ such that:

$$\chi = \chi_i$$

and so $\chi = \operatorname{Ind}_N^G \varphi$ is irreducible, as required.

□

Here is a nice StackExchange post regarding the inner product of characters.

3.3 Example: Characters of Dihedral Groups

We now how that if:

$$G = D_{2m+1}$$

for $m \geq 1$, then G has:

- m irreducible characters of degree 2
- 2 linear characters (which are automatically irreducible)

In particular, we can define:

$$\langle r, s \mid r^{2m+1} = e_G = s^2, s^{-1}rs = r^{-1} \rangle$$

whereby:

- s corresponds to a **rotation**
- r corresponds to a **rotation** by $\frac{2\pi}{2m+1}$

Now, let

$$N = \langle r \rangle$$

N is normal, since $|G/N| = 2$. Moreover, N corresponds to the group of rotations, and it is abelian (since it is cyclic and generated by r); in particular

$$N \cong C_{2m+1}$$

Now, since N is abelian, $N' = \{e_G\}$, so by:

> *Let G be a **finite group**. Then, G has*
>
> $$|G/G'|$$
>
> ***distinct complex linear characters**.*
> *(Lemma 5.10)*

N has $2m + 1$ (linear) characters. In particular, for each $r^k \in N, k \in [0, 2m]$, we can identify a corresponding linear character φ^k. In particular, each φ^k must send r to a $(2m + 1)th$ root of unity (since r has order $2m + 1$ in N, and by Lemma 5.6 **linear characters** correspond to **homomorphisms** $\varphi : N \to \mathbb{C}^\times$). In particular, defining:

$$\omega = e^{\frac{2i\pi}{2m+1}}$$

we have that:

$$\varphi^k(r) = \omega^k$$

Now, we want to use the theory we have developed when using induced characters. For this, we would like to be able to use Corollary 6.22:

> *Let $N \triangleleft G$ be a **normal subgroup** of the finite group G. Suppose that φ is an **irreducible character** of N, such that:*
>
> $$\forall x \in G \setminus N, \qquad \varphi^x \neq \varphi$$
>
> *Then, the **induced character***
>
> $$\operatorname{Ind}_N^G \varphi$$
>
> *is **irreducible**.*
> *(Corollary 6.22)*

Suppose that $x \in G \setminus N$. Then, for some $j \in [0, 2m]$, we have that $x = sr^j$ so:

$$(sr^j)^{-1} r^k (sr^j) = r^{-j} s^{-1} r^k s r^j = r^{-j} r^{-k} r^{-j} = r^{-k}$$

which implies that:

$$(\varphi^i)^x(r^k) = \varphi^i(r^{-k}) = \omega^{-ik} = \varphi^{-i}(r^k) = \varphi^{2m+1-i}(r^k)$$

Thus, for any $x \in G \setminus N$, we have that that:

$$(\varphi^i)^x \neq \varphi^i$$

(except for the trivial homomorphism $\varphi^0 = \mathbb{1}$). In particular, $\{\varphi^1, \ldots, \varphi^m\}$ define m irreducible characters of N (via the induction $\operatorname{Ind}_N^G \varphi^i$). To this end, define:

$$\chi_i = \operatorname{Ind}_N^G \varphi^i$$

Now, N partitions G into 2 cosets via:

$$G = sN \sqcup rN$$

since:

- sN contains all the **reflections**:
- rN contains all the **rotations** (trivially):

if we use:

> *Let $N \triangleleft G$ be a **normal subgroup** of the **finite group** G. If φ is a **character** of N (so that for some $\mathbb{C}N$ module W, we have that $\varphi = \chi_W$), then:*
>
> 1. *φ^x is a **character** of N*
> 2. *if*
>
> $$\{x_1, \ldots, x_m\}$$
>
>   *defines a **complete set of left coset representatives** for N in G, then:*
>
> $$\operatorname{Res}_N^G \operatorname{Ind}_N^G \varphi = \sum_{i=1}^{m} \varphi^{x_i}$$
>
> *(Proposition 6.21)*

which in particular implies that we have the decomposition:

$$\mathrm{Res}_N^G \mathrm{Ind}_N^G(\varphi^i) = (\varphi^i)^s + (\varphi^i)^r = \varphi^i + \varphi^{-i}$$

where we have used the work above where we showed that:

$$(\varphi^i)^s(r^k) = \varphi^{-i}(r^k)$$

alongside the (trivial) fact that:

$$(\varphi^i)^r(r^k) = \varphi^i(r^{-1}r^k r^k) = \varphi^i(r^k)$$

In particular, since the restricted character takes characters in G, and restricts them to N, it follows that each χ_i must be a degree 2 character in G (since it decomposes into 2 irreducibles in G, and we have that $\varphi^i \neq \varphi^{-i}$).

Lastly, we have that:

$$G/N \cong C_2$$

Again, since C_2 is abelian, it has 2 linear representations, and these correspond to homomorphisms:

$$\chi : C_2 \to \mathbb{C}^\times$$

of which there are only 2:

$$\alpha(1) = 1 \qquad \alpha(-1) = -1$$

and

$$\alpha(1) = -1 \qquad \alpha(-1) = 1$$

In particular, we can inflate these into linear characters of G (and these won't be equal to any of the χ_i, since χ_i has degree 2, whereas the inflated cahracters are linear).

Using:

> *Let*
>
> $$\chi_1, \dots, \chi_r$$
>
> *be a* ***complete list*** *of* ***characters*** *of the complex* ***irreps*** *of a* ***finite gorup*** G*. Then:*
>
> $$|G| = \sum_{i=1}^{r} \chi_i(1)^2$$
>
> *(Proposition 5.7)*

we see that:

$$\sum_{i=1}^{m} \chi_i(1)^2 + 1^2 + 1^2 = 4m + 2 = |G|$$

so these must be all the irreducible characters of G.