Introduction to Representation Theory - Week 1 - Representations of Finite Groups

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Based on the notes by Konstantin Ardakov, Section 1

1 Introduction to Representations

1.1 Definition: General Linear Group of Vector Space

> *Let V be a **vector space** over a field k. The **general linear group** of V is the group of **automorphisms** of V:*
>
> $$GL(V) = \{\varphi : V \to V \mid \varphi \textit{ is an } \textbf{invertible k-linear map}\}$$
>
> *(Notation 1.1)*

1.2 Definition: Finite Group Representations

> *Let G be a **finite group** with V a finite dimensional **vector space** over k.*
> *A **representation** of G on V is a **group homomorphism**:*
>
> $$\rho : G \to GL(V)$$
>
> *The **degree** of ρ is $\dim(V)$.*
> *(Definition 1.2)*

1.3 Definition: Trivial Representation

> *The **trivial representation** of a group G on V is the **identity automorphism**:*
>
> $$1 : G \to GL(V)$$
>
> *whereby:*
>
> $$\forall g \in G, \forall v \in V, \quad 1(g)(v) = v$$
>
> *(Definition 1.16)*

1.4 Examples of Representations

1.4.1 Representations from Geometry

- let $G = \{e, g\} = \langle g \rangle$ act on $V = k$ via negation, such that $\rho(e) = 1, \rho(g) = -1$. This yields a representation of order 1.
- let $G = D_3 = \{e, g, g^2, h, gh, g^2h\}$ and let $k = \mathbb{R}$. G acts on $V = \mathbb{R}^2$ as the symmetries of a triangle; this induces a representation

$$\rho : D_3 \to GL(\mathbb{R}^2)$$

where:

- $\rho(g)$ is a rotation by $\frac{2\pi}{3}$ about the origin
- $\rho(h)$ is a reflection in the y-axis

- for a regular n-gon, $\rho : D_n \to GL(\mathbb{R}^2)$ is a degree 2 representation
- let $X \subset \mathbb{R}^3$ be the set of vertices of a cube centered at the origin. Let:

$$G = \mathrm{Stab}_{SO_3(\mathbb{R})}(X)$$

(so G contains all 3-dimensional rotations about the origin which map the cube to itself). It can be shown that $G \cong S_4$, which induces a representation:

$$\rho : S_4 \to GL(\mathbb{R}^3)$$

of degree 3.

1.4.2 Permutation Representations of Sets

1.4.2.1 Definition: Free Vector Space on a Set

Let X be a ***finite set****. The* ***free vector sapce*** *on X is a* ***vector space*** *with* ***basis*** *X, and whose underlying set is:*

$$kX = \left\{ \sum_{x \in X} a_x x \ | a_x \in k \right\}$$

(here this is just a ***formal linear combination****; as such, the sum is just notation, and addition may not even be defined amongst elements of X)*
Addition *and* ***scalar multiplication*** *is as would be expected.*
(Definition 1.4)

1.4.2.2 Definition: The Permutation Representation of a Set

Let G be a ***finite group*** *acting on a* ***finite set*** *X. Each $g \in G$ defines a permutation of X:*

$$\rho_g : X \to X$$

via:

$$\rho_g(x) = g \cdot x$$

This extends (uniquely) to an ***invertible*** *(since g is invertible) linear map:*

$$\rho_g \in GL(kX)$$

via:

$$\rho_g\left(\sum_{x \in X} a_x x\right) = \sum_{x \in X} a_x (g \cdot x)$$

In particular, by properties of group actions:

$$\rho : G \to GL(kX) \qquad g \mapsto \rho_g$$

is a ***representation****, called the* ***permutation representation*** *associated with X.*

One can check that ρ_g is indeed a group homomorphism. For any $x \in X$, consider $g, h \in G$. Then:

$$\rho_{gh}(x) = (gh) \cdot x = g \cdot (h \cdot x) = \rho_g(\rho_h(x)) \implies \rho_{gh} = \rho_g \circ \rho_h$$

1.4.3 Representations from Galois Theory

- let F be a finite field extension of $\mathbb{Q}$, such that $[F : \mathbb{Q}] < \infty$
- from Galois Theory, $G = Gal(F : \mathbb{Q})$ (the group of all automorphisms of F over $\mathbb{Q}$) is a finite group
- the inclusion $G \hookrightarrow GL(F)$ thus gives a $\mathbb{Q}$-linear representation of G of degree $\deg_{\mathbb{Q}}(F)$
- for example, if $F = SF_{\mathbb{Q}}(t^3 - 2) = \mathbb{Q}(\sqrt[3]{2}, e^{2\pi/3})$, then $G \cong S_3$ and this gives a degree 6 representation $\rho : S_3 \to GL(\mathbb{Q}(\sqrt[3]{2}, e^{2\pi/3}))$

1.5 Definition: Faithful Representations

A ***representation***

$$\rho : G \to GL(V)$$

is ***faithful*** *if*

$$\ker(\rho) = \{e_G\}$$

(Definition 1.7)

1.5.1 Remark: Why are representations interesting for group theorists?

- let $\rho : G \to GL(V)$
- by the First Isomorphism Theorem for groups:

$$G/\ker(\rho) \cong \mathrm{im}(\rho)$$

- if ρ is faithful, then its kernel is trivial, so:

$$G \cong \mathrm{im}(\rho) \leq GL(V) \cong GL_n(k)$$

  As we will see, working with G will involve using matrix representations, which are easy to work with.
- if ρ isn't faithful, then $\ker(\rho)$ will be a proper normal subgroup of ρ, and still $G/\ker(\rho) \cong \mathrm{im}(\rho)$

2 Matrix Representations

2.1 Definition: Matrix Representation of Finite Groups

*Let G be a **finite group**. A **matrix representation** is a **group homomorphism***

$$\rho : G \to GL_n(k)$$

where

$$GL_n(k) = M_n(k)^\times$$

is the group of invertible matrices under matirx multiplication.
(Definition 1.8)

2.2 From Automorphisms to Matrices

2.2.1 Definition: Matrices from Linear Maps

Let

$$\mathcal{B} = \{v_1, \ldots, v_n\}$$

*be a **basis** for V.*
Let

$$\phi : V \to V$$

*be a **linear map**.*
*The **matrix of ϕ with respect to $\mathcal{B}$** is ${}_{\mathcal{B}}[\phi]_{\mathcal{B}}$, an $n \times n$ matrix whose entries a_{ij} are given by the coefficients:*

$$\phi(v_j) = \sum_{i=1}^{n} a_{ij} v_i$$

(Definition 1.9)

2.2.2 Remark: Link Between Representations and Matrix Representations

*Let V be a **vector space** with **basis** $\mathcal{B}$. Then:*

1. $\phi \mapsto {}_{\mathcal{B}}[\phi]_{\mathcal{B}}$ *is a **group isomorphism**, such that*

$$GL(V) \cong GL_n(k)$$

2. ***representations*** $\rho : G \to GL(V)$ *lead to **matrix representations*** $\rho_{\mathcal{B}} : G \to GL_n(k)$ *via:*

$$\forall g \in G, \quad \rho_{\mathcal{B}}(g) = {}_{\mathcal{B}}[\rho(g)]_{\mathcal{B}}$$

3. ***matrix representations*** $\sigma : G \to GL_n(k)$ *lead to **representations*** $\underline{\sigma} : G \to GL(k^n)$ *via:*

$$\forall g \in G, v \in k^n \quad \underline{\sigma}(g)(v) = \sigma(g)v$$

where $\sigma(g)v$ is matrix multiplication of $\sigma(g)$ (an $n \times n$ matrix) by a column vector $v \in k^n$

2.2.3 Example: Permutation Representations

- consider $G = S_3$ acting on the set $X = \{e_1, e_2, e_3\}$ by permuting the indices
- this yields a degree 3 permutation $\rho : G \to GL(kX)$ (since $|X| = 3$ and X is a basis)
- we can compute the effect of the representation on some elements of S_3:
  - if $g = (1\ 2\ 3)$, then:

$$\begin{aligned}(1\ 2\ 3) \cdot e_1 &= 0 \cdot e_1 + 1 \cdot e_2 + 0 \cdot e_3 \\ (1\ 2\ 3) \cdot e_2 &= 0 \cdot e_1 + 0 \cdot e_2 + 1 \cdot e_3 \\ (1\ 2\ 3) \cdot e_3 &= 1 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3\end{aligned}$$

    so its corresponding matrix is:

$$\rho_X((1\ 2\ 3)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

    where notice, this has the effect of permuting the columns of the identity matrix according to the permutation
  - similarly

$$\rho_X((1\ 2)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3 G-Stability

3.1 Definition: Intertwining Operator

Consider 2 representations:

$$\rho : G \to GL(V) \qquad \sigma : G \to GL(W)$$

A ***homomorphism*** *or* ***intertwining operator*** *is a linear map:*

$$\varphi : V \to W$$

such that:

$$\forall g \in G,\ \sigma(g) \circ \varphi = \varphi \circ \rho(g)$$

If φ is ***bijective****, then it is an isomorphism.*
(Definition 1.12)

An alternative way of thinking about the intertiwing operator is as the mapping required such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle \rho(g)}\downarrow & & \downarrow{\scriptstyle \sigma(g)} \\ V & \xrightarrow[\varphi]{} & W \end{array}$$

3.2 Definition: Equivalent Matrix Representations

Matrix representations

$$\rho_1 : G \to GL_n(k) \qquad \rho_2 : G \to GL_n(k)$$

*are **equivalent** if:*

$$\exists A \in GL_n(k) \; : \; \forall g \in G, \rho_2(g) = A\rho_1(g)A^{-1}$$

If ρ_1, ρ_2 are equivalent, this translates to equality of the following linear maps in $GL(k^n)$:

$$\underline{\rho_2(g)} \circ \underline{A} = \underline{A} \circ \underline{\rho_1(g)}$$

The converse remains true.
(Definition 1.13)

Note, when underlining matrix representations, we refer to the corresponding linear map, as per:

*Let V be a **vector space** with **basis** $\mathcal{B}$. Then:*

1. *$\phi \mapsto {}_{\mathcal{B}}[\phi]_{\mathcal{B}}$ is a **group isomorphism**, such that*

$$GL(V) \cong GL_n(k)$$

2. ***representations** $\rho : G \to GL(V)$ lead to **matrix representations** $\rho_{\mathcal{B}} : G \to GL_n(k)$ via:*

$$\forall g \in G, \quad \rho_{\mathcal{B}}(g) = {}_{\mathcal{B}}[\rho(g)]_{\mathcal{B}}$$

3. ***matrix representations** $\sigma : G \to GL_n(k)$ lead to **representations** $\underline{\sigma} : G \to GL(k^n)$ via:*

$$\forall g \in G, v \in k^n \quad \underline{\sigma}(g)(v) = \sigma(g)v$$

where $\sigma(g)v$ is matrix multiplication of $\sigma(g)$ (an $n \times n$ matrix) by a column vector $v \in k^n$

3.3 G-Stability

For readability, if ρ is some representation, we denote $\rho(g) = \rho_g$.

3.3.1 Definition: G-Stable Subspace

Let $\rho : G \to GL(V)$ be a ***representation****, and let U be a* ***linear subspace*** *of V.*
U is ***G-stable*** *if:*

$$\forall u \in U, \forall g \in G,\ \rho_g(u) \in U$$

(Definition 1.14, a))

3.3.2 Definition: Subrepresentation Afforded by Subspace

Let $\rho : G \to GL(V)$ be a ***representation****, and let U be a* ***linear subspace*** *of V.*
Suppose that U is ***G-Stable****.*
The ***subrepresentation of*** *ρ* ***afforded by*** *U is:*

$$\rho_U : G \to GL(U)$$

given by:

$$\forall w \in U, \forall g \in G,\ \rho_U(g)(w) = \rho_g(w)$$

(Definition 1.14, b))

3.3.3 Definition: Quotient Subrepresentation Afforded by Subspace

Let $\rho : G \to GL(V)$ be a ***representation****, and let U be a* ***linear subspace*** *of V.*
Suppose that U is ***G-Stable****.*
The ***quotient representation of*** *ρ* ***afforded by*** *U is:*

$$\rho_{V/U} : G \to GL(V/U)$$

given by:

$$\forall v + U \in V/U, \forall g \in G,\ \rho_{V/U}(g)(v + U) = \rho_g(v) + U$$

(Definition 1.14, c))

3.3.4 Remark: Short Exact Sequence

We can represent this sequence of vector spaces as a ***short exact sequence****:*

$$O \to A \to B \to C \to O$$

Here:

- *C is the quotient of B by A*
- *A is the kernel of the canonical mapping $B \to C$*

In our case, we'd have:

$$O \to U \to V \to V/U \to O$$

3.4 Lemma: G-Stable Subspaces from Homomorphisms

Let

$$\varphi : V \to W$$

be a ***homomorphism*** *between the representations:*

$$\rho : G \to GL(V) \qquad \sigma : G \to GL(W)$$

Then:

1. $\ker(\varphi)$ *is a* ***G-stable*** *subspace of V*
2. $\operatorname{im}(\varphi)$ *is a* ***G-stable*** *subspace of W*

(Lemma 1.15)

Proof. Both kernels and images are subspaces of V, W respectively. Recall, a subspace is G-stable if:

Let $\rho : G \to GL(V)$ be a ***representation****, and let U be a* ***linear subspace*** *of V.*
U is ***G-stable*** *if:*

$$\forall u \in U, \forall g \in G,\ \rho_g(u) \in U$$

(Definition 1.14, a))

Let $g \in G, k \in \ker(\varphi) \subseteq V$. We need to show that:

$$\rho_g(k) \in \ker(\varphi)$$

Then, using the definition of homomorphism:

$$\varphi(\rho_g(k)) = \sigma_g(\varphi(k)) = \sigma_g(1_W) = 1_W \iff \rho_g(k) \in \ker(\varphi)$$

where we use that σ_g is an invertible linear map. Thus, $\ker(\varphi)$ is indeed a **G-stable** subspace of V.

Let $g \in G, m \in \text{im}(\varphi) \subseteq W$. We need to show that:

$$\sigma_g(m) \in \text{im}(\varphi)$$

Since $m \in \text{im}(\varphi), \exists v \in V : \varphi(v) = m$. Then, using the definition of homomorphism:

$$\sigma_g(m) = \sigma_g(\varphi(v)) = \varphi(\rho_g(v)) \in \text{im}(\varphi)$$

Thus, $\text{im}(\varphi)$ is indeed a **G-stable** subspace of W.

□

3.5 Theorem: First Isomorphism Theorem for Representations

*Let $\rho : G \to GL(V)$ be a **representation**, and let U be a **linear subspace** of V.*
*Then, there is a **natural isomorphism**:*

$$V/\ker(\varphi) \cong \text{im}(\varphi)$$

*between the G-**representations** $\rho_{V/\ker(\varphi)}$ and $\sigma_{\text{im}(\varphi)}$.*
(Lemma 1.15)

3.6 Example: Trivial Subrepresentations Don't Imply Trivial Representations

- let k be the **finite field** $\mathbb{F}_p$ and $G \cong C_p$ have generator g
- let $\rho : G \to GL_2(k)$ be the matrix representation given by:

$$\rho(g^i) = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

  where $i \in [0, p-1]$. This can be easily verified to show that $\rho(g^i g^j) = \rho(g^i)\rho(g^j)$.
- the standard basis for $V = \mathbb{F}_p^2$ si given by:

$$\left\{ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

- notice, $\langle v_1 \rangle$ is G-stable, since for any $\alpha \in \mathbb{F}_p$:

$$\rho(g^i)(\alpha v_1) = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \in \langle v_1 \rangle$$

- in fact, this shows that the **subrepresentation of ρ afforded by U**, ρ_U, is trivial, since any $g^i \in G$ maps into the identity automorphism in $GL(\langle v \rangle)$:

$$\begin{pmatrix} \alpha \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

- similarly, consider the action of G on the quotient space $\mathbb{F}_p^2 / \langle v_1 \rangle$

$$\rho(g^i)(\alpha v_1) = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 \\ \beta \end{pmatrix} + \langle v_1 \rangle \right) = \left(\begin{pmatrix} \beta i \\ \beta \end{pmatrix} + \langle v_1 \rangle \right) = \left(\begin{pmatrix} 0 \\ \beta \end{pmatrix} + \langle v_1 \rangle \right)$$

- again, the **quotient representation of ρ afforded by U** is trivial
- however, ρ itself is not trivial!
- this gives us the short exact sequence:

$$O \to 1 \to \mathbb{F}_p^2 \to 1 \to O$$

3.7 Definition: Irreducible Representations

*The **representation***

$$\rho : G \to GL(V)$$

*is **irreducible/simple** if:*

1. *V is not the **zero vector space***
2. *if U is a G**-stable subspace** of V, then either:*
   - $U = \{0\}$
   - $U = V$

(Definition 1.18)

***Irreducible representations** are the atoms of **representation theory**: finding all the irreducible representations up to isomorphism is a major goal of representation theory.*

3.8 Maschke's Theorem

3.8.1 Definition: G-Stable Complement

Let

$$\rho : G \to GL(V)$$

be a ***representation****, and let* U *be a* G***-stable subspace****.*
A G***-stable complement*** *for* U *in* G *is a* G***-stable subspace*** W *such that:*

$$V = U \oplus W$$

where recall, this means that:

- $U + W = V$
- $U \cap W = \{0\}$

(Definition 1.19)

3.8.2 Example: G-Stable Complement in Permutation Representation of S_3

- consider the **permutation representation** of S_3 when acting on kX, where $X = \{e_1, e_2, e_3\}$
- let
$$U = \langle e_1 + e_2 + e_3 \rangle$$
- U is a subspace of kX, and since the generator of U is symmetric in the indices of e_1, e_2, e_3, S_3 fixes every $u \in U$, so U is S_3-stable (and affords a trivial subrepresentation of V)
- can we find a non-trivial subrepresentation? Consider
$$W = \{a_1e_1 + a_2e_2 + a_3e_3 \mid a_1 + a_2 + a_3 = 0\}$$
- W is also S_3-stable (due to the condition $a_1 + a_2 + a_3 = 0$ holding irrespective of how we permute e_1, e_2, e_3)
- we claim that W is a S_3-stable complement to U
  - notice, $U \cap W = \emptyset$, since U contains sums of all 3 basis vectors, but W can never contain such a sum (or multiples thereof)
  - to show that $U + W = kX$, it is sufficient to show that we can generate a basis for kX from U, W. For sake of argument, lets consider constructing e_1. For some $m \in k$, a general element in $U + W$ is given by:
  $$m(e_1 + e_2 + e_3) + a_1e_1 + a_2e_2 + a_3e_3 = (m + a_1)e_1 + (m + a_2)e_2 + (m + a_3)e_3$$
  Solving simultaneous equations and using the fact that $a_1 + a_2 + a_3 = 0$, we get that:
  $$m = -a_3 = -a_2 \quad a_1 - a_3 = 1 \quad a_1 = -a_2 - a_3$$
  It follows that:
  $$-3a_3 = 1$$

Thus, assuming that $\text{char}(k) \neq 3$, we can always solve this system. For example, using $m = -1, a_1 = 2, a_2 = a_3 = 1$)

A perhaps easier way of reaching the same conclusion is to notice that:

$$\dim(U) = 1 \quad \dim(W) = 2$$

but $\dim(V) = 3$. *Since* $U, W \leq V$, *it suffices to show that* $U \cap W$ *is trivial. Indeed, assume that:*

$$\lambda(e_1 + e_2 + e_3) \in U \cap W$$

In particular, this requires that $3\lambda = 0$, *which, if* $\text{char}(k) \neq 0$ *implies that* $\lambda = 0$, *so* $U \cap W = \{0\}$.

- moreover, we can construct a basis for W via:

$$\mathcal{B} = \{v_1 = e_1 - e_2, v_2 = e_2 - e_3\}$$

- S_3 is generated by the permutations (1 2) and (1 2 3); thus, the degree 2 matrix representation:

$$\sigma = {}_{\mathcal{B}}[\rho_W]_{\mathcal{B}} : G \to GL_2(k)$$

  is determined by its effect on $\mathcal{B}$

- we compute:

$$\begin{aligned}(1\ 2\ 3) \cdot (e_1 - e_2) &= e_2 - e_3 = v_2 \\ (1\ 2\ 3) \cdot (e_2 - e_3) &= e_3 - e_1 = -v_1 - v_2\end{aligned}$$

  so:

$$\sigma((1\ 2\ 3)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

  Similarly:

$$\begin{aligned}(1\ 2) \cdot (e_1 - e_2) &= e_2 - e_1 = -v_1 \\ (1\ 2) \cdot (e_2 - e_3) &= e_1 - e_3 = v_1 + v_2\end{aligned}$$

  so:

$$\sigma((1\ 2\ 3)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

3.8.3 Theorem: Maschke's Theorem

Turns out, we can always find G-stable complements.

*Let G be a **finite group**, $|G| \neq 0$ in k (i.e $|G|$ isn't a multiple of $\text{char}(k)$). Let U be a G**-stable subspace** of a **finite-dimensional** G-representation V. Then, U admits **at least** one G**-stable complement** W in V.*
(Theorem 1.21)

Proof. Start with a basis for U. We can extend it to a basis for V, which yields a linear complement Z for U in V:

$$V = U \oplus Z$$

Generally Z won't be G-stable, but we can use it to generate a G-stable complement.

Let $\pi : V \to V$ be the projection map for $V = U \oplus Z$:

$$\forall u \in U, z \in Z, \quad \pi(u + z) = u$$

Define a new linear map $\varphi : V \to V$ via:

$$\forall v \in V, \quad \varphi(v) = \frac{1}{|G|} \sum_{x \in G} x \cdot (\pi(x^{-1} \cdot v))$$

where we've used the notation:

$$\rho(g)(v) = g \cdot v$$

for some representation $\rho : G \to GL(V)$. Notice, φ is linear: it is a sum of compositions of linear mappings

We claim that φ is a homomorphism of representations. Consider:

$$|G|\varphi(g \cdots v) = \sum_{x \in G} x \cdot (\pi(x^{-1} \cdot (g \cdot v)))$$

Now, make the substitution $y^{-1} = x^{-1}g$. Then, $x = gy$, and the sum can be made to run over all $y \in G$:

$$|G|\varphi(g \cdots v) = \sum_{y \in G} (gy) \cdot (\pi(y^{-1} \cdot v)) = g \cdot \left(\sum_{y \in G} y \cdot (\pi(y^{-1} \cdot v)) \right) = g \cdot |G|\varphi(v)$$

where we've used properties of group actions (from the fact that ρ is a homomorphism, so $g \cdot v = \rho(g)(v)$ defines a group actions). Thus, after cancelling $|G|$:

$$\varphi(g \cdot v) = g \cdot \varphi(v)$$

which is precisely the definition of φ being a representation homomorphism.

Now, let $u \in U$, then:

$$\begin{aligned}
\varphi(u) &= \frac{1}{|G|} \sum_{x \in G} x \cdot (\pi(x^{-1} \cdot u)) \\
&= \frac{1}{|G|} \sum_{x \in G} x \cdot (x^{-1} \cdot u) \quad \textit{(since } U \textit{ is } G\textit{-stable, } x^{-1} \cdot u \in U, \textit{ so } \pi \textit{ does nothing)} \\
&= u
\end{aligned}$$

To recap:

- $\varphi : V \to V$ is a representation homomorphism
- $\forall u \in U, \varphi(u) = u$

Now, let $W = \ker(\varphi) \leq U$. By:

> *Let*
> $$\varphi : V \to W$$
> *be a* ***homomorphism*** *between the representations:*
> $$\rho : G \to GL(V) \qquad \sigma : G \to GL(W)$$
> *Then:*
>
> 1. $\ker(\varphi)$ *is a* ***G-stable*** *subspace of* V
> 2. $\mathrm{im}(\varphi)$ *is a* ***G-stable*** *subspace of* W
>
> *(Lemma 1.15)*

W is a G-stable subspace of V. Moreover, by the rank-nullity theorem:

$$\dim(W) = \mathrm{nullity}(\varphi) = \dim(V) - \mathrm{rank}(\varphi)$$

Moreover, $\mathrm{im}(\pi) = \pi(V) = U$, so by definition of φ, $\varphi(V) = \mathrm{im}(V) \subseteq U$ (since U is G-stable, then $\pi(x^{-1} \cdot v) \in U$, so $x \cdot \pi(x^{-1} \cdot v) \in U$). But $\varphi(U) = U$, so $U = \varphi(U) \subseteq \varphi(V)$ implies that $\varphi(V) = U$ so $\mathrm{im}(\varphi) = U$ and $\mathrm{rank}(\varphi) = \dim(U)$. Putting it all together:

$$\dim(V) = \dim(W) + \dim(U)$$

It remains to show that $W \cap U$ is empty, but this follows easily: assume that $v \in W \cap U$. Then:

$$0 \overset{\text{since } v \in W \leq V}{=} \varphi(v) \overset{\text{since } v \in U}{=} v$$

Thus, $W \cap U$ is empty, as required.

Hence, $V = U \oplus W$, and $W = \ker(\varphi)$ is a G-stable complement to U in V.

□

3.8.4 Definition: Completely Reducible Representation

Let:

$$\rho : G \to GL(V)$$

be a ***representation****.*
ρ is ***completely reducible*** *if there exist G*-***stable subspaces***
$U_1, \ldots, U_m \leq V$ such that:

$$V = U_1 \oplus \ldots \oplus U_m$$

and the ***subrepresentation*** *of G afforded by each U_i is* ***irreducible****, or if $V = \{0\}$.*
(Definition 1.23)

3.8.5 Corollary: Representations of Finite Groups are Completely Reducible

Let G be a ***finite group****, and suppose that:*

$$\text{char}(k) \not| |G|$$

Then, every ***finite dimensional representation***

$$\rho : G \to GL(V)$$

is ***completely reducible****.*
(Corollary 1.24)

Proof. We induce on $n = \dim(V)$.

(1)

If $n = 0$, then $V = \{0\}$ and by definition, ρ is completely reducible.

(2)

Assume that if $\dim(V) = k$, then ρ is completely reducible.

(3)

Let $\dim(V) = k + 1$. Pick U_1 to be a non-zero, G-stable subspace of V of smallest dimension (if no such U_1 exists, then V will already be completely reducible).

We claim that U_1 is irreducible. Consider some $S \leq U_1$. Assume that $0 < \dim(S) \leq \dim(U_1)$. If U_1 is not irreducible, then S will be a G-stable subspace of V, and have smaller dimension than U_1, which is a contradiction. Thus, either $S = \{0\}$ of $\dim(S) = \dim(U_1) \implies V = S$, so U_1 is indeed irreducible.

By Maschke's Theorem, U_1 admits a G-stable complement W. But $\dim(W) < \dim(V)$, so by induction, there exist G-stabel irreducible subspaces $U_2, \ldots, U_m$ such that:

$$W = U_2 \oplus \ldots U_m \implies V = U_1 \oplus U_2 \oplus \ldots U_m$$

as required.

□