Introduction to Representation Theory - Week 5 - Character Theory

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Contents

Based on the notes by Konstantin Ardakov, Section 5

1 Useful Remarks

1.1 Definition: G-Stable Subspace

For readability, if ρ is some representation, we denote $\rho(g) = \rho_g$.

> *Let $\rho : G \to GL(V)$ be a* ***representation****, and let U be a* ***linear subspace*** *of V.*
> *U is* ***G-stable*** *if:*
> $$\forall u \in U, \forall g \in G,\ \rho_g(u) \in U$$
> *(Definition 1.14, a))*

1.2 Definition: Irreducible Representations

> *The* ***representation***
> $$\rho : G \to GL(V)$$
> *is* ***irreducible/simple*** *if:*
>
> 1. *V is not the* ***zero vector space***
> 2. *if U is a G-****stable subspace*** *of V, then either:*
>   - $U = \{0\}$
>   - $U = V$
>
> *(Definition 1.18)*

- *From now on, we shall call irreducible representations* ***irreps****.*
- *We denote the identity automorphism of V via 1_V*
- *We also restrict ourselves to work over the field of complex numbers, such that the only group rings we consider will be of the form $\mathbb{C}G$.*

2 Characters

2.1 Definition: Character

*Let V be a **vector space** over $\mathbb{C}$, and let*

$$\rho : G \to GL(V)$$

*be a **complex representation** of G.*

---

*The **character** of ρ is the function:*

$$\chi_\rho : G \to \mathbb{C}$$

where:

$$\chi_\rho(g) = \mathrm{tr}(\rho(g))$$

(Definition 5.1)

2.1.1 Definition: Character Over Group Ring Modules

Since we identify representations of G with kG-modules, we have alternative notation for characters.

*If V is a $\mathbb{C}G$-module, we write χ_V to denote the **character** of the **representation** afforded by V.*

2.1.2 Definition: Degree of a Character

*The **degree** of χ_ρ is the **degree** of ρ (which is the **dimension** of V).*

2.1.3 Definition: Linear Character

*If χ_V has degree 1, then it is a **linear character**.*

2.2 Definition: Class Function

A ***class function*** *is a* ***function***

$$f : G \to \mathbb{C}$$

which is ***constant*** *on* ***conjugacy classes*** *of G:*

$$\forall g, x \in G, \qquad f(xgx^{-1}) = f(g)$$

We denote the ***space*** *of all* ***class functions*** *on G via $\mathcal{C}(G)$.*
(Definition 5.2)

2.2.1 Lemma: Characters are Class Functions

Let V be a ***finite dimensional*** *kG-module. Then, $\chi_V \in \mathcal{C}(G)$.*
(Lemma 5.3)

Proof. Let $\rho : G \to GL(V)$ be the representation corresponding to the character χ_V. Then:

$$\begin{aligned}
\chi_\rho(xgx^{-1}) &= \mathrm{tr}(\rho(x)\rho(g)\rho(x)^{-1}) \\
&= \mathrm{tr}(\rho(g)\rho(x)^{-1}\rho(x)) \\
&= \mathrm{tr}(\rho(g)) \\
&= \chi_\rho(g)
\end{aligned}$$

where we have used the property that the trace of the product of $A, B, C \in GL(V)$ satisfies:

$$\mathrm{tr}(ABC) = \mathrm{tr}(C(AB)) = \mathrm{tr}((BC)A)$$

□

2.2.2 Lemma: Space of Characters is a Commutative Ring

The ***vector space*** *$\mathcal{C}(G)$ is in fact a* ***commutative ring****, whereby* ***ring multiplication*** *is defined* ***pointwise****:*

$$\forall g \in G, \qquad (\phi\psi)(g) = \phi(g)\psi(g) = \psi(g)\phi(g) = (\psi\phi)(g)$$

2.3 Definition: Character Tables

> *Let G be a **finite group**, and consider:*
>
> - *a set*
> $$\{g_1, \dots, g_s\}$$
> *of **representatives** for the **conjugacy classes** of G*
> - *a collection*
> $$V_1, \dots, V_r$$
> *of **representatives** for the **isomorphism classes** of **simple** $\mathbb{C}G$**-modules** (which correspond to the **irreps** of the representation)*
>
> *The **character table** of G is the $r \times s$ array whose (i, j)th entry is $\chi_{V_i}(g_j)$.*
> *(Definition 5.4)*

- **Under what conditions are character tables square?**
  - recall Corollary 3.16:

> *Let G be a **finite group**, with k an **algebraically closed field** and $|G| \neq 0$ in k. Then:*
> $$r_k(G) = s(G)$$
> *(Corollary 3.16)*

  where recall that:
    * $s(G)$ is the **number of conjugacy classes** in G
    * for **finite groups** $r_k(G)$ denotes the number of **isomorphism classes** of **irreducible** k**-representations** of G (which we identify with **simple** $\mathbb{C}G$-modules.
  - hence, in this setting, it follows that we **always** have that $r = s$, so the character table is **always** square
  - as an additional pointer, recall that:

*Let G be a **finite group** with **conjugacy classes**:*

$$C_1, \dots, C_2$$

Then,

$$\{\hat{C}_1, \dots, \hat{C}_2\}$$

*is a **basis** for $Z(kG)$ as a **vector space**, and thus:*

$$\dim(Z(kG)) = s(G)$$

(Proposition 3.15)

- **Are character tables well-defined? That is, do they depend on the choice of representative V_i or g_j?**
  - the **trace** is a well-defined mapping, irrespective of **basis**
  - moreover, the **character** is a **class function**
  - thus, for any choice of **representative** V_i, g_j, the **character** table is always the same

3 Properties of Characters

3.1 Lemma: Basic Character Properties

Let

$$\rho : G \to GL(V)$$

be a ***finite dimensional representation****. Then:*

1. *The* ***character*** *of the* ***trivial conjugacy class*** *is the* ***degree*** *of the* ***irrep****:*

$$\chi_V(1) = \dim(V)$$

2.

$$\chi_V(g) = \chi_V(1) = \dim(V) \iff \rho(g) = 1_V$$

3. *If*

$$\dim(V) = 1$$

then χ *is a* ***group homomorphism***

4. *If:*
   - G *is* ***abelian***
   - V *is* ***irreducible***

then

$$\dim(V) = 1$$

(Lemma 5.6)

Proof.

(1) The identity conjugacy class contains as a representative the identity of G. Then, $\rho(1) = 1_V$, which as a matrix is the identity matrix, and whose trace is the dimension of the vector space.

(2)

$$\begin{aligned} & \chi_V(g) = \chi_V(1) \\ \iff & \operatorname{tr}(\rho(g)) = \operatorname{tr}(\rho(1)) \\ \iff & \operatorname{tr}(\rho(g)) = \dim(V) \end{aligned}$$

Now, notice that $\rho(g)^{|G|} = 1_V$, so $\rho(g)$ is a diagonalisable matrix...

(3)

If $\dim(V) = 1$, then $\rho(g)$ is just a non-zero complex number in $\mathbb{C}$ (so that $\rho(g) \in \mathbb{C}^\times$). Thus, this defines a group homomorphism:

$$\rho : G \to \mathbb{C}^\times \cong GL(\mathbb{C})$$

(4)

Recall Schur's Lemma:

> *Suppose k is **algebraically closed**. Let V be a **simple module** over a **finite dimensional** k**-algebra** A.*
> *Then, every A**-module endomorphism** of V is given by the action of some **scalar** $\lambda \in K$, such that:*
>
> $$\text{End}_A(V) = k1_V$$
>
> *(Theorem 3.6)*

Let ρ be a representation. We claim that $\rho(g)$ defines an endomorphism (G-module endomorphism) of V. Then, by irreducibility of V, this implies that:

$$\exists \lambda \in \mathbb{C} \ : \ \rho(g) = \lambda \in \mathbb{C}^\times$$

so that ρ is 1-dimensional, so $\dim(V) = 1$.

To this end, for any $g, h \in G$ and $v \in V$, we have that:

$$\rho(gh)(v) = (gh) \cdot v = g \cdot (h \cdot v) = \rho(g)(h \cdot v)$$

Since G is abelian, we also have that:

$$\rho(gh)(v) = (hg) \cdot v = h \cdot (g \cdot v) = h \cdot (\rho(g)(v))$$

In particular, this shows that:

$$\rho(g)(h \cdot v) = h \cdot (\rho(g)(v))$$

so in particular, $\rho(g)$ defines a G-linear module endomorphism of V, as required.

□

3.1.1 Example: Character Table of C_3

- if $G = C_3 = \langle x \rangle$, since G is abelian, each element constitutes its own **conjugacy class**, so the **character table** will be 3×3

	e	x	x^2
$\mathbb{1}$			
χ			
χ^2			

- the trivial representation will always have character 1, and characters on the identity are equal to $\dim(V)$, for each V (Lemma 5.6 (1) above), so we have:

	e	x	x^2
$\mathbb{1}$	1	1	1
χ	1		
χ^2	1		

- again by Lemma 5.6 (3), since each $\dim(V) = 1$, then the characters define a group homomorphism

$$\varphi : C_3 \to \mathbb{C}^\times$$

Group homomorphisms are defined by where they send generators; in particular, since each element in G has order 3, each entry in the character table must correspond to a cube root of 1

- since each character represents a distinct homomorphism, we have that:

	e	x	x^2
$\mathbb{1}$	1	1	1
χ	1	ω	ω^2
χ^2	1	ω^2	ω

where:

$$\omega = e^{2\pi i/3}$$

3.2 Proposition: Order of Group from Characters

Let

$$\chi_1, \ldots, \chi_r$$

be a ***complete list*** *of* ***characters*** *of the complex* ***irreps*** *of a* ***finite gorup*** *G. Then:*

$$|G| = \sum_{i=1}^{r} \chi_i(1)^2$$

(Proposition 5.7)

Proof. Let V_i be the simple kG-module associated to the character χ_i. Then, using Lemma 5.6, (1):

Let

$$\rho : G \to GL(V)$$

be a ***finite dimensional representation****. Then:*

1. *The* ***character*** *of the* ***trivial conjugacy class*** *is the* ***degree*** *of the* ***irrep****:*

$$\chi_V(1) = \dim(V)$$

2. $$\chi_V(g) = \chi_V(1) = \dim(V) \iff \rho(g) = 1_V$$

3. *If*

$$\dim(V) = 1$$

   then χ *is a* ***group homomorphism***

4. *If:*
   - G *is* ***abelian***
   - V *is* ***irreducible***

   then

$$\dim(V) = 1$$

(Lemma 5.6)

we get that:

$$\chi_i(1) = \dim(V_i)$$

Finally, by Corollary 3.20:

Suppose that k *is* ***algebraically closed****.*
Let G *be a* ***finite group*** *such that* $|G| \neq 0$ *in* k*, and let*

$$V_1, \ldots, V_r$$

be a complete list of ***pairwise nonisomorphic simple*** kG***-modules****.*
Then:

1. kG *(as a* kG*-module) is such that:*

$$kG \cong V_1^{\dim(V_1)} \oplus \ldots \oplus V_r^{\dim(V_r)}$$

2. $$|G| = \sum_{i=1}^{r} \dim(V_i)^2$$

(Corollary 3.20)

part (2), the result follows.

□

3.3 Counting Complex Linear Characters

3.3.1 Definition: Inflated Representation

Let $N \triangleleft G$ be a ***normal subgroup*** *of the* ***finite group*** *G. Let:*

$$\rho : G/N \to GL(V)$$

be a ***representation****.*

The ***inflated representation*** *of G is:*

$$\dot{\rho} : G \to GL(V)$$

where:

$$\forall g \in G, \qquad \dot{\rho}(g) = \rho(gN)$$

(Definition 5.8)

3.3.2 Definition: Derived Subgroup

Let G be a ***finite group****. The* ***derived subgroup*** *G' of G is the* ***subgroup*** *generated by the* ***commutators*** *in G:*

$$G' = \left\langle [x, y] = xyx^{-1}y^{-1} \mid x, y \in G \right\rangle$$

(Definition 5.9)

3.3.3 Proposition: Properties of Commutators and the Derived Subgroup

Let G be a group, and G' its derived subgroup.

1. ***Inverses*** *and* ***conjugates*** *of* ***commutators*** *are* ***commutators****:*

$$[x,y]^{-1} = [y,x] \qquad z[x,y]z^{-1} = [zxz^{-1}, zyz^{-1}]$$

2. *G' is a* ***normal subgroup*** *of G:*

$$G' \triangleleft G$$

3. *Let N be a subgroup of G. Then:*

$$N \triangleleft G \text{ and } G/N \text{ is abelian} \quad \iff \quad G' \subseteq N$$

   In particular, G' is the ***smallest normal subgroup*** *of G such that G/N is* ***abelian****.*

4. *G is* ***abelian*** $\iff$

$$G' = \{e_G\}$$

Proof. See these notes on Group Theory.

□

3.3.4 Lemma: Number of Complex Linear Characters from Derived Subgroup

Let G be a ***finite group****. Then, G has*

$$|G/G'|$$

distinct complex linear characters*.*
(Lemma 5.10)

Proof. Assume that $\chi : G \to \mathbb{C}$ is a complex linear character. By Lemma 5.6 (3)

Let

$$\rho : G \to GL(V)$$

be a ***finite dimensional representation****. Then:*

1. *The* ***character*** *of the* ***trivial conjugacy class*** *is the* ***degree*** *of the* ***irrep****:*

$$\chi_V(1) = \dim(V)$$

2. $$\chi_V(g) = \chi_V(1) = \dim(V) \iff \rho(g) = 1_V$$

3. *If*

$$\dim(V) = 1$$

then χ is a ***group homomorphism***

4. *If:*
   - *G is* ***abelian***
   - *V is* ***irreducible***

   then

$$\dim(V) = 1$$

(Lemma 5.6)

χ is a group homomorphism:

$$\chi : G \to \mathbb{C}^\times$$

Now, $\mathbb{C}^\times$ is abelian, so $\mathrm{im}(\chi) \leq \mathbb{C}^\times$ is abelian. By the First Isomorphism Theorem, it thus follows that:

$$G/\ker(\chi) \cong \mathrm{im}(\varphi)$$

so $G/\ker(\chi)$ is abelian. Thus, by properties of the derived subgroups, we must have that:

$$G' \subseteq \ker(\chi)$$

Moreover, since G/G' is abelian, it has $|G/G'|$ linear characters (using (4) of Lemma 5.6, as G/G' is abelian).

We claim that each (distinct) linear character χ of G corresponds to a linear character from G/G'. To do this, consider the Universal Property of Factor Groups where:

- $\pi : G \to G/G'$ is the canonical map:

$$\pi(g) = gG'$$

- $\chi : G \to \mathbb{C}^\times$ is a 1-dimensional character, viewed as a group homomorphism, which has

$$G' \subseteq \ker(\chi)$$

Then, the Universal Property tells us that:

$$\exists! \ \varphi : G/G' \to \mathbb{C}^\times$$

such that:

$$\chi = \varphi \circ \pi : G \to \mathbb{C}^\times$$

In particular, this implies that for each complex linear character χ of G, there is a (unique) corresponding character φ of G/G'. In particular, this correspondence is bijective: it is a mapping between a finite set of characters, and it is injective by definition.

Hence, since there are $|G/G'|$ linear characters in G/G', it follows that G also has $|G/G'|$ complex linear characters, as required.

□

3.3.5 Example 5.5: Character Table of S_3

- in S_n the conjugacy classes are given by cycle type; in particular, for S_3, we can identify 3 representatives:

$$\iota \qquad (1\ 2) \qquad (1\ 2\ 3)$$

- as always, the first row contains 1s (corrsesponding to the trivial representation):

	ι	(1 2 3)	(1 2)
$\mathbb{1}$	1	1	1
χ_2			
χ_3			

  where we've organise the conjugacy representatives in increasing order of conjugacy class size, and the characters are organised in decreasing order of degree

- now, the **derived subgroup** of S_3 will be A_3, since:

$$S_3/A_3 \cong C_2$$

  which is abelian, so $S_3' \subseteq A_3$. $A_3 = \langle (1\ 2\ 3) \rangle$ is cyclic, and so, simple, so this would force that either $S_3' = A_3$ or $S_3' = \{\iota\}$. The latter can't be the case, as S_3 isn't abelian

- now, it follows that by Lemma 5.10, there are $|S_3/A_3| = 2$ distinct complex linear characters; we have already found the trivial one; the second one can be "pulled back" from the inflated representation of S_3/A_3

- indeed, the linear representation in $S_3/A_3 \to C_2$ corresponds to a group homomorphism:

$$C_2 \to \mathbb{C}^\times$$

- since $A_3 = \langle (1\ 2\ 3) \rangle$, we have that $(1\ 2\ 3) \in A_3$, $(1\ 2\ 3)A_3 \in S_3/A_3$ maps to $1 \in C_2$ under the isomorphism, so the character of (1 2 3) in S_3 will be 1; similarly,, and since $C_2 \to \mathbb{C}^\times$ must be a homomorphism, the character of (1 2) will be -1

- hence, we get:

	ι	(1 2 3)	(1 2)
$\mathbb{1}$	1	1	1
χ_2	1	1	-1
χ_3			

Alternatively, we could've identified the existence of the **sign character**:

$$\chi_2 : S_3 \to \{\pm 1\}$$

which maps even permutations to 1, and odd permutations to -1

- from Proposition 5.7, we know that:

$$|S_3| = 6 = \mathbb{1}(1)^2 + \chi_2(1)^2 + \chi_3(1)^2 \implies \chi_3(1) = 2$$

	ι	(1 2 3)	(1 2)
$\mathbb{1}$	1	1	1
χ_2	1	1	-1
χ_3	2		

- lastly, in Example 1.20, we looked at the **permutation representation** of S_3. In particular, we let S_3 act on a set $X = \{e_1, e_2, e_3\}$. We found a 2-dimensional G-stable subspace of $\mathbb{C}X$ via:

$$W = \{a_1e_1 + a_2e_2 + a_3e_3 \mid a_1 + a_2 + a_3 = 0\}$$

where:

$$W = \langle v_1 = e_1 - e_2, v_2 = e_2 - e_3 \rangle$$

We also found that, in matrix form, the matrix representation σ of S_3 afforded by W was given by:

$$\sigma((1\ 2\ 3)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\sigma((1\ 2\ 3)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

- since this is a 2-dimensional representation, this must be the one we are missing, so computing the traces of $\sigma((1\ 2\ 3)), \sigma((1\ 2\ 3))$ we obtain the final character table for S_3:

	ι	(1 2 3)	(1 2)
$\mathbb{1}$	1	1	1
χ_2	1	1	-1
χ_W	2	-1	0

3.3.6 Example 5.11: Linear Characters in A_4

- A_4 has a **normal subgroup** of order 4:

$$V_4 = \{\iota, (1\ 2)(3\ 4), (1\ 4)(2\ 3), (1\ 3)(2\ 4)\}$$

known as the **Klein four-group**

- notice that:

$$|A_4/V_4| = \frac{12}{4} = 3 \implies A_4/V_4 \cong C_3$$

- since C_3 is abelian:

$$A_4' \leq V_4 \qquad |A_4'| \in \{1, 2, 4\}$$

- since A_4 is not abelian, A_4' is non-trivial
- the **subgroups** of order 2 are generated by each non-identity element, but none of these are normal, since, for example:

$$(1\ 4)(2\ 3)(1\ 2)(3\ 4)(1\ 4)(2\ 3) = (1\ 4)(2\ 3) \notin \{\iota, (1\ 2)(3\ 4)\}$$

- thus, the only possibility is that $A_4 = V_4$
- this then implies that A_4 admits 3 distinct linear characters, which are inflated as characters from representations of $A_4/V_4 \cong C_3$

4 Class Function Orthogonality

4.1 Definition: Inner Product on Class Functions

*Let G be a **finite group**. The **inner product on class functions** is the map:*

$$\langle -, - \rangle : \mathcal{C}(G) \times \mathcal{C}(G) \to \mathbb{C}$$

defined by:

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \psi(g)$$

In particular, this mapping satisfies, for $\varphi, \psi \in \mathcal{C}(G), \lambda \in \mathbb{C}$:

- ***sesquilinearity***

$$\langle \lambda\varphi, \psi \rangle = \overline{\lambda} \langle \varphi, \psi \rangle$$
$$\langle \varphi, \lambda\psi \rangle = \lambda \langle \varphi, \psi \rangle$$

- ***additivity*** *in both variables*
- ***antisymmetry***

$$\langle \varphi, \psi \rangle = \overline{\langle \psi, \varphi \rangle}$$

- ***positivity***

$$\langle \varphi, \varphi \rangle \geq 0$$

*with **equality** if and only if*

$$\varphi = 0$$

(Definition 5.12)

4.2 Notation for Conjugacy Classes

4.2.1 Definition: Conjugacy Class and Centralisers

*Let G be a **finite group** and let $g \in G$. Then:*

1. *We denote the **conjugacy class** of $g \in G$ via:*

$$g^G = \{g^x = x^{-1}gx \mid x \in G\}$$

2. *We denote the **centraliser** of $g \in G$ via:*

$$C_G(g) = \{x \in G \mid gx = xg\}$$

*where the **centraliser** of g is the set of all $x \in G$ which commute with g.*

(Definition 5.16)

4.2.2 Lemma: Order of Group from Centraliser

$$\forall g \in G, \qquad |G| = |g^G||C_G(g)|$$

(Lemma 5.17)

Proof. Apply the Orbit-Stabiliser Theorem to the action of G on itself defined by conjugation. Then, the orbit of $g \in G$ is its conjugacy class, and the stabiliser of g is its centraliser. ☐

4.3 Towards Class Function Orthogonality

4.3.1 Definition: Invariant Submodules

*Let V be a $\mathbb{C}G$-module. The **invariant submodule** of V is:*

$$V^G = \{v \in V \mid \forall g \in G,\ g \cdot v = v\}$$

---

*V^G is the **largest subspace** of V which is fixed by G.*
(Definition 5.19)

4.3.2 Proposition: Fixed Point Formula

*Let G be a **finite group** and let V be a **finite dimensional** $\mathbb{C}G$-module. Then:*

$$\dim(V^G) = \langle \mathbb{1}, \chi_V \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

(Proposition 5.20)

Proof. Define the principal idempotent of $\mathbb{C}G$ to be:

$$e = \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{C}G$$

To see it is an idempotent, notice that:

$$ge = \frac{1}{|G|} \sum_{g' \in G} gg' = \frac{1}{|G|} \sum_{h \in G} h = e$$

and similarly:

$$ge = eg$$

Here we have used the uniqueness of products in groups. e is also idempotent, since:

$$e^2 = \frac{1}{|G|} \sum_{g \in G} ge = \left(\frac{1}{|G|} \sum_{g \in G} 1 \right) e = \frac{|G|}{|G|} e = e$$

By

*Recall, A decomposes into **left ideals**:*

$$A = B_1 \oplus \ldots \oplus B_r$$

*In fact, each B_i is a **two-sided ideal** of A.*
(Lemma 3.11)

we decompose V by using the idempotent e, whereby $\{e, 1-e\}$ defines an orthogonal set of idempotents, and thus generate two-sided ideals $e \cdot V, (1-e) \cdot V$ such that:

$$V = e \cdot V \oplus (1-e) \cdot V$$

We now claim that:

$$e \cdot V = V^G$$

Firstly, assume that $g \in G$. Then:

$$g \cdot (e \cdot v) = (ge) \cdot v = e \cdot v \implies e \cdot V \leq V^G$$

On the other hand, if $v \in V^G$, by definition:

$$\forall g \in G, \quad g \cdot v = v$$

so in particular:

$$(|G|e) \cdot v = \left(\sum_{g \in G} g \right) \cdot v = \sum_{g \in G} (g \cdot v) = |G|v \implies e \cdot v = v$$

so $v \in e \cdot V$ and $V^G \leq e \cdot V$. Thus, as required:

$$e \cdot V = V^G$$

Lastly, we can identiy the action of $e \in \mathbb{C}G$ on V with a linear map:

$$e_V : V \to V$$

via:

$$v \mapsto e \cdot v$$

It is clear that:

$$\mathrm{im}(e_V) = e \cdot V$$

Now, here's a useful fact:

Let P be an ***idempotent linear map****. Then, P has eigenvalues 0 and 1, and the* ***algebraic multiplicity*** *of 1 is:*

$$\mathrm{tr}(P)$$

Proof. If v is an eigenvector with eigenvalue λ, then:

$$\lambda v = Pv = P^2 v = P(\lambda v) = \lambda(Pv) = \lambda^2 v$$

so:

$$\lambda^2 - \lambda = 0 \implies \lambda \in \{0, 1\}$$

Moreover, since P is idempotent, it has a minimal polynomial:

$$p(t) = t^2 - t = t(t-1)$$

so it is diagonalisable:

$$P = A \Lambda A^{-1}$$

where Λ is a diagonal matrix containing eigenvalues. Then:

$$\mathrm{tr}(P) = \mathrm{tr}(A \Lambda A^{-1}) = \mathrm{tr}(A^{-1} A \Lambda) = \mathrm{tr}(\Lambda)$$

But $\mathrm{tr}(\Lambda)$ counts the number of times 1 appears as an eigenvalue of P: it is the algebraic multiplicity. □

In particular, since e_V is idempotent, it follows that it has $\text{tr}(e_V)$ non-zero eigenvalues. These correspond to a set of $\text{tr}(e_V)$ linearly independent eigenvectors which will span the image of e_V. In other words:

$$\dim(e \cdot V) = \text{tr}(e_V)$$

Hence:

$$\begin{aligned}
\dim(V^G) &= \dim(e \cdot V) \\
&= \text{tr}(e_V) \\
&= \text{tr}\left(\frac{1}{|G|}\sum_{g \in G} \rho(g)\right) \quad \text{(since } e_V \text{ a linear map is equivalent to the action of } e \text{ on } v \in V \text{ via } \rho(g)) \\
&= \frac{1}{|G|}\sum_{g \in G} \text{tr}(\rho(g)) \quad \text{(by linearity of the trace)} \\
&= \frac{1}{|G|}\sum_{g \in G} \chi_V(g)
\end{aligned}$$

□

4.3.3 Proposition: Equalities in Class Functions

Here we consider how the character acts on the vector space constructions we saw last week:

- *the* ***dual vector space****:*
$$V^* = \{\textit{linear } f : V \to k\}$$
- *the* ***external direct sum****:*
$$V \oplus W = V \times W$$
- *the* ***tensor product****:*
$$V \otimes W$$
- *the* ***hom vector space****:*
$$\text{Hom}(V, W)$$
- *the* ***symmetric square****:*
$$S^2V = \left\langle vw = \frac{1}{2}(v \otimes w + w \otimes v) \mid v, w \in V \right\rangle$$
- *the* ***alternating square****:*
$$\Lambda^2V = \left\langle v \wedge w = \frac{1}{2}(v \otimes w - w \otimes v) \mid v, w \in V \right\rangle$$

*Let G be a **finite group**, with V, W as **finite dimensional** $\mathbb{C}G$-modules. Then:*

1. $$\chi_{V^*} = \overline{\chi_V}$$
2. $$\chi_{V \oplus W} = \chi_V + \chi_W$$
3. $$\chi_{V \otimes W} = \chi_V \chi_W$$
4. $$\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V}\chi_W$$
5. $$\chi_{S^2V}(g) = \frac{1}{2}\left(\chi_V(g)^2 + \chi_V(g^2)\right)$$
6. $$\chi_{\Lambda^2V}(g) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right)$$

(Proposition 5.21)

4.3.4 Proposition: Properties of Homomorphisms over $\mathbb{C}G$-Modules

*Let V, W be **finite diemnsional** $\mathbb{C}G$-modules. Then:*

1. $$\mathrm{Hom}_{\mathbb{C}G}(V, W) = \mathrm{Hom}(V, W)^G$$
2. $$\langle \chi_V, \chi_W \rangle = \dim\left(\mathrm{Hom}_{\mathbb{C}G}(V, W)\right)$$

(Proposition 5.22)

Proof.

(1)

Let $f \in \text{Hom}(V, W)$. Then, f is fixed by the G-action iff:

$$\forall g \in G, v \in V \qquad (g \cdot f)(v) = g \cdot f(g^{-1} \cdot v) = f(v)$$

If we denote with $g_V \in GL(V), g_W \in GL(W)$ the action of g on V, W respectively, then the above is equivalent to having:

$$\forall g \in G, v \in V \qquad g_W(f(g_V^{-1}(v)) = f(v)$$

or equivalently:

$$g_W \circ f \circ g_V^{-1} = f \implies g_W \circ f = f \circ g_V$$

But now using Definition 1.12:

> *Consider 2 representations:*
>
> $$\rho : G \to GL(V) \qquad \sigma : G \to GL(W)$$
>
> *A* ***homomorphism*** *or* ***intertwining operator*** *is a linear map:*
>
> $$\varphi : V \to W$$
>
> *such that:*
>
> $$\forall g \in G,\ \sigma(g) \circ \varphi = \varphi \circ \rho(g)$$
>
> *If φ is* ***bijective****, then it is an isomorphism.*
> *(Definition 1.12)*

it follows that f is a $\mathbb{C}G$-homomorphism:

$$f \in \text{Hom}_{\mathbb{C}G}(V, W)$$

as required.

(2)

We apply definitions. Using:

> *Let G be a* ***finite group*** *and let V be a* ***finite dimensional*** *$\mathbb{C}G$-module.*
> *Then:*
>
> $$\dim(V^G) = \langle \mathbb{1}, \chi_V \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$
>
> *(Proposition 5.20)*

and part 4 of Proposition 5.21:

> $$\chi_{\text{Hom}(V,W)} = \overline{\chi_V} \chi_W$$

it follows that:

$$\begin{aligned}\dim\left(\text{Hom}(V,W)\right)^G &= \frac{1}{|G|}\sum_{g\in G}\chi_{\text{Hom}(V,W)}(g)\\ &= \frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}\chi_W(g)\\ &= \langle \chi_V, \chi_W\rangle\end{aligned}$$

□

4.4 Row Orthogonality

4.4.1 Theorem: Row Orthogonality in Character Table

One extremely useful fact about character tables is that their rows are orthogonal, according to the character inner product.

> *Let φ, ψ be* ***irreducible characters*** *of the* ***finite group*** *G. Then:*
>
> $$\langle \varphi, \psi\rangle = \begin{cases}1, & \varphi = \psi\\ 0, & \varphi \neq \psi\end{cases}$$
>
> *(Theorem 5.13)*

Proof. Let V, W be simple $\mathbb{C}G$-modules, whose characters are:

$$\varphi = \chi_V \qquad \psi = \chi_W$$

(these are irreducible characters, since V, W are simple/irreducible).

By Schur's Lemma

> *Suppose k is* ***algebraically closed****. Let V be a* ***simple module*** *over a* ***finite dimensional*** *k**-algebra** A.*
> *Then, every A**-module endomorphism** of V is given by the action of some* ***scalar*** *$\lambda \in K$, such that:*
>
> $$\text{End}_A(V) = k1_V$$
>
> *(Theorem 3.6)*

alongside

> *Let V, W be* ***simple** A**-modules***. *Then, every* ***non-zero,** A**-linear map***
> $$\varphi : V \to W$$
> *is an* ***isomorphism***.
> *(Lemma 2.13)*

it follows that we must have:

$$\dim\left(\mathrm{Hom}_{\mathbb{C}G}(V, W)\right) = \begin{cases} 1, & V \cong W \\ 0, & V \ncong W \end{cases}$$

since:

- if $V \ncong W$, by Lemma 2.13 any map V, W must be the 0 map
- if $V \cong W$, by Schur's Lemma, any $\mathbb{C}G$-isomorphism $V \to W$ must respect the fact that the endomorphisms in V, W are given by scalars, so $\mathrm{Hom}_{\mathbb{C}G}(V, W)$ itself must be a space spanned by the identity map (and is thus 1 dimensional)

Then, by

> *Let V, W be* ***finite dimensional*** *$\mathbb{C}G$-modules. Then:*
>
> *1.*
> $$\mathrm{Hom}_{\mathbb{C}G}(V, W) = \mathrm{Hom}(V, W)^G$$
> *2.*
> $$\langle \chi_V, \chi_W \rangle = \dim\left(\mathrm{Hom}_{\mathbb{C}G}(V, W)\right)$$
> *(Proposition 5.22)*

we have that:

$$\langle \varphi, \psi \rangle = \langle \chi_V, \chi_W \rangle = \dim\left(\mathrm{Hom}_{\mathbb{C}G}(V, W)\right) \in \{0, 1\}$$

Now, suppose that $\chi_V = \chi_W$. Then:

$$\begin{aligned} \langle \chi_V, \chi_W \rangle &= \|\chi_V\|^2 \\ &= \frac{1}{|G|} \sum_{g \in G} |\chi_V(g)|^2 \\ &\geq \frac{(\dim(V))^2}{|G|} \\ &> 0 \end{aligned}$$

where in the penultimate step, we have used that when the character is evaluated at the identity group element we get back the vector space dimension (Lemma 5.6, 1)). In particular, if $\chi_V = \chi_W$ this forces:

$$\langle \chi_V, \chi_W \rangle = 1$$

On the other hand, if $\chi_V \neq \chi_W$, $V \ncong W$ (as isomorphic representations have the same character), so:

$$\langle \varphi, \psi \rangle = \dim\left(\mathrm{Hom}_{\mathbb{C}G}(V, W)\right) = 0$$

as required.

□

4.4.2 Corollary: Module Isomorphism from Character Equality

> *Let V, W be* ***finite dimensional*** *kG-modules. Then:*
>
> $$V \cong W \iff \chi_V = \chi_W$$
>
> *(Corollary 5.14)*

Proof. Let

$$\chi_1, \ldots, \chi_r$$

be the complete list of characters of the complex irreps of G, and let V_i be the simple kG-module with character χ_i.

By Mashcke's Theorem

> *Let*
>
> $$\rho : G \to GL(V)$$
>
> *be a* ***representation****, and let U be a* ***G-stable subspace****.*
> *A* ***G-stable complement*** *for U in G is a* ***G-stable subspace*** *W such that:*
>
> $$V = U \oplus W$$
>
> *where recall, this means that:*
>
> - $U + W = V$
> - $U \cap W = \{0\}$
>
> *(Definition 1.19)*

V can be written as a direct sum of simple kG-modules. Since, up to isomorphism, $V_1, \ldots, V_r$ are the only such simple kG-modules:

$$\exists a_i \in \mathbb{Z}^+ \ : \ V \cong V_1^{a_1} \oplus \ldots \oplus V_r^{a_r}$$

We call a_i the multiplicity of V_i in V.

By the correspondence between simple modules and irreps, we get that:

$$\chi_V = \sum_{i=1}^{r} a_i \chi_i$$

By

Let φ, ψ be ***irreducible characters*** *of the* ***finite group*** *G. Then:*

$$\langle \varphi, \psi \rangle = \begin{cases} 1, & \varphi = \psi \\ 0, & \varphi \neq \psi \end{cases}$$

(Theorem 5.13)

we then get that:

$$\langle \chi_i, \chi_V \rangle = \left\langle \chi_i, \sum_{i=1}^{r} a_i \chi_i \right\rangle = \sum_{i=1}^{r} a_i \delta_{ij} = a_i$$

Then, if $\chi_V = \chi_W$ we must have that:

$$\exists b_i \in \mathbb{Z}^+ \ : \ W \cong V_1^{b_1} \oplus \ldots \oplus V_r^{b_r}$$

as kG-modules. But then:

$$a_i = \langle \chi_i, \chi_V \rangle = \langle \chi_i, \chi_W \rangle = b_i$$

so in fact:

$$V \cong W$$

Conversely, if $V \cong W$, we trivially have that:

$$\chi_V = \chi_W$$

as required.

□

4.4.3 Corollary: Orthonormal Basis for Class Functions

The ***irreducible characters*** *of G form an* ***orthonormal basis*** *for $\mathcal{C}(G)$.*
(Corollary 5.15)

Proof. By orthonormality of rows in the character table, we know that

$$\langle \chi_i, \chi_j \rangle = \delta_{i,j}$$

Thus, the chracters χ_i are pairwise orthogonal elements of the inner product space $\mathcal{C}(G)$ (since characters are class functions). Now:

$$\dim(\mathcal{C}(G)) = s(G) = r_{\mathbb{C}}(G) = r$$

so $\{\chi_i\}_{i \in [1,r]}$ forms a linearly independent set of r elements, which are orthonormal. Thus, its an orthonormal basis.

□

4.4.4 Example: Character Table of A_4

- consider $G = A_4$
- we first look for **conjugacy classes** (which aren't as simple as for S_n). In A_n, conjugacy classes from S_n either stay the same, or they split into 2 separate conjugacy classes (if lengths of the cycle type are distinct odd numbers - see this link)
- in particular, this implies that A_4 has 4 conjugacy classes, with representatives:

$$\iota \qquad g_2 = (1\ 2)(3\ 4) \qquad g_3 = (1\ 2\ 3) \qquad g_4 = (1\ 3\ 2)$$

- we also need to compute the number of elements in each **conjugacy class**. Indeed:
  - ι yields the trivial conjugacy class, so:
  $$|\iota^G| = 1$$
  - the conjugacy class of g_2 is identical to that of S_4. It have:
  $$\frac{1}{2}\left(\frac{4\times 3}{2} \times \frac{2\times 1}{2}\right) = 3$$
  elements, so:
  $$|g_2^G| = 3$$
  - the conjugacy class of g_3 contains half the elements in A_4 as it did in S_4. In S_4 the cycles of shape 3 had:
  $$\frac{4\times 3\times 2}{3} = 8$$
  elements, so:
  $$|g_3^G| = |g_4^G| = 4$$
  - the last precomputation we make is in figuring out the **linear characters**. For this, we need to find the **derived subgroup** of A_4. But we already saw in the example above (Example 5.11) that:
  $$A_4' = V_4$$
  and so A_4 has 3 linear characters, inflated from characters $C_3 \to \mathbb{C}^\times$
  - in fact, since V_4 is generated by g_2, then the inflation of characters from C_3 means that we can "copy" the character table for C_3 into that for A_4, where we identify
  $$C_3 \cong \{\iota V_4 = g_2 V_4 \cong e, g_3 V_4 \cong x, g_4 V_4 \cong x^2\}$$

	e	x	x^2
$\mathbb{1}$	1	1	1
χ	1	ω	ω^2
χ^2	1	ω^2	ω

Thus, we have most of the character table done:

	ι	g_2	g_3	g_4
$\mathbb{1}$	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	d	a	b	c

– Proposition 5.7

> *Let*
> $$\chi_1, \dots, \chi_r$$
> *be a* ***complete list*** *of* ***characters*** *of the complex* ***irreps*** *of a* ***finite gorup*** *G. Then:*
> $$|G| = \sum_{i=1}^{r} \chi_i(1)^2$$
> *(Proposition 5.7)*

tells us that:

$$12 = 1^2 + 1^2 + 1^2 + d^2 \implies d = 3$$

so:

	ι	g_2	g_3	g_4
$\mathbb{1}$	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	d	a	b	c

– we can complete the table by exploiting **row orthogonalityñ**, which tells us that:

$$0 = \langle \chi_1, \chi_4 \rangle = |\iota^G|(1 \cdot 1) + |g_2^G|(1 \cdot a) + |g_3^G|(1 \cdot b) + +|g_4^G|(1 \cdot c) = 3(1 + a) + 4(b + c)$$
$$0 = \langle \chi_1, \chi_4 \rangle = |\iota^G|(1 \cdot 1) + |g_2^G|(1 \cdot a) + |g_3^G|(\omega \cdot b) + +|g_4^G|(\omega^2 \cdot c) = 3(1 + a) + 4\omega(b + c\omega)$$
$$0 = \langle \chi_1, \chi_4 \rangle = |\iota^G|(1 \cdot 1) + |g_2^G|(1 \cdot a) + |g_3^G|(\omega^2 \cdot b) + +|g_4^G|(\omega \cdot c) = 3(1 + a) + 4\omega(b\omega + c)$$

This yields:

$$b + c = \omega(b + c\omega) \qquad b + c = \omega(b\omega + c)$$

which in turn implies that:

$$b\omega + c = b + c\omega \;\therefore\; (\omega - 1)(b - c) = 0$$

Since $\omega = e^{2\pi/3} \neq 1$, we must have that $b = c$. Moreover, from the first equality:

$$3 + 3a + 8b = 0$$

implies that b, c must be real. Thus, if we return to:

$$b + c = \omega(b + c\omega) \implies 2b = \omega b(1 + \omega)$$

we see that the LHS is real, and the RHS is complex, which implies that $b = 0 = c$. This then forces $a = -1$.

– thus, the finalised character table is:

	ι	g_2	g_3	g_4
$\mathbb{1}$	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

4.5 Theorem: Column Orthogonality in Character Table

As a consequence of row orthogonality, we also have column orthogonality.

---

*Let G be a **finite group**, and let*

$$\chi_1, \dots, \chi_R$$

*be **irreducible characters** of G.*

If $g, h \in G$, then:

$$\sum_{i=1}^{R} \overline{\chi_i(g)}\chi_i(h) = \begin{cases} |C_G(g)|, & g^G = h^G \\ 0, & otherwise \end{cases}$$

---

*In other words, taking the **dot product** of **columns** in the **character table** will alwyas be 0.*
(Theorem 5.23)

Proof. Let

$$\{g_1, \dots, g_r\}$$

be a complete list of representatives of the conjugacy classes.

Now, *define* the following:

$$x_{i,j} = \chi_i(g_j)c_j \qquad c_j = \sqrt{\frac{|g_j^G|}{|G|}}$$

Notice, $x_{i,j}$ is the (i,j)th entry of the character table for G, scaled by a factor c_j. Now, if we compute the (complex) inner product of the scaled rows of the character table

$$\begin{aligned}\sum_{j=1}^{r} \overline{x_{i,j}} x_{k,j} &= \sum_{j=1}^{r} \overline{\chi_i(g_j) c_j} \chi_k(g_j) c_j \\ &= \sum_{j=1}^{r} c_j^2 \overline{\chi_i(g_j)} \chi_k(g_j) \\ &= \frac{1}{|G|} \sum_{j=1}^{r} |g_j^G| \overline{\chi_i(g_j)} \chi_k(g_j) \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \chi_k(g) \\ &= \langle \chi_i, \chi_k \rangle \\ &= \delta_{i,k}\end{aligned}$$

But now, if we define the $r \times r$ matrix:

$$X = (x_{i,j})$$

the above says that X is a unitary matrix, since:

$$\delta_{i,k} = \sum_{j=1}^{r} \overline{x_{i,j}} x_{k,j} = (\overline{X} X^T)_{ik}$$

In particular:

$$\overline{X} X^T = I \implies \overline{X}^T X = I$$

But then:

$$\forall j, k \in [1, r], \quad (\overline{X}^T X)_{j,k} = \sum_{i=1}^{r} \overline{x_{i,j}} x_{i,k} \implies c_j c_k \sum_{i=1}^{r} \overline{\chi_i(g_j)} \chi_i(g_k) = \delta_{j,k}$$

In particular, if $j \neq k$, this column product is 0; otherwise, when $j = k$, we have that:

$$\sum_{i=1}^{r} \overline{\chi_i(g_j)} \chi_i(g_k) = \frac{1}{c_j^2} = \frac{|G|}{|g_j^G|} = |C_G(g_j)|$$

by the Orbit-Stabilizer Theorem.

□

4.5.1 Example: Character Table for S_4

- let $G = S_4$
- the **conjugacy classes** are defined by **cycle type**, so we have representatives

$$g_1 = \iota \qquad g_2 = (1\ 2)(3\ 4) \qquad g_3 = (1\ 2\ 3) \qquad g_4 = (1\ 2) \qquad g_5 = (1\ 2\ 3\ 4)$$

- to compute the order of the conjugacy classes:
  -
  $$|g_1^G| = 1 \implies |C_G(g_1)| = \frac{24}{1} = 24$$
  -
  $$|g_2^G| = \frac{1}{2}\left(\frac{4 \times 3}{2} \times \frac{2 \times 1}{2}\right) = 3 \implies |C_G(g_2)| = \frac{24}{3} = 8$$

–
$$|g_3^G| = \frac{4 \times 3 \times 2}{3} = 8 \implies |C_G(g_3)| = \frac{24}{8} = 3$$

–
$$|g_4^G| = \frac{4 \times 3}{2} = 6 \implies |C_G(g_4)| = \frac{24}{6} = 4$$

–
$$|g_5^G| = \frac{4 \times 3 \times 2 \times 1}{4} = 6 \implies |C_G(g_5)| = \frac{24}{6} = 4$$

- now, we have the following chain of normal subgroups:

$$V_4 \lhd A_4 \lhd S_4$$

where it is also the case that $V_4 \lhd S_4$. Moreover, we have that

$$S_4/V_4 \cong S_3$$

Moreover, if:

$$f : S_3 \hookrightarrow S_4 \qquad g : S_4 \twoheadrightarrow S_4/V_4$$

we know that f (the inclusion) is injective, and g (the canonical map) is surjective, so their **composition**:

$$g \circ f : S_3 \to S_4/V_4$$

is **injective**. Since it is an injective morphism between 2 groups of order 6 (finite), the two groups must be isomorphic.

- in particular, we can **inflate** the character table for S_4, by using that of S_3:

	ι	(1 2 3)	(1 2)
$\mathbb{1}$	1	1	1
χ_2	1	1	-1
χ_W	2	-1	0

- to this end, recall that

$$V_4 = \{\iota, (1\ 2)(3\ 4), (1\ 4)(2\ 3), (1\ 3)(2\ 4)\}$$

so in particular:

$$\iota V_4 = g_2 V_4 \cong \iota \in S_3$$

and thus we inflate to get:

g	ι	g_2	g_3	g_4	g_5
$\|g^G\|$	1	3	8	6	6
$\|C_G(g)\|$	24	8	3	4	4
$\mathbb{1}$	1	1	1	1	1
χ_2	1	1			
χ_3	2	2			
χ_4					
χ_5					

- moreover, since:

$$S_4/A_4 \cong C_2$$

and A_4 contains the even permutations, in particular:

$$\iota A_4 = g_2 A_4 = g_3 A_4 \cong 1 \in C_2$$

and since characters in C_2 will be homomorphisms (by dimension 1), we must have that:

$$g_4 A_4 = g_5 A_5 \cong -1 \in C_2$$

so

g	ι	g_2	g_3	g_4	g_5
$\lvert g^G \rvert$	1	3	8	6	6
$\lvert C_G(g) \rvert$	24	8	3	4	4
$\mathbb{1}$	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2			
χ_4					
χ_5					

- then, comparing entries with the character table of S_3, it follows that:

g	ι	g_2	g_3	g_4	g_5
$\lvert g^G \rvert$	1	3	8	6	6
$\lvert C_G(g) \rvert$	24	8	3	4	4
$\mathbb{1}$	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4					
χ_5					

In fact, this tells us that g_3 forms its own conjugacy class in S_4/V_4 (3-cycles), and g_4, g_5 are in the same conjugacy class in S_4/V_4 (2-cycles)

- this is all we need to fill in the rest of the table: we can now use column orthogonality!

g	ι	g_2	g_3	g_4	g_5
$\lvert g^G \rvert$	1	3	8	6	6
$\lvert C_G(g) \rvert$	24	8	3	4	4
$\mathbb{1}$	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	d_4	w_4	x_4	y_4	z_4
χ_5	d_5	w_5	x_5	y_5	z_5

- firstly (by Proposition 5.7):

$$|G| = 24 = 1^2 + 1^2 + 2^2 + d_4^2 + d_5^2 \iff d_4 = d_5 = 3$$

g	ι	g_2	g_3	g_4	g_5
$\lvert g^G \rvert$	1	3	8	6	6
$\lvert C_G(g) \rvert$	24	8	3	4	4
$\mathbb{1}$	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3	w_4	x_4	y_4	z_4
χ_5	3	w_5	x_5	y_5	z_5

- then using Column Orthogonality with the first 2 columns:

  $$1 + 2 + 4 + 3w_4 + 3w_5 = 0$$

  and column orthogonality of the second column with itself:

  $$1 + 1 + 4 + |w_4|^2 + |w_5|^2 = |G_G(g_2)| = 8$$

  Together, these imply that:

  $$w_4 + w_5 = -2 \qquad |w_4|^2 + |w_5|^2 = 2$$

  This can be shown to be the case if and only if:

  $$w_4 = w_5 = -1$$

- applying column orthogonality to the third column yields:

  $$1 + 1 + 1 + |x_4|^2 + |x_5|^2 = 3 \implies |x_4|^2 + |x_5|^2 = 0$$

  But since these terms are non-negative, this is only possible if:

  $$x_4 = x_5 = 0$$

  Updating the character table:

g	ι	g_2	g_3	g_4	g_5
$\lvert g^G \rvert$	1	3	8	6	6
$\lvert C_G(g) \rvert$	24	8	3	4	4
$\mathbb{1}$	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3	-1	0	y_4	z_4
χ_5	3	-1	0	y_5	z_5

- applying column orthogonality to the last 2 columsn individually yields:

  $$1 + 1 + |y_4|^2 + |y_5|^2 = 4 = 1 + 1 + |z_4|^2 + |z_5|^2$$

  so in particular:

  $$|y_4|^2 + |y_5|^2 = 2 = |z_4|^2 + |z_5|^2$$

- then, by column orthogonality of the first and fourth columns:

  $$1 + -1 + 3y_4 + 3y_5 = 0 \implies 3(y_4 + y_5) = 0$$

  which is true if and only if:

  $$y_4 = -y_5$$

  which in particular implies that:

  $$|y_4|^2 + |y_5|^2 = 2 \implies 2|y_4|^2 = 2 \implies |y_4| = |y_5| = 1$$

- similarly, by column ofrthogonality of the first and fifth columns:
- then, by column orthogonality of the first and fourth columns:

  $$1 + -1 + 3z_4 + 3z_5 = 0 \implies 3(z_4 + z_5) = 0$$

  so again:

  $$z_4 = -z_5 \qquad |z_4| = |z_5| = 1$$

- now, we know that $\chi_4(g_4)$ and $\chi_5(g_4)$ are **traces** of matrices. Moreover, since $g_4 = (1\ 2)$, for any representation:

  $$\rho : S_4 \to GL(V)$$

  $\Gamma = \rho(g_4)$ will be a morphism of order 2. In particular, if $v \in V$ is some eigenvector:

  $$\Gamma v = \lambda v \implies (\Gamma^2)v = \lambda^2 v$$

  so in fact we have that:

  $$v = \lambda^2 v \iff \lambda^2 = 1$$

  Thus, since the trace is the sum of eigenvalues, and the eigenvalues must be ± 1 in particular we know that:

  $$\chi_4(g_4), \chi_5(g_4) \in \mathbb{R}$$

  which in turn forces:

  $$y_4 \in \{1, -1\} \qquad y_5 \in \{1, -1\}$$

WLOG, we may pick $y_4 = 1$,m which forces $y_5 = -1$. Moreover, since the characters must be different (i.e two rows/columns can't be identical), this in turn forces $z_4 = -1, z_5 = 1$, so the finalised table is:

g	ι	g_2	g_3	g_4	g_5
$\lvert g^G \rvert$	1	3	8	6	6
$\lvert C_G(g) \rvert$	24	8	3	4	4
$\mathbb{1}$	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3	-1	0	1	-1
χ_5	3	-1	0	-1	1