Networks - Week 4 - Non-Linear Evolving Network Models

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Contents

Here we explore generating a ***dynamically evolving network****, by using a* ***non-linear Markov Model****.*

We use the following notation:

- $A(t)$*: $n \times n$ binary (with 0 diagonal), symmetric* ***adjacency matrix*** *for the network at time t*
- $G_{A(t)}$*: the corresponding graph, whose edges appear/disappear over time*
- S*: real, symmetric, $n \times n$ matrices with elements in $[0, 1]$ and 0 diagonals. The* ***expected value*** $\langle B \rangle$ *of a graph B must lie in S*

1 Markov Network Evolution

1.1 Definition: Conditional Stochastic Network

*Consider a **random graph** $A(t)$, whose edges evolve **independently** over time, although they are **conditionally dependent** on the **current state** of the network.*

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*Since **edges** evolve **independently**, we can generate such **random graphs** from its **expected value**, which is conditioned on the **previous state**:*

$$\langle A(t+\delta t) \mid A(t) \rangle = A(t) + \delta t F(A(t))$$

where:

- *this holds as $\delta t \to 0$*
- *F is a **real matrix-valued function**, taking values in S*

More specifically, we typically fix F as:

$$F(A(t)) = -A(t) \circ \omega(A(t)) + (\mathbf{1} - A(t)) \circ \alpha(A(t))$$

where:

- *$\mathbf{1}$ is the **adjacency matrix** of the **clique** (all edges present, except for 0 diagonal)*
- *$\circ$ is the **Hadamard Product** (elementwise multiplication)*

- **How can we interpret the role of ω, α in F?**
  - ω acts on $-A(t)$, so it is used to **eliminate** edges in A, by assigning **edge death probabilities**
  - α acts on $(\mathbf{1} - A(t))$, so it is used to **add** edges in A, by assigning **edge birth probabilities** to the **complement** of A (edges which aren't present in A can be added)

1.2 Definition: Markov Process for Dynamic Networks

The above framework is perhaps too complex. In practice, we can consider using a discrete set of matrices instead.

*Let $\{A_k \mid k \in [1,K]\}$ be a set of **adjacency matrices**, representing a **discrete time evolving network** with value A_k at time t_k. We assume **edges evolve independently**, but that each network is conditionally independent on the previous one. By the **edge independence** assumption, the distribution of A_k is determined by the expected value $\langle A_{k+1} \mid A_k \rangle$.*

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*A **Markov Process** on the A_k is defined by:*

$$\langle A_{k+1} \mid A_k \rangle = A_k \circ (\mathbf{1} - \tilde{\omega}(A_k)) + (\mathbf{1} - A_k) \circ \tilde{\alpha}(A_k)$$

where again $\tilde{\omega}, \tilde{\alpha}$ are functions with image in S

1.2.1 Definition: Markov Network with Triadic Closure Dynamic

***Triadic closure** is the process by which if two **unconnected** people hav ea friend in common at step k, they are more likely to be **connected** at step $k+1$.*

Mathematically, we model this via:

$$\tilde{w}(A_k) = \gamma\,\mathbf{1} \qquad \tilde{\alpha}(A_k) = \delta\,\mathbf{1} + \epsilon A_k^2 \circ 1$$

where:

- $\gamma \in [0,1]$
- $\delta, \epsilon > 0$, *such that:*
$$\delta + \epsilon(n-2) < 1$$

*Then, this yields a **Markov Process with Triadic Closure Dynamic** given by:*

$$\langle A_{k+1} \mid A_k \rangle = A_k \circ (1-\gamma)\,\mathbf{1} + (\mathbf{1} - A_k) \circ (\delta\,\mathbf{1} + \epsilon A_k^2 \circ \mathbf{1})$$

*This is an **ergodic dynamic**: given enough time, it can visit **all** states.*

- **Why is $\tilde{\omega}$ defined thus?**
  - the **triadic closure dynamic** doesn't make assumptions about **death rate** of edges
  - this $\tilde{\omega}$ assumes that deaths occur **uniformly randomly**, with probability γ

- **Why is $\tilde{\alpha}$ defined thus?**
  - there are 2 terms:
    - $\delta\,\mathbf{1}$ gives a **uniformly random** birth rate of edges
    - $\epsilon A_k^2 \circ \mathbf{1}$ increases chance of an edge appearing, if the two vertices aren't connected, but share a common vertex (given by A_k^2, entry (i,j) contains vertices shared by i,j)
- **Why do we require that δ, ϵ satisfy $\delta + \epsilon(n-2) < 1$?**
  - consider entry i, j of
$$\delta\,\mathbf{1} + \epsilon A_k^2 \circ \mathbf{1}$$
  - this is given by:
$$\delta + \epsilon(\text{number of walks of length 2 between } i \text{ and } j$$
    (this number of walks encodes whether 2 unconnected verteices have a common friend)
  - since we want $\tilde{\omega}$ to lie in S, we require that this sum is **always** in the range $[0,1]$
  - the **maximum** number of walks of length 2 possible between two edges is $n-2$ (whereby two vertices can use every other vertex to get to each other)
  - hence, we require that:
$$\delta + \epsilon(n-2) < 1$$
  - since $\delta, \epsilon > 0$, we ensure that this sum is always in $[0,1]$
- **Despite being ergodic, what is the behaviour of the above Markov process in practice?**
  - the **average behaviour** over time of the process is given by its **expected value**
  - solutions will tend to spend most time close to states with **high edge density**

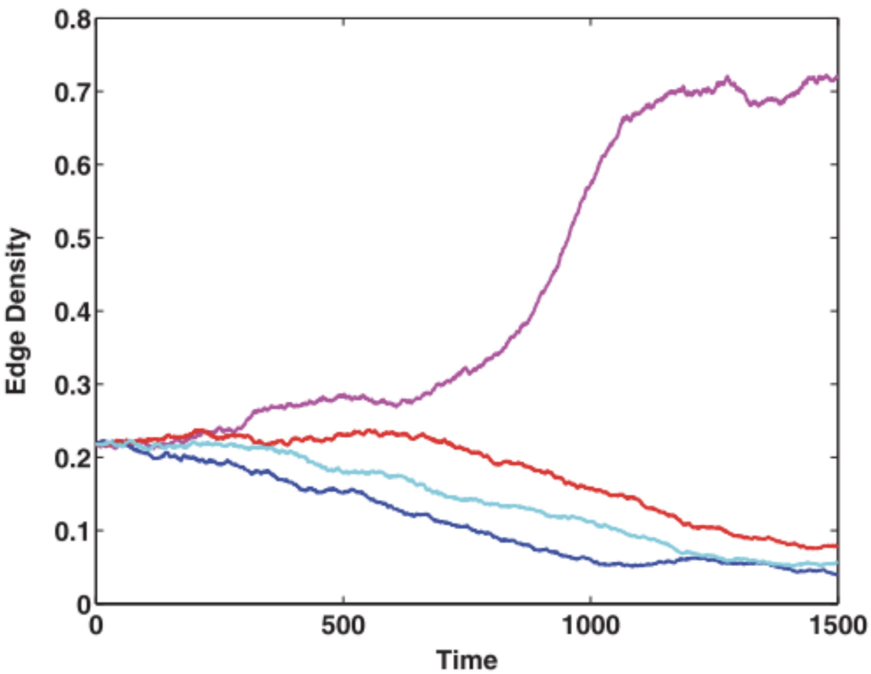

Figure 1: Simulations of the above Markov Process with $(n, \delta, \epsilon, \gamma) = (100, 4\times10^{-4}, 5\times10^{-4}, 1\times10^{-2})$. At each step, A_k is generated as a random graph from its expected value. Notice, despite being ergodic, graphs tend towards 2 distinct edge distributions. In theory, there is a small change that the graph generated is a **clique** or an **empty graph**, but other graphs are much more likely.

2 Mean Field Dynamic Approximation

2.1 Definition: Mean Field Dynamic Approximation for Markov Networks

The ***Mean Field Dynamic Approximation*** *approximates each* A_k*, with its own* ***expectation****:*

$$\langle A_{k+1} \mid A_k \rangle = p_k \, \mathbf{1}, \quad p_{k+1} \in [0, 1]$$

as an ***Erdös-Rényi Random Graph*** *with* ***edge density*** p_k*. We replace the right hand side with:*

$$\langle A_k \rangle = p_k \, \mathbf{1} \quad \left\langle A_k^2 \circ \mathbf{1} \right\rangle = (n-2) p_k^2 \, \mathbf{1}$$

Hence, we obtain a recursion:

$$p_{k+1} = p_k(1-\gamma) + (1-p_k)(\delta + (n-2)\epsilon p_k^2)$$

If

- δ *is* ***small***
- $\omega < \epsilon(n-2)/4$

then this iteration has ***3 fixed points****:*

- *2* ***stable fixed points*** *at:*

$$x_0 = \frac{\delta}{\gamma} + \delta^{\in} \qquad x_1 = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\gamma}{\epsilon}(n-2)} + \mathcal{O}(\delta)$$

- *1* ***unstable fixed point***

$$x_2 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\gamma}{\epsilon}(n-2)} + \mathcal{O}(\delta)$$

- **How does the mean field dynamic approximation differ from the actual Markov model?**
  - the key difference is that the **approximation** is **deterministic**, whereas the **Markov** model is **stochastic**
  - in practice, in short/medium scales, the approximation is really good

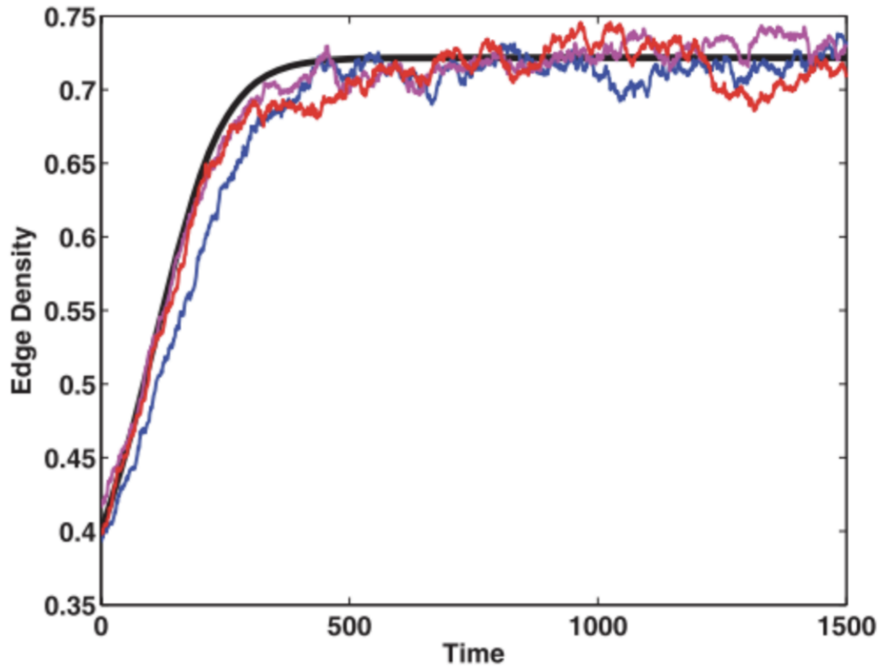

Figure 2: Squiggly lines corresponding to 3 Markov chain networks. The solid curve is the mean field evolution. Notice how the fit is fairly good. However, in the long run, stochasticity will lead to observable divergence between the 2 modles.

- **What sort of system is this network modelling?**
  - consider a group of friends, and let $\delta \to 0$
  - if **triadic closure** doesn't really apply, the system will approach the lower equilibrium, where the friendship rate is determined by γ
  - if in turn there is a lot of socialisation, the equilibrium will shift towards to higher value for edge density