Introduction to Representation Theory - Week 2 - Representations and Modules

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October 2023

Contents

Based on the notes by Konstantin Ardakov, Section 2

1 Rings and Modules

1.1 Definition: Ring

*A **ring** $(R, +, \cdot)$ consists of an **abelian group** $(R, +)$, alongside an **associative multiplication** operation:*

$$\cdot : R \times R \to R$$

*which satisfies **distributivity**:*

- $$a \cdot (b + c) = a \cdot b + a \cdot c$$
- $$(a + b) \cdot c = a \cdot c + b \cdot c$$

and contains an identity element 1_R:

$$1_R \cdot a = a = a \cdot 1_R$$

(Definition 2.1)

1.2 Definition: Module

*Let R be a **ring**.*
*An R-**Module** $(M, \cdot)$ consists of an **abelian group** $(M, +)$ alongside a **left** R-**action** (i.e scalar multiplication):*

$$\cdot : R \times M \to M$$

which satisfies:

- $$r \cdot (m + n) = (r \cdot m) + (r \cdot n)$$
- $$(r \cdot s) \cdot m = r \cdot (s \cdot m)$$
- $$1_R \cdot m = m$$

1.3 Group Rings

Before, for some set X and field k, we defined kX as the free vector space on X. Now, we define a similar concept, but with the underlying set being a group. This helps define a ring from a group.

1.3.1 Definition: Group Ring

*Let G be a **finite group**. The **group ring** of G is the **vector space** kG (where k is a field).*
Ring addition (and scalar multiplication by elements of k) are formally defined. Multiplication in the ring is defined by:

$$\left(\sum_{x\in G} a_x x\right) \cdot \left(\sum_{x\in G} a_x x\right) = \sum_{g\in G}\left(\sum_{x\in G}(a_x b_{x^{-1}g}\right) g$$

(Definition 2.2)

- **How is a group G embedded within kG?**
  - any $g \in G$ has a natural embedding within kG (it is a linear combination of g with scalar multiple 1)
  - G itself is a **subgroup** of the group of units $kG^{\times}$

2 Representations as Modules (and Viceversa)

2.1 Proposition: Bijection Between Representations and Modules

*Let V be a **vector space** and G be a **group**.*

1. *If*
$$\rho : G \to GL(V)$$
*is a **representation**, then V is a **left** kG**-module**, where left multiplication of kG on elements of V is given by:*
$$\forall a_x \in k, v \in V, \quad \left(\sum_{x \in G} a_x x \right) \cdot v = \sum_{x \in G} a_x \rho_x(v)$$
where $\rho_x := \rho(x)$

2. *If V is a **left** kG**-module**, there is a representation*
$$\rho : G \to GL(V)$$
defined by:
$$\forall g \in G, v \in V, \quad \rho_g(v) = g \cdot v$$

3. *1) & 2) define a **bijection** between **representations** $\rho : G \to GL(V)$ and left multiplication in kG**-modules** $kG \times V \to V$*

2.2 Useful Module Theorems

The bijection above gives us that we can think about G-representations and kG-modules interchangeably. In particular, theorems from modules translate to theorems for representations.

2.2.1 Theorem: First Isomorphism Theorem

Let
$$\varphi : V \to W$$
*be a kG**-module homomorphism**. Then:*
$$\operatorname{im}(\varphi) \cong V / \ker(\varphi)$$

2.2.2 Theorem: Second Isomorphism Theorem

*Suppose $U, W \leq V$ are kG-**submodules**. Then:*

$$(U+W)/U \cong W/(U \cap W)$$

2.2.3 Theorem: Third Isomorphism Theorem

*If $U \leq W \leq V$ are kG-**submodules**, then:*

$$\frac{V/U}{W/U} \cong V/W$$

2.2.4 Theorem: Correspondence Theorem

*Let $W \leq V$ be kG-**modules**. Then, there is a **canonical bijection**:*

$$\{\textit{submodules } U \leq V \textit{ containing } W\} \to \{\textit{submodules of } V/W\}$$

given by:

$$U \mapsto U/W$$

2.3 Properties in Representations Apply in Modules

2.3.1 Definition: Irreducible/Simple Modules

*An A-**module** M is **irreducible/simple** if $M \neq \{0\}$ and if $N \leq M$ is an A-**submodule**:*

- $N = \{0\}$
- *or* $N = M$

2.3.2 Definition: kG-Linear Maps

*A kG-**linear map** is a **homomorphism of representations**.*

Recall:

Consider 2 representations:

$$\rho : G \to GL(V) \qquad \sigma : G \to GL(W)$$

*A **homomorphism** or **intertwining operator** is a linear map:*

$$\varphi : V \to W$$

such that:

$$\forall g \in G,\ \sigma(g) \circ \varphi = \varphi \circ \rho(g)$$

*If φ is **bijective**, then it is an isomorphism.*
(Definition 1.12)

2.3.3 Definition: Completely Reducible Modules

*An A-**module** M is **completely reducible** if either:*

- $M = \{0\}$
- *M is a **direct sum** of finitely many **simple** submodules:*

$$M = N_1 \oplus \ldots \oplus N_m$$

3 Free Modules

3.1 Definition: Free A-Modules

*Let A be a **ring**.*
*The **free A-module of rank 1**, denote ${}_AA$, is the **abelian group** A equipped with **left multiplication** by A:*

$$\forall a, b \in A, \quad a \cdot b = ab$$

(Example 2.7)

3.1.1 Exercise: Simple Free A-Modules

1. **Show that ${}_AA$ is a simple module *if and only if* A is a division ring (that is, if A is a ring in which ever element has a multiplicative inverse).**

3.2 Definition: Left Ideals

> *Let ${}_AA$ be the **free A-module**. An A-**submodule** of ${}_AA$ is a **left ideal**.*

3.3 Definition: Left Regular Representations

> *Let $A = kG$ be the **free group**. The **representation***
>
> $$\rho : G \to GL(kG)$$
>
> *(which corresponds to the **free kG-module of rank 1**) is called the **left regular representation**.*
> *(Definition 2.8)*

3.4 Definition: Semisimple Rings

> *Let A be a **ring**. A is **semisimple** if ${}_AA$ is **completely reducible**.*
>
> ---
>
> *That is, if ${}_AA$ "factors" as a direct sum of **simple** submodules.*
> *(Definition 2.9)*

3.4.1 Examples: Semisimple Rings

- if k is a field, k is **semisimple**
- if A is a division ring, A is semisimple
- if G is a finite group with $|G| \neq 0$ in k, the **group ring** kG is **semisimple** (this is a **corollary of Maschke's Theorem**)
  - in particular, Maschke's Theorem translated to kG states that if M is a two-sided, simple ideal, then there exists some other two-sided, simple ideal N such that:

    $$kG = M \oplus N$$

  - translated to the language of modules, if kG (as a free module) has a simple kG-submodule M, then there exists some other simple kG-submodule N such that:

    $$kG = M \oplus N$$

  - that is, kG is **semisimple**

4 Cyclicity in Modules

4.1 Definition: Cyclic Modules

*Let V be an A-**module**. V is **cyclic** if it is generated by a single element $v \in V$:*

$$V = \{a \cdot v \mid a \in A\} = Av$$

(Definition 2.11)

4.2 Definition: Annihilators

*Let V be an A-**module**. The **annihilator** of $v \in V$ is the **submodule**:*

$$\operatorname{ann}_A(v) = \{a \in A \mid av = 0\}$$

(Definition 2.11)

4.2.1 Proposition: Simple Modules are Cyclic

*Every **simple module** is **cyclic**.*

Proof. Let M be an A-module. If A is simple, then any submodule B is either $\{0\}$ or $N = M$. For any $m \in M$, Am is a submodule of M, so either $Am = \{0\}$ or $Am = M$. If its the latter, we are done. It can never be the former, since $1_A \in A \implies m \in Am$. □

5 Isomorphisms in Modules

5.1 Lemma: Isomorphism Between Cyclic Modules and Annihilator Quotients

*Let V be a **cyclic** A-**module**, such that for $v \in V$, $Av = V$. Then:*

$$Av \cong A/\operatorname{ann}_A(v)$$

(Lemma 2.12)

Proof. Define an A-module homomorphism:

$$\varphi : A \to V$$

via:

$$a \mapsto a \cdot v$$

φ is surjective, as V is cyclic. Moreover, by definition:

$$\ker(\varphi) = \operatorname{ann}_A(v)$$

Thus, by the Frist Isomorphism Theorem for Modules:

$$Av \cong A/\operatorname{ann}_A(v)$$

□

5.2 Lemma: Isomorphic Linear Maps in Simple Modules

*Let V, W be **simple** A**-modules**. Then, every **non-zero,** A**-linear map***

$$\varphi : V \to W$$

*is an **isomorphism**.*
(Lemma 2.13)

Proof. By properties of modules:

$$\ker(\varphi) \leq V \qquad \operatorname{im}(\varphi) \leq W$$

φ is non-zero, so $\exists v \in V \ : \ \varphi(v) \neq 0$, so in particular:

$$\ker(\varphi) < V$$

For the same reason, $\operatorname{im}(\varphi) \neq \{0\}$. But now, by assumption V, W are simple, which forces:

$$\ker(\varphi) = \{0\} \qquad \operatorname{im}(\varphi) = W$$

Thus, φ is injective and surjective, and thus defines a module isomorphism.

□

5.3 Proposition: Simple Modules for Semisimple Rings

*Let A be a **semisimple ring**. Then, A has only **finitely** many **simple** A**-modules**, up to **isomorphism**.*
(Proposition 2.14)

Proof. Since A is semisimple, it can be decomposed into simple A-submodules $V_i \leq A$:

$$_A A = V_1 \oplus \ldots \oplus V_r$$

Now, let V be any (simple) A-module, and pick some non-zero $v \in V$.

Define an A-module map:

$$\varphi : A \to V$$

via:

$$a \mapsto a \cdot v$$

Further define:

$$\varphi_i : V_i \to V$$

as φ restricted to V_i. Notice, since V_i, V are simple, by

> *Let V, W be **simple A-modules**. Then, every **non-zero, A-linear map***
>
> $$\varphi : V \to W$$
>
> *is an **isomorphism**.*
> *(Lemma 2.13)*

it follows that, if *at least* one φ_i is non-zero, then V will be isomorphic to *at least* one of the V_i.

Thus, assume for contradiction that $\forall i, \varphi_i = 0$. In particular, if $a \in A$, then we can decompose it as:

$$a = a_1 + \ldots + a_m$$

where $a_i \in V_i$. Moreover:

$$\varphi(a) = \varphi_1(a_1) + \ldots + \varphi_r(a_r) = 0$$

which implies that φ is the 0 map. But this is impossible, since:

$$\varphi(1_A) = v \neq 0$$

by assumption. Hence, at least one of the φ_i is non-zero, and so, any A-module V which is simple is isomorphic to one of the simple submodules $V_1, \ldots, V_r$.

□

5.4 Theorem: Irreducible Representations of Finite Groups

> *Let G be a **finite group**, such that $|G| \neq 0$ in k. Then, G has only **finitely** many **irreducible representations** ,up to isomorphism.*
> *(Theorem 2.15)*

Proof. By Maschke's Theorem, kG always has some irreducible G-stable complement, so kG is semisimple (as $|G| \neq 0$). By the Proposition above:

> *Let A be a **semisimple ring**. Then, A has only **finitely** many **simple** A-**modules**, up to **isomorphism**.*
> *(Proposition 2.14)*

kG has finitely many simple modules up to isomorphism, and by:

> *Let V be a **vector space** and G be a **group**.*
>
> 1. *If*
> $$\rho : G \to GL(V)$$
> *is a **representation**, then V is a **left** kG-**module**, where left multiplication of kG on elements of V is given by:*
> $$\forall a_x \in k, v \in V, \quad \left(\sum_{x \in G} a_x x\right) \cdot v = \sum_{x \in G} a_x \rho_x(v)$$
> *where $\rho_x := \rho(x)$*
> 2. *If V is a **left** kG-**module**, there is a representation*
> $$\rho : G \to GL(V)$$
> *defined by:*
> $$\forall g \in G, v \in V, \quad \rho_g(v) = g \cdot v$$
> 3. *1) & 2) define a **bijection** between **representations** $\rho : G \to GL(V)$ and left multiplication in kG-**modules** $kG \times V \to V$*

there is a bijective mapping between kG-modules and representations of G.

□

6 Schur's Lemma

6.1 Building Definitions

6.1.1 Definition: Number of Isomorphism Classes

> *For finite groups G, $r_k(G)$ denotes the number of **isomorphism classes** of **irreducible** k-**representations** of G.*
> *(Definition 2.16)*

6.1.2 Definition: Centre of Rings

*Let A be a **ring**. Its **centre** is a **commutative unital subring** of A, defined by:*

$$Z(A) = \{z \in A \mid \forall a \in A, az = za\}$$

(Definition 3.1)

6.1.3 Definition: Endomorphism Ring of Module

*Let A be a **ring**, and V an A-**module**.*
*The **endomorphism ring of** V, denoted $\text{End}_A(V)$, is the ring consisting of A-**module endomorphisms***

$$\varphi : V \to V$$

where:

- ***ring addition** is pointwise addition of **homomorphisms***
- ***ring multiplication** is **composition***

6.1.4 Remark: Central Elements Give Endomorphisms

*Take any $z \in Z(A)$, and define an **endomorphism**:*

$$z_V : V \to V$$

via:

$$v \mapsto z \cdot v$$

*We can check that z_V is indeed an **endomorphism**:*

$$\begin{aligned} z_V(a \cdot v) &= z \cdot (a \cdot v) \\ &= (za) \cdot v \\ &= (az) \cdot v \\ &= a \cdot z_V(v) \end{aligned}$$

6.1.5 Definition: Opposite Ring

> *Let A be a **ring**. The **opposite ring** to A (denoted A^{op}) has the same underlying **abelian group** as A, but it has a new multiplication:*
>
> $$a * b = b \cdot a$$
>
> *(Definition 3.3)*

6.1.6 Definition: k-Algebra

> *We say that A is a k**-algebra** if it contains k as a **central subfield**. Moreover, k**-linear ring homomorphisms** are **homomorphisms of** k**-algebras**.*
>
> ---
>
> *If A is a **semisimple ring**, we say that A is a **semisimple** k**-algebra**. (Definition 3.5)*

- **Is it possible to have a commutative subfield which isn't central?**
  - consider the **quaternions** $\mathbb{H}$ (which are a non-commutative division ring)
  - the complex numbers $\mathbb{C}$ are a subfield of $\mathbb{H}$
  - however, they aren't central (since $\mathbb{H}$ itself isn't commutative
  - thus, $\mathbb{H}$ is **not** a $\mathbb{C}$-algebra
  - in other words: non-commutative division rings can contain subfields which aren't central, so **centrality** is key int he definition

6.2 Lemma: Module Endomorphisms

*Let A be a **ring**. Then:*

1. *for each $a \in A$ **right multiplication** by a defines an A**-module endomorphism**:*

$$r_a : {}_A A \to_A A$$

given by:

$$b \mapsto ba$$

2. *Every A**-module endomorphism***

$$\omega : {}_A A \to_A A$$

is of this form

3. *The map:*

$$A^{op} \to \mathrm{End}_A({}_A A)$$

given by:

$$a \mapsto r_a$$

*is an **isomorphism** of rings*

(Lemma 3.4)

6.3 Theorem: Schur's Lemma

*Suppose k is **algebraically closed**. Let V be a **simple module** over a **finite dimensional** k**-algebra** A.*
*Then, every A**-module endomorphism** of V is given by the action of some **scalar** $\lambda \in K$, such that:*

$$\mathrm{End}_A(V) = k 1_V$$

(Theorem 3.6)

Proof. By

> *Let V be a **cyclic** A**-module**, such that for $v \in V$, $Av = V$. Then:*
>
> $$Av \cong A/\operatorname{ann}_A(v)$$
>
> *(Lemma 2.12)*

since V is simple it is isomorphic to a quotient module of A, so in particular V is also a finite dimensional k-vector space.

Let $\varphi : V \to V$ be an A-module endomorphism. Then, φ has at least one eigenvalue $\lambda \in k$ (by algebraic closure of k, the roots of the characteristic polynomial lie in k).

Now, consider the mapping:

$$\varphi - \lambda 1_V : V \to V$$

This has non-zero kernel (if v is the eigenvector associated with λ, then $v \in \ker(\varphi - \lambda 1_V)$), so it can't be injective, and thus, isn't an isomorphism. Hence, by the contrapositive of:

> *Let V, W be **simple** A**-modules**. Then, every **non-zero,** A**-linear map***
>
> $$\varphi : V \to W$$
>
> *is an **isomorphism**.*
> *(Lemma 2.13)*

since $\varphi - \lambda 1_V$ is not an isomorphism ,it must be the zero map, which implies that:

$$\varphi = \lambda 1_V$$

as required.

□