Introduction to Representation Theory - Week 4 - Generating New Representations

Antonio León Villares

November 2023

Contents

1 Useful Remarks

1.1 Remark: Representations as Vector Spaces

*Let V be a **vector space** and G a **finite group**. Recall, we can identify a **representation***

$$\rho : G \to GL(V)$$

*with a **group action** over the **vector space***

$$g \cdot v = \rho(g)(v)$$

*In particular, if we can construct **new vector spaces** from old ones, this allows us to construct **new representations**.*

1.2 Lemma: Linear G-Actions Induce kG-Modules

*Let V be a **vector space** and let*

$$G \times V \to V$$

*be a G**-action** on V.*
*This action extends to a kG**-module** structure on V **if and only if** the G**-action** on V is **linear**:*

$$\forall g \in G, v, w \in V, \lambda \in k, \qquad g \cdot (v + \lambda w) = (g \cdot v) + \lambda(g \cdot w)$$

(Lemma 4.1)

2 Generating New Representations

2.1 Definition: External Direct Sum Representation

*Let V, W be G**-representations**. The **external direct sum** is the **vector space***

$$V \oplus W = V \times W$$

*which is again a G**-representation** via:*

$$\forall g \in G, v \in V, w \in W, \qquad g \cdot (v, w) = (g \cdot v, g \cdot w)$$

(Definition 4.2)

- **In what way is this definition consistent with what we've met in linear algebra?**
  - notice, if $U = V \oplus W$, this typically means that:
    1. for any $u \in U$, we can write it as $u = v + w$ for some $v \in V, w \in W$
    2. $V \cap W = \{0\}$
  - to this respect, if we think of:

$$V' = \{(v, 0) \mid v \in V\} \subseteq V \times W$$

$$W' = \{(0, w) \mid w \in V\} \subseteq V \times W$$

then we do indeed see that:

$$V' \oplus W' = V \times W$$

2.2 Definition: Dual Representation

*Let V be a G-**representation**. The **dual representation** is the space:*

$$V^* = \{\text{linear } f : V \to k\}$$

G acts on V^* via:

$$\forall g \in G, f \in V^*, v \in V, \qquad (g \cdot f)(v) = f(g^{-1} \cdot v)$$

(Definition 4.3)

- **Why is there an inverse in the definition of the action?**
  - if g^{-1} weren't present, this wouldn't define a left G-action
  - for example, if we had $(g \cdot f)(v) = f(g \cdot v)$ then:

$$((gh) \cdot f)(v) = f((gh) \cdot v) = f(g \cdot (h \cdot v))$$

but:

$$(g \cdot (h \cdot f))(v) = (h \cdot f)(g \cdot v) = f(h \cdot (g \cdot v)$$

so

$$((gh) \cdot f)(v) \neq (g \cdot (h \cdot f))(v)$$

2.2.1 Lemma: Isomorphism of G-Representations from Double Dual

Leet V be a ***finite dimensional*** *G-representation. The* ***natural*** *isomorphism from V to its double dual:*

$$\tau : V \to V^{**}$$

given by

$$\forall f \in V^*, v \in V, \qquad \tau(v)(f) = f(v)$$

is an ***isomorphism*** *of G-representations.*
(Lemma 4.5)

2.3 Definition: Hom Representation

Let V, W be ***G-representations***. *The* ***vector space*** $\mathrm{Hom}(V, W)$ *of all* ***linear maps***

$$f : V \to W$$

admits a ***linear G-action*** *via:*

$$\forall g \in G, f \in \mathrm{Hom}(V, W), v \in V, \qquad (g \cdot f)(v) = gf(g^{-1} \cdot v)$$

(Definition 4.4)

2.4 Tensor Products

2.4.1 Definition: Tensor Product

Let V, W be ***vector spaces***, *with bases:*

$$\mathcal{V} = \{v_1, \ldots, v_m\} \subset V$$

$$\mathcal{W} = \{w_1, \ldots, w_n\} \subset W$$

The ***tensor product*** *$V \otimes W$ of V, W is the* ***free vector space*** *on the set of* ***formal symbols***

$$\{v_i \otimes w_j \mid i \in [1, m], j \in [1, n]\}$$

(Definition 4.6)

2.4.2 Definition: Elementary Tensor

*Let V, W be **vector spaces**, with bases:*

$$\mathcal{V} = \{v_1, \dots, v_m\} \subset V$$

$$\mathcal{W} = \{w_1, \dots, w_n\} \subset W$$

If:

$$v = \sum_{i=1}^{m} \lambda_i v_i \in V \qquad w = \sum_{j=1} \mu_j w_j \in W$$

*the **elementary tensor** is:*

$$v \otimes w = \sum_{i=1}^{m} \sum_{j=1} \mu_j \lambda_i (v_i \otimes w_j) \in V \otimes W$$

(Definition 4.6)

2.4.3 Remark: Properties of Tensor Product

*Let V, W be **vector spaces**. Then:*

1. $$\dim(V \otimes W) = (\dim(V))(\dim(W))$$
2. ***Elementary tensors span** $V \otimes W$)*
3. *Not every element of $V \otimes W$ is an **elementary tensor***

2.4.4 Lemma: Bases for Tensor Product

The definition makes it clear that the tensor product is not "natural": it depends on the choice of basis. This lemma goes to show that this isn't really an issue, since such tensor products will be isomorphic.

Let

$$\mathcal{V}' = \{v_1', \dots, v_m'\} \subset V \qquad \mathcal{W}' = \{w_1', \dots, w_n'\} \subset W$$

be other bases for V, W. Then:

$$X' = \{v_i' \otimes w_j' \mid i \in [1, m], j \in [1, n]\}$$

*is a **basis** for $V \otimes W$.*
(Lemma 4.8)

Proof. We can distribute elementary tensors in $V \otimes W$:

$$\forall v, v' \in V, w, w' \in W, \qquad (v + v') \otimes (w + w') = (v \otimes w) + (v \otimes w') + (v' \otimes w) + (v' \otimes w')$$

$$\forall v \in V, w \in W, \lambda \in k, \qquad (\lambda v) \otimes w = \lambda(v \otimes w) = v \otimes (\lambda w)$$

In particular, in V, W, we can write each v_i as a linear combination in $\mathcal{V}'$, and each w_i as a linear combination in $\mathcal{W}'$. Hence, each of the original basis vectors $v_i \otimes w_j$ will lie in the span of X'. Hence, since X' spans $V \otimes W$, and $|X'| = mn$, it follows that X' is a linearly independent spanning set, and thus, defines a basis. □

In particular, this is saying that if we have bases:

$$\mathcal{V} = \{v_1, \ldots, v_m\} \subset V \qquad \mathcal{W} = \{w_1, \ldots, w_n\} \subset W$$

$$\mathcal{V}' = \{v'_1, \ldots, v'_m\} \subset V \qquad \mathcal{W}' = \{w'_1, \ldots, w'_n\} \subset W$$

and we define:

$$V \otimes' W = \text{free vector space on symbols } v'_i \otimes' v'_j$$

then there is an isomorphism:

$$V \otimes' W \cong V \otimes W$$

given by:

$$v'_i \otimes' v'_j \mapsto v'_i \otimes v'_j$$

so in fact the two tensor products are isomorphic.

2.4.5 Remark: Canonical Map in Tensor Product

Let V, W be ***vector spaces****. Then, there is a* ***canonical map****:*

$$\otimes : V \times W \to V \otimes W$$

defined by:

$$(v, w) \mapsto v \otimes w$$

which is ***bilinear***

2.4.6 Lemma: Universal Property of Tensor Product

*Let V, W, U be **vector spaces**. Then, for every **bilinear map***

$$b : V \times W \to U$$

*there is a **unique linear map***

$$\tilde{b} : V \otimes W \to U$$

such that:

$$b = \tilde{b} \cdot \otimes$$

In other words:

$$\forall v, w \in V, \qquad b(v, w) = \tilde{b}(v \otimes w)$$

(Lemma 4.9)

Proof. We begin by proving existence.

Fix bases for V, W:

$$\mathcal{V} = \{v_1, \ldots, v_m\} \subset V \qquad \mathcal{W} = \{w_1, \ldots, w_n\} \subset W$$

Let $b : V \times W \to U$ be a bilinear map. Define a map:

$$\tilde{b} : V \otimes W \to U$$

via:

$$\tilde{b}(v_i \otimes w_j) = b(v_i, w_j)$$

Now, if:

$$v = \sum_{i=1}^{m} \lambda_i v_i \in V \qquad w = \sum_{j=1} \mu_j w_j \in W$$

Then:

$$\begin{aligned}
\tilde{b}(v \otimes w) &= \tilde{b}\left(\sum_{i=1}^{m} \sum_{j=1} \mu_j \lambda_i (v_i \otimes w_j)\right) \\
&= \sum_{i=1}^{m} \sum_{j=1} \mu_j \lambda_i b(v_i \otimes w_j) \\
&= \sum_{i=1}^{m} \sum_{j=1} \mu_j \lambda_i b(v_i, w_j) \\
&= b\left(\sum_{i=1}^{m} \lambda_i v_i, \sum_{j=1} \mu_j w_j\right) \\
&= b(v, w)
\end{aligned}$$

Hence, $b = \tilde{b} \cdot \otimes$ on each of the basis elements of $V \otimes W$, so we have shown existence.

Now, assume there exists some different linear map:

$$c : V \otimes W \to U$$

such that:

$$b(v, w) = c(v \otimes w)$$

Then, in particular:

$$c(v_i \otimes w_j) = b(v_i, w_j)$$

since c must send basis elements to basis elements. But then:

$$c(v_i \otimes w_j) = b(v_i, w_j) = \tilde{b}(v_i \otimes w_j)$$

so in fact $c = \tilde{b}$, since they agree on the basis elements.

□

2.4.7 Definition: Tensor Product Representation

> *Let V, W be **finite dimensional** kG-modules (where for finite groups kG-modules are just G **vector spaces**).*
> *Define a G-**action** on the **tensor product** $V \otimes W$ via:*
>
> $$\forall g \in G, v \in V, w \in W, \qquad g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$$

- **Does the above define a G-representation?**
  - using Lemma 4.1 above:

> *Let V be a **vector space** and let*
>
> $$G \times V \to V$$
>
> *be a G-**action** on V.*
> *This action extends to a kG-**module** structure on V **if and only if** the G-**action** on V is **linear**:*
>
> $$\forall g \in G, v, w \in V, \lambda \in k, \qquad g \cdot (v + \lambda w) = (g \cdot v) + \lambda(g \cdot w)$$
>
> *(Lemma 4.1)*

alongside the properties of tensor products, this shows that the above defines a G-**representation**

3 Working with Tensor Products

3.1 Lemma: Isomorphism Between Tensor Product and Homs

Let V, W be ***finite dimensional*** *kG-modules. Then, there is an* ***isomorphism*** *of kG-modules*

$$V^* \otimes W \cong \mathrm{Hom}(V, W)$$

(Lemma 4.11)

Proof. Notice, we can define a map:

$$\forall f \in V^*, w \in W, \qquad b(f, w) : V \to W$$

via:

$$\forall v \in V, \qquad b(f, w)(v) = f(v)w$$

Since $f(v) \in k$ and f is linear, this map is linear. Hence, we have defined a map:

$$b : V^* \times W \to \mathrm{Hom}(V, W)$$

Moreover, b is bilinear:

$$\begin{aligned} b(\lambda f_1 + f_2, \mu w_1 + w_2)(v) &= (\lambda f_1 + f_2)(v)(\mu w_1 + w_2) \\ &= (\lambda f_1(v) + f_2(v))(\mu w_1 + w_2) \\ &= \lambda\mu f_1(v)w_1 + \lambda f_1(v)w_2 + \mu f_2(v)w_1 + f_2(v)w_2 \end{aligned}$$

as required. Thus, by the Universal Property of Tensor Products

Let V, W, U be ***vector spaces****. Then, for every* ***bilinear map***

$$b : V \times W \to U$$

there is a ***unique linear map***

$$\tilde{b} : V \otimes W \to U$$

such that:

$$b = \tilde{b} \cdot \otimes$$

In other words:

$$\forall v, w \in V, \qquad b(v, w) = \tilde{b}(v \otimes w)$$

(Lemma 4.9)

we have that there exists a unique linear map:

$$\alpha : V^* \otimes W \to \text{Hom}(V, W)$$

such that:

$$\forall f \in V^*, w \in W, v \in V, \qquad \alpha(f \otimes w)(v) = f(v)w$$

Now, we claim that α defines an isomorphism of kG-modules. Firstly, we show that it is a kG-module homomorphism. To this end, let:

$$\mathcal{V} = \{v_1, \ldots, v_n\} \subset V$$
$$\mathcal{V}^* = \{v_1^*, \ldots, v_n^*\} \subset V^*$$
$$\mathcal{W} = \{w_1, \ldots, w_m\} \subset W$$

be bases for V, V^*, W respectively. It is sufficient to verify that α is a kG-module homomorphism when acting on elementary tensors (since any element of $V^* \otimes W$ will be a linear combination of these elementary tensors, so linearity of the homomorphism will be preserved). Since α is linear:

$$\alpha(f_1 \otimes w_1 + f_2 \otimes w_2) = \alpha(f_1 \otimes w_1) + \alpha(f_2 \otimes w_2)$$

It remains to show that it is kG linear. To this end, let

$$\rho = \sum_{g \in G} a_g g \in kG$$

and consider for some $v \in V$:

$$\begin{aligned}
\alpha(\rho \cdot (f \otimes w))(v) &= \alpha\left(\left[\sum_{g \in G} a_g g\right] \cdot (f \otimes w)\right)(v) \\
&= \alpha\left(\sum_{g \in G} a_g (g \cdot f \otimes w)\right)(v) \\
&= \sum_{g \in G} a_g \alpha[(g \cdot f) \otimes (g \cdot w)](v) \\
&= \sum_{g \in G} a_g [g \cdot f](v)[g \cdot w] \\
&= \sum_{g \in G} a_g f(g^{-1} \cdot v)[g \cdot w] \\
&= \sum_{g \in G} a_g g \cdot [f(g^{-1} \cdot v)w] \\
&= \sum_{g \in G} a_g g \cdot [\alpha(f \otimes w)(g^{-1} \cdot v)] \\
&= [\rho \cdot [\alpha(f \otimes w)]](v)
\end{aligned}$$

Hence, α defines a kG-module homomorphism.

It remains to show it is bijective. Since α is a homomorphism of finite dimensional kG-modules, it is sufficient to show that α is injective, from which surjectivity follows. Indeed:

$$f \otimes w \in \ker(\alpha) \iff \forall v \in V \alpha(f \otimes w)(v) = f(v)w = 0 \iff f = 0 \textit{ and } w = 0$$

so the kernel is trivial, and so, α is injective.

Alternatively, one can construct a direct inverse

$$\beta : \mathrm{Hom}(V, W) \to V^* \otimes W$$

via:

$$\beta(f) = \sum_{i=1}^{n} v_i^* \otimes f(v_i)$$

We verify that this indeed defines an inverse for $f \in \mathrm{Hom}(V, W), v \in V$ we have that:

$$\begin{aligned}(\alpha \circ \beta)(f)(v) &= \alpha[\beta(f)](v) \\ &= \sum_{i=1} .\alpha[v_i^* \otimes f(v_i)](v) \\ &= \sum_{i=1} .v_i^*(v) f(v_i) \\ &= f\left(\sum_{i=1} .v_i^*(v) v_i\right) \\ &= f(v)\end{aligned}$$

since by definition $v_i^*(v)$ will be the coefficients in the linear expansion of v in terms of basis elements v_i.

Similarly:

$$\begin{aligned}(\beta \circ \alpha)(f \otimes w) &= \beta[\alpha(f \otimes w)] \\ &= \sum_{i=1}^{n} v_i^* \otimes [\alpha(f \otimes w)(v_i)] \\ &= \sum_{i=1}^{n} v_i^* \otimes [f(v_i)w] \\ &= \sum_{i=1}^{n} [f(v_i)v_i^*] \otimes w \\ &= \left(\sum_{i=1}^{n} f(v_i)v_i^*\right) \otimes w \\ &= f \otimes w\end{aligned}$$

since the $f(v_i)$ are the constants defining f as a linear combination of the v_i^*, and using the bilinearity of the tensor product.

□

3.2 Decomposing Tensor Products

3.2.1 Definition: Symmetric Square

Let V be a ***finite dimensional vector space*** *and assume that* $\text{char}(k) \neq 2$. *Then:*

$$\forall v, w \in V,\ vw := \frac{1}{2}(v \otimes w + w \otimes v) \in V \otimes V$$

The ***symmetric square*** *of V is the* ***subspace*** *of $V \otimes V$ generated by all such vw:*

$$S^2V = \langle vw \mid v, w \in V \rangle$$

Notice that:

$$\forall v, w \in V,\ vw = wv$$

(Definition 4.12)

3.2.2 Definition: Alternating Square

Let V be a ***finite dimensional vector space*** *and assume that* $\text{char}(k) \neq 2$. *Then:*

$$\forall v, w \in V,\ v \wedge w := \frac{1}{2}(v \otimes w - w \otimes v) \in V \otimes V$$

The ***alternating square*** *of V is the* ***subspace*** *of $V \otimes V$ generated by all such $v \wedge w$:*

$$\Lambda^2V = \langle v \wedge w \mid v, w \in V \rangle$$

Notice that:

$$\forall v, w \in V,\ v \wedge w = -w \wedge v$$

(Definition 4.12)

3.2.3 Lemma: Decomposing Tensor Squares

Let

$$\dim(V) = n \qquad \mathrm{char}(k) \neq 2$$

Then:

1.
$$V \otimes V = S^2V \oplus \Lambda^2 V$$

2.
$$\dim(S^2V) = \frac{n(n+1)}{2} \qquad \dim(\Lambda^2 V) = \frac{n(n-1)}{2}$$

3. *If V is a G-**representation**, then so are S^2V, Λ^2V via:*

$$\forall g \in G, v, w \in V, \qquad g\cdot(vw) = (g\cdot v)(g\cdot w) \quad g\cdot(v\wedge w) = (g\cdot v)\wedge(g\cdot w)$$

(Lemma 4.13)

Proof.

(1)

Let $S_2 = \langle\sigma\rangle$ be the cyclic group or order 2. Since $\mathrm{char}(k) \neq 2$, we admit division by 2, so define:

$$e_1 = \frac{\iota + \sigma}{2} \qquad e_2 = \frac{\iota - \sigma}{2}$$

where ι is the identity permutation, and $e_1, e_2 \in kS_2$. Notice:

$$e_1^2 = \frac{\iota^2 + 2\iota\sigma + \sigma^2}{4} = \frac{2(\iota+\sigma)}{4} = e_1$$

$$e_1e_2 = \frac{\iota^2 - \sigma^2}{2} = 0$$

$$e_2^2 = \frac{\iota^2 - 2\iota\sigma + \sigma^2}{2} = \frac{2(\iota-\sigma)}{4} = e_2$$

so $\{e_1, e_2\}$ forms an orthogonal, idempotent set.

Now, from Lemma 3.11:

*Recall, A decomposes into **left ideals**:*

$$A = B_1 \oplus \ldots \oplus B_r$$

*In fact, each B_i is a **two-sided ideal** of A.*
(Lemma 3.11)

and the proof of:

> *Let A be a* ***finite dimensional semisimple** k**-algebra**, and suppose that k is* ***algebraically closed***. *Then:*
>
> $$\dim(Z(A)) \leq r$$
>
> *(Proposition 3.9)*

we have an ideal decomposition of kS_2 using the orthogonal idempotent set, via:

$$kS_2 = ke_1 \oplus ke_2$$

In particular, this allows us to decompose any kS_2-module M into even and odd elements:

$$M = e_1 M \oplus e_2 M = \{m \in M \mid \sigma m = m\} \oplus \{m \in M \mid \sigma m = -m\}$$

Now, S_2 will act on $V \otimes V$ via:

$$\forall v, w \in V, \qquad \sigma \cdot 8v \otimes w) = w \otimes v$$

Thus, we have that:

$$e_1 \cdot (V \otimes V) = S^2 V \qquad e_2 \cdot (V \otimes V) = \Lambda^2 V$$

Hence:

$$V \otimes V = S^2 V \oplus \Lambda^2 V$$

as required.

(2)

Let:

$$\{v_1, \dots, v_n\}$$

be a basis for V. Then:

$$\{v_i \otimes v_j \mid 1 \leq i, j \leq n\}$$

spans $V \otimes V$, so

$$\{e_1 \cdot (v_i \otimes v_j) \mid 1 \leq i, j \leq n\} \text{ spans } e_1 \cdot (V \otimes V) = S^2 V$$

$$\{e_2 \cdot (v_i \otimes v_j) \mid 1 \leq i, j \leq n\} \text{ spans } e_2 \cdot (V \otimes V) = \Lambda^2 V$$

But now we have that:

$$e_1 \cdot (v_i \otimes v_j) = \frac{v_i \otimes v_j + v_j \otimes v_i}{2} = v_i v_j = v_j v_i$$

so

$$\{v_i v_j \mid 1 \leq i \leq j \leq n\} \text{ spans } S^2 V$$

This set has $\frac{n(n+1)}{2}$ (think of each $v_i v_j$ as an element of a matrix; the set contains all elements in the upper triangular part of the matrix, including the main diagonal, by symmetry), which implies that:

$$\dim(S^2 V) \leq \frac{n(n+1)}{2}$$

Similarly:

$$e_2 \cdot (v_i \otimes v_j) = \frac{v_i \otimes v_j - v_j \otimes v_i}{2} = v_i \wedge v_j$$

Thus, since $e_2 \cdot (v_i \otimes v_j) = 0$ when $i = j$, it follows that

$$\{v_i \wedge v_j \mid 1 \leq i < j \leq n\} \text{ spans } \Lambda^2 V$$

and

$$\dim(S^2V) \leq \frac{n(n-1)}{2}$$

But now, $\dim(V \otimes V) = n^2$ and since $V \otimes V = S^2V \oplus \Lambda^2 V$ this forces equality, which is the desired result.

(3)

G, S_2 act on $V \otimes V$ via:

$$\begin{aligned}
\sigma \cdot (g \cdot (v \otimes w)) &= \sigma(g \cdot v \otimes g \cdot w) \\
&= g \cdot w \otimes g \cdot v \\
&= g \cdot (w \otimes v) \\
&= g \cdot (\sigma \cdot (v \otimes w))
\end{aligned}$$

Thus, the two actions commute pointwise. In particular, the G-action will preserve every S_2-submodule of $V \otimes V$, which means that $S^2V, \Lambda^2 V$ are G-stable, and thus, inherit a linear G-action from $V \otimes V$, as required.

□