Introduction to Partial Differential Equations - Week 9 - The Wave Equation

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1 Proposition: The Wave Equation

Let $u(t,x), t \in \mathbb{R}, \underline{x} \in \mathbb{R}^n$ represent the "shape" of an oscillating body. The ***wave equation*** *is:*

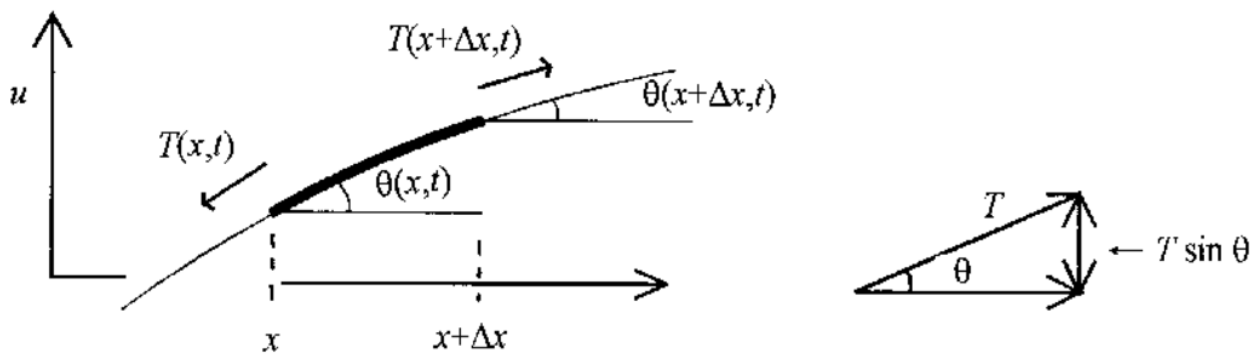
$$-u_{tt} + c^2 \Delta u = 0$$

where c^2 is called the ***speed****.*

To derive the wave equation, we can think of a piece of string in 1 dimension, whose shape is given by $u(t,x)$. In particular, we consider applying **Newton's Second Law**:

$$\sum F = ma$$

to a section of infinitesimal length Δx of the string. We have the following diagram:



The only non-negligible forces acing on the section of string are provided by the tension T at the endpoints x and $x + \Delta x$. These are dependent on the angle which the string makes θ. Overall, Newton's Second Law tells us (using $a = u_{tt}$):

$$\underbrace{T(x+\Delta x, t)\sin(\theta(x+\Delta x, t)) - T(x,t)\sin(\theta(x,t))}_{net\ force} = \underbrace{\rho \Delta x}_{mass} \underbrace{u_{tt}}_{acceleration}$$

where ρ represents the density of the string.

Now, if we divide through by Δx:

$$\rho u_{tt} = \frac{T(x+\Delta x, t)\sin(\theta(x+\Delta x, t)) - T(x,t)\sin(\theta(x,t))}{\Delta x}$$

Now, assuming θ will be small, then $\sin(\theta) \approx \tan(\theta)$. Moreover:

$$\tan(\theta(x,t)) = \frac{u(x+\Delta x, t) - u(x,t)}{\Delta x} = \frac{\partial u}{\partial x}$$

Hence, we can write:

$$\rho u_{tt} = \frac{\partial u}{\partial x} \frac{T(x+\Delta x, t)\frac{\partial u}{\partial x} - T(x,t)\frac{\partial u}{\partial x}}{\Delta x}$$

so taking $\Delta x \to 0$ and using the definition of partial derivative:

$$\rho u_{tt} = \frac{\partial d}{\partial dx}\left(T\frac{\partial u}{\partial x}\right)$$

Thinking of T as constant, and rearranging by defining $c^2 = \frac{T}{\rho}$ we get:

$$-u_{tt} + c^2 u_{xx} = 0$$

as required.

2 Solving the Wave Equation: 1+1 Spacetime Dimensions

2.1 Well-Posed Problems

- **What is a well-posed problem?**
  - a PDE is **well-posed** if:
    - a **solution** exists provided suitable **data**
    - the solution is **unique**
    - the solution depends **continuously** on the data
- **What is the global Cauchy problem?**
  - the **wave equation**, over $1+n$ **spacetime** dimensions, and over an **infinite interval**:

$$\begin{cases} -u_{tt}(t,\underline{x}) + \Delta_x u(t,\underline{x}) = 0, & t \in \mathbb{R}, \underline{x} \in \mathbb{R}^n \\ u(0,\underline{x}) = f(\underline{x}), & \underline{x} \in \mathbb{R}^n \\ u_t(0,\underline{x}) = g(\underline{x}), & \underline{x} \in \mathbb{R}^n \end{cases}$$

  - we need to prescribe 2 initial conditions, since there are 2 time derivatives involved
  - the **global Cauchy problem** is **well-posed**
- **How can we generate a well-posed problem on a finite interval?**
  - in the cases of $1+1$ **spacetime** dimensions, we might be interested in solutions u over **finite intervals** of x
  - the **Cauchy data** will be:

$$\begin{cases} -u_{tt}(t,x) + \Delta_x u(t,x) = 0, & t \in \mathbb{R}, x \in [0,L] \\ u(0,x) = f(x), & x \in [0,L] \\ u_t(0,x) = g(x), & x \in [0,L] \end{cases}$$

  - due to the **finiteness** of $[0,L]$, we need to provide additional information to generate a **well-posed problem**:
    1. **Dirichlet Data**:
    $$u(t,0) = a(t) \qquad u(t,L) = b(t) \qquad t > 0$$
    2. **Neumann Data**:
    $$u_x(t,0) = a(t) \qquad u_x(t,L) = b(t) \qquad t > 0$$
    3. **Robin Data**:
    $$u_x(t,0) - ku(t,0) = a(t) \qquad u_x(t,L) + ku(t,L) = b(t) \qquad t > 0, k \in \mathbb{R}^+$$
    4. **Mixed Data**: one kind of data at $x = 0$, and another one at $x = L$

2.2 Theorem: d'Alembert's Formula

Assume that:

$$f \in C^2(\mathbb{R}) \qquad g \in C^1(\mathbb{R})$$

Then, the ***unique*** *solution* $u(t,x)$ *to the* ***wave equation****:*

$$\begin{cases} -u_{tt}(t,x) + c^2 u_{xx}(t,x) = 0 \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$$

satisfies $u \in C^2([0,\infty) \times \mathbb{R})$ *and can be represented by:*

$$u(t,x) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{z=x-ct}^{z=x+ct} g(z)dz$$

This is ***d'Alembert's formula***

Proof. We begin by noticing that $A(x-ct), B(x+ct)$ are solutions to the wave equation, provided that A, B are twice differentiable with respect to t, x. Defining $z = x - ct$ and $u(t,x) = A(x-ct)$:

$$u_t = (-c)A'(z) \implies u_{tt} = c^2 A''(z)$$

$$u_x = A'(z) \implies u_{xx} = A''(z)$$

so:

$$-u_{tt} + c^2 u_{xx} = -c^2 A''(z) + c^2 A''(z) = 0$$

Similarly, if we let $z = x - ct$ and $u(t,x) = B(x+ct)$:

$$u_t = cA'(z) \implies u_{tt} = c^2 A''(z)$$

$$u_x = A'(z) \implies u_{xx} = A''(z)$$

so:

$$-u_{tt} + c^2 u_{xx} = -c^2 A''(z) + c^2 A''(z) = 0$$

Moreover, by linearity of the wave equation, we expect that:

$$\alpha A(z) + \beta B(z)$$

is also a solution.

We now try to derive **d'Alembert's formula**. To do so, without loss of generality assume $c = 1$ (we can just redefine $t \equiv c\tau$). Moreover, consider a change of variables (to **null coordinates**):

$$q(t,x) = x - t \qquad p(t,x) = x + t$$

Then:

$$u_t = u_p p_t + u_q q_t = u_p - u_q$$

$$u_{tt} = u_{pp} p_t + u_{pq} q_t - u_{qq} q_t - u_{qp} p_t = u_{pp} + -2u_{pq} + u_{qq}$$

and:

$$u_x = u_p p_x + u_q q_x = u_p + u_q$$

$$u_{xx} = u_{pp} p_x + u_{pq} q_x + u_{qq} q_x + u_{qp} p_x = u_{pp} + 2u_{pq} + u_{qq}$$

Hence, subtracting, we get:

$$-u_{tt} + u_{xx} = 4u_{pq}$$

But if u satisfies the wave equation $-u_{tt} + u_{xx} = 0$, which means that:

$$u_{pq} = 0$$

But notice, this just says that:

$$\frac{\partial}{\partial q}\left(\frac{\partial u}{\partial p}\right) = 0 \iff \frac{\partial u}{\partial p} = H(p)$$

where H is a function which only depends on p.

Now, we can think of t, x as functions of p, q:

$$p + q = 2x \implies x = \frac{1}{2}(p+q)$$

$$p - q = 2t \implies t = \frac{1}{2}(p-q)$$

so:

$$u_p = u_t t_p + u_x x_p = \frac{1}{2}(u_x + u_t)$$

$$u_q = u_t t_q + u_x x_q = \frac{1}{2}(u_x - u_t)$$

Thus, we have that:

$$H(p(t,x)) = u_p(t,x) = \frac{1}{2}(u_x(t,x) + u_t(t,x))$$

But now notice, if $(\tau, y) \in \mathbb{R} \times \mathbb{R}$, then:

$$p(\tau, y) = y + \tau = 0 + (y + \tau) = p(0, y + \tau)$$

Hence, it follows that:

$$u_p(\tau, y) = u_p(0, y+\tau) = \frac{1}{2}(u_x(0, y+\tau) + u_t(0, y+\tau)) = \frac{1}{2}\left(f'(y+\tau) + g(y+\tau)\right)$$

by using the initial conditions that u must satify.

Similarly, we can have:

$$u_{pq} = u_{qp} = 0 \iff \frac{\partial u}{\partial q} = K(q)$$

and $q(\tau, y) = q(0, y - \tau)$ so:

$$u_q(\tau, y) = \frac{1}{2}\left(f'(y-\tau) - g(y-\tau)\right)$$

Now, we have that:

$$u_p - u_q = \frac{1}{2}(u_x + u_t) - \frac{1}{2}(u_x - u_t) = u_t$$

so coming back to t, x coordinates from τ, y:

$$u_t = \frac{1}{2}\left(f'(x+t) - f'(x-t) + g(x+t) + g(x-t)\right)$$

If we integrate with respect to t, from 0 to t (using τ as a dummy variable):

$$\begin{aligned}
u(t,x) &= \int_0^t u_\tau d\tau \\
&= \int_0^\tau \frac{1}{2}\left(f'(x+\tau) - f'(x-\tau) + g(x+\tau) + g(x-\tau)\right) d\tau \\
&= \frac{1}{2}\left[f(x+\tau) - f(x-\tau)\right]_0^t + \frac{1}{2}\int_0^t (g(x+\tau) + g(x-\tau)) d\tau \\
&= \frac{1}{2}\left[f(x+t) - f(x-t) + f(x) - f(x)\right] + \frac{1}{2}\int_{z=x-t}^{z=x+t} g(z) dz \\
&= \frac{1}{2}\left(f(x+t) - f(x-t)\right) + \frac{1}{2}\int_{z=x-t}^{z=x+t} g(z) dz
\end{aligned}$$

which is d'Alembert's Formula.

The technique of using variables p, q also works to solve more general equations. For instance:

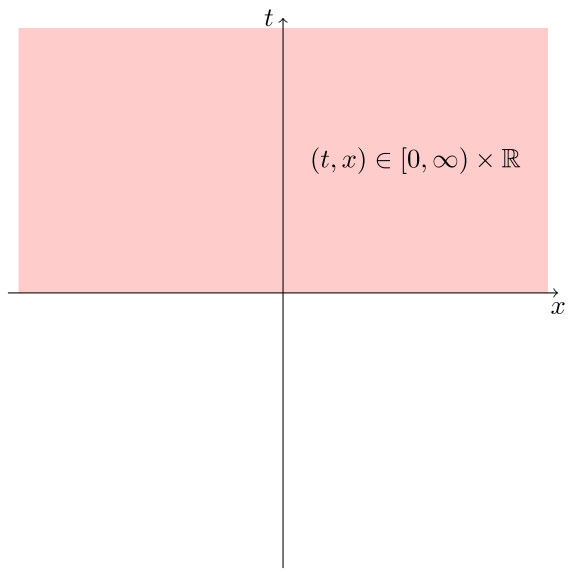
$$u_{tt} - u_{xx} = a(u_t + u_x)$$

can be solved by using the same substitution; we can then use an integrating factor to solve for u. However, the solution will no longer be a sum of travelling waves.

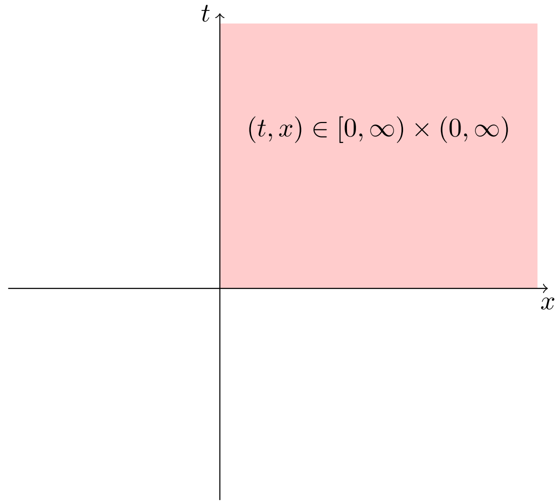
□

2.2.1 Corollary: d'Alembert's Formula for a Half-Plane

d'Alembert's formula above applies to solutions u where $x \in \mathbb{R}$:



However, we can easily adapt it to work even if $x \in (0, \infty)$*:*



Let:

- $f \in C^2([0,\infty))$
- $g \in C^1([0,\infty))$
- $f(0) = g(0) = 0$

Then, the ***unique*** *solution to the following initial + boundary value problem:*

$$\begin{cases} -u_{tt}(t,x) + u_{xx}(t,x) = 0, & (t,x) \in [0,\infty) + \times(0,\infty)] \\ u(t,0) = 0, & t \in [0,\infty) \\ u(0,x) = f(x), & x \in (0,\infty) \\ u_t(0,x) = g(x), & x \in (0,\infty) \end{cases}$$

satisfies:

$$u \in C^2([0,\infty) \times [0,\infty))$$

Moreover, we have that:

- *if* $0 \le ct \le x$*:*

$$u(t,x) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{z=|x-ct|}^{z=x+ct} g(z)dz$$

- *if* $0 \le x \le ct$*:*

$$u(t,x) = \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{z=|x-ct|}^{z=x+ct} g(z)dz$$

The key is that we can extend this to the above problem by considering **odd** extensions of our functions:

$$\tilde{u}(t,x) = \begin{cases} u(t,x), & t \ge 0, x \ge 0 \\ -u(t,-x), & t \ge 0, x \le 0 \end{cases}$$

$$\tilde{f}(x) = \begin{cases} f(x), & x \ge 0 \\ -f(-x), & x \le 0 \end{cases}$$

$$\tilde{g}(x) = \begin{cases} g(x), & x \ge 0 \\ -g(-x), & x \le 0 \end{cases}$$

This is applicable since $f(0) = g(0) = 0$, so $\tilde{f}, \tilde{g}$ will be continuous everywhere. Now, $\tilde{u}, \tilde{f}, \tilde{g}$ define a standard wave equation problem, with solution given by d'Alembert's Formula:

$$\tilde{u}(t,x) = \frac{1}{2}\left[\tilde{f}(x+ct) + \tilde{f}(x-ct)\right] + \frac{1}{2c}\int_{z=x-ct}^{z=x+ct} \tilde{g}(z)dz$$

so clearly u, f, g satisfy the wave equation on the quarter plane.

The explicit expression for u can then be found by decomposing the d'Alembert formula above in terms of $\tilde{u}, \tilde{f}, \tilde{g}$.

3 Solving the Wave Equation: 1+3 Spacetime Dimensions

We now seek to find an analogue for the wave equation in the physically relevant case: 1 + 3 spacetime dimensions, where $u(t, x) \in C^2([0, \infty) \times \mathbb{R}^3)$.

3.1 Proposition: Spherical Averages for Wave Equation

Let:

$$u(t,x) \in C^2([0,\infty) \times \mathbb{R}^3)$$

be a solution to the 1+3 dimensional ***global Cauchy problem****:*

$$\begin{cases} -u_{tt}(t,\underline{x}) + \Delta u(t,\underline{x}) = 0, & (t,\underline{x}) \in [0,\infty) \times \mathbb{R}^3 \\ u(0,\underline{x}) = f(\underline{x}), & \underline{x} \in \mathbb{R}^3 \\ u_t(0,\underline{x}) = g(\underline{x}), & \underline{x} \in \mathbb{R}^3 \end{cases}$$

For each $r > 0$, define the ***spherically averaged quantities****:*

$$U(t,r;\underline{x}) = \frac{1}{4\pi r^2} \int_{\partial B_r(\underline{x})} u(t,\underline{\sigma}) d\underline{\sigma} = \frac{1}{4\pi} \int_{\underline{\omega} \in \partial B_1(\underline{0})} u(t,\underline{x} + r\underline{\omega}) d\underline{\omega}$$

$$F(r;\underline{x}) = \frac{1}{4\pi r^2} \int_{\partial B_r(\underline{x})} f(\underline{\sigma}) d\underline{\sigma}$$

$$G(r;\underline{x}) = \frac{1}{4\pi r^2} \int_{\partial B_r(\underline{x})} g(\underline{\sigma}) d\underline{\sigma}$$

and their related modifications:

$$\tilde{U}(t,r;\underline{x}) = rU(t,r;\underline{x})$$

$$\tilde{F}(r;\underline{x}) = rF(r;\underline{x})$$

$$\tilde{G}(r;\underline{x}) = rG(r;\underline{x})$$

Then, for fixed $\underline{x} \in \mathbb{R}^3$:

$$\tilde{U}(t,r;\underline{x}) \in C^2([0,\infty) \times [0,\infty))$$

is a solution to the IVP + BVP for the ***one-dimensional wave equation****:*

$$\begin{cases} -\tilde{U}_{tt}(t,r;\underline{x}) + \tilde{U}_{rr}(t,r;\underline{x}) = 0, & (t,r) \in [0,\infty) \times [0,\infty) \\ \tilde{U}(t,0;\underline{x}) = 0, & t \in [0,\infty) \\ \tilde{U}(0,r;\underline{x}) = \tilde{F}(r;\underline{x}), & r \in (0,\infty) \\ \tilde{U}_t(0,r;\underline{x}) = \tilde{G}(r;\underline{x}), & r \in (0,\infty) \end{cases}$$

Moreover:

$$\lim_{r \to 0} U(t,r;\underline{x}) = u(t,\underline{x})$$

Before the proof, we recap spherical coordinates in $\mathbb{R}^3$:

$$\underline{\sigma} = (r, \theta, \phi) \in [0, \infty) \times [0, \pi) \times [0, 2\pi)$$

If we consider a ***sphere*** *centered at* $\underline{p} = (p^1, p^2, p^3)$*, then the Cartesian coordinate for some point* $\underline{x} = (x^1, x^2, x^3)$ *is given by:*

$$x^1 = p^1 + r \sin\theta \cos\phi$$
$$x^2 = p^2 + r \sin\theta \sin\phi$$
$$x^3 = p^3 + r \cos\theta$$

Moreover, when integrating, we have that:

$$d\underline{x} = r^2 \sin\theta dr d\theta d\phi$$

and if we integrate over some surface parametrised by $\underline{\omega} = (\theta, \phi) \in \partial B_1(\underline{0})$*:*

$$d\underline{\sigma} = r^2 d\underline{\omega} = r^2 \sin\theta d\theta d\phi$$

Proof. We want to show that $\tilde{U}$ satisfies the one-diemnsional wave equation. For this, we need to compute:

$$\tilde{U}_{tt} \qquad \tilde{U}_{rr}$$

---

(1) $\tilde{U}_{rr}$

We first notice that:

$$\partial_r[u(t, \underline{x} + r\underline{\omega})] = (\nabla u) \cdot \underline{\omega}$$

Thus, and using the fact that we can differentiate under the integral defining U:

$$\begin{aligned} U_r &= \frac{1}{4\pi} \int_{\underline{\omega} \in \partial B_1(\underline{0})} u_r(t, \underline{x} + r\underline{\omega}) d\underline{\omega} \\ &= \frac{1}{4\pi} \int_{\underline{\omega} \in \partial B_1(\underline{0})} (\nabla u) \cdot \underline{\omega} \; d\underline{\omega} \\ &= \frac{1}{4\pi r^2} \int_{\partial B_r(\underline{x})} (\nabla u) \cdot \underline{\hat{N}}(\underline{\sigma}) \; d\underline{\sigma} \end{aligned}$$

where we have applied the relation $d\underline{\sigma} = r^2 d\underline{\omega}$, alongisde the fact that $\underline{\omega}$ is the unit, outward, normal vector to $B_r(\underline{x})$.

Thus, if we apply the Divergence Theorem:

$$U_r = \frac{1}{4\pi r^2} \int_{B_r(\underline{x})} \Delta_y u(t, \underline{y}) \; d\underline{y}$$

Now, if we have a continuous function h on $\mathbb{R}^3$, and we use $(\rho, \underline{\omega})$ to denote spherical coordinates centered at $\underline{x}$. Then:

$$\begin{aligned}
\partial_r \int_{B_r(\underline{x})} h(\underline{y}) d\underline{y} &= \partial_r \int_0^r \int_{\underline{w} \in \partial B_r(\underline{x})} \rho^2 h(\rho, \underline{x} + \rho\underline{\omega}) d\underline{\omega} d\rho \\
&= \int_{\underline{w} \in \partial B_r(\underline{x})} \int_0^r \partial_r(\rho^2 h(\rho, \underline{x} + \rho\underline{\omega})) d\rho d\underline{\omega} \\
&= \int_{\underline{w} \in \partial B_r(\underline{x})} \left[\rho^2 h(\rho, \underline{x} + \rho\underline{\omega})\right]_0^r d\underline{\omega} \\
&= \int_{\underline{w} \in \partial B_r(\underline{x})} r^2 h(r, \underline{x} + r\underline{\omega}) d\underline{\omega} \\
&= \int_{\underline{w} \in \partial B_r(\underline{x})} h(\underline{\sigma}) d\underline{\sigma}
\end{aligned}$$

Using this then we have that:

$$\begin{aligned}
U_r &= \frac{1}{4\pi r^2} \int_{B_r(\underline{x})} \Delta_y u(t, \underline{y}) \ d\underline{y} \\
\implies r^2 U_r &= \frac{1}{4\pi} \int_{B_r(\underline{x})} \Delta_y u(t, \underline{y}) \ d\underline{y} \\
\implies \partial_r(r^2 U_r) &= \partial_r \left[\frac{1}{4\pi} \int_{B_r(\underline{x})} \Delta u(t, \underline{y}) \ d\underline{y} \right] \\
\implies \partial_r(r^2 U_r) &= \frac{1}{4\pi} \int_{\partial B_r(\underline{x})} \Delta u(t, \underline{\sigma}) \ d\underline{\sigma}
\end{aligned}$$

In other words:

$$\partial_r(r^2 U_r) = 2rU_r + r^2 U_{rr} = \frac{1}{4\pi} \int_{\partial B_r(\underline{x})} \Delta_y u(t, \underline{\sigma}) \ d\underline{\sigma}$$

Moreover:

$$\tilde{U}_{rr} = \partial_r^2(rU) = \partial_r(U + rU_r) = U_r + U_r + rU_{rr} = 2U_r + rU_{rr}$$

In other words:

$$\tilde{U}_{rr} = \frac{1}{r} \left(\partial_r(r^2 U_r)\right) = \frac{1}{4\pi r} \int_{\partial B_r(\underline{x})} \Delta_y u(t, \underline{\sigma}) \ d\underline{\sigma}$$

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(2) $\tilde{U}_{tt}$

Again, given:

$$U(t, r; \underline{x}) = \frac{1}{4\pi r^2} \int_{\partial B_r(\underline{x})} u(t, \underline{\sigma}) d\underline{\sigma}$$

we differentiate under the integral twice with respect to t to get:

$$U_{tt} = \frac{1}{4\pi r^2} \int_{\partial B_r(\underline{x})} u_{tt}(t, \underline{\sigma}) d\underline{\sigma}$$

Using the fact that u satisfies the wave equation:

$$-u_{tt} + \Delta u = 0 \implies U_{tt} = \frac{1}{4\pi r^2} \int_{\partial B_r(\underline{x})} \Delta u(t, \underline{\sigma}) d\underline{\sigma}$$

But then:

$$rU_{tt} = \tilde{U}_{tt} = \frac{1}{4\pi r} \int_{\partial B_r(\underline{x})} \Delta u(t, \underline{\sigma}) d\underline{\sigma}$$

Hence, we have shown that:

$$\tilde{U}_{tt} = \frac{1}{4\pi r} \int_{\partial B_r(\underline{x})} \Delta u(t, \underline{\sigma}) d\underline{\sigma} = \tilde{U}_{rr}$$

In other words, $\tilde{U} = rU$ satisfies the one-dimensional wave equation, as required.

We now need to verify that it satisfies the initial/boundary conditions:

$$\begin{cases} -\tilde{U}_{tt}(t, r; \underline{x}) + \tilde{U}_{rr}(t, r; \underline{x}) = 0, & (t, r) \in [0, \infty) \times [0, \infty) \\ \tilde{U}(t, 0; \underline{x}) = 0, & t \in [0, \infty) \\ \tilde{U}(0, r; \underline{x}) = \tilde{F}(r; \underline{x}), & r \in (0, \infty) \\ \tilde{U}_t(0, r; \underline{x}) = \tilde{G}(r; \underline{x}), & r \in (0, \infty) \end{cases}$$

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(1) $\tilde{U}(0, r; \underline{x}) = \tilde{F}(r; \underline{x})$

$$\begin{aligned} \tilde{U}(0, r; \underline{x}) &= rU(0, \underline{x}) \\ &= r \left(\frac{1}{4\pi r^2} \int_{\partial B_r(\underline{x})} u(0, \underline{\sigma}) d\underline{\sigma} \right) \\ &= r \left(\frac{1}{4\pi r^2} \int_{\partial B_r(\underline{x})} f(\underline{\sigma}) d\underline{\sigma} \right) \\ &= rF(r; \underline{x}) \end{aligned}$$

where we have used the fact that $u(0, \underline{x}) = f(\underline{x})$ when u solves the 1+3 global Cauchy problem.

(2) $\tilde{U}_t(0, r; \underline{x}) = \tilde{G}(r; \underline{x})$

$$\begin{aligned} \tilde{U}_t(0, r; \underline{x}) &= rU_t(0, \underline{x}) \\ &= r \left(\frac{1}{4\pi r^2} \int_{\partial B_r(\underline{x})} u_t(0, \underline{\sigma}) d\underline{\sigma} \right) \\ &= r \left(\frac{1}{4\pi r^2} \int_{\partial B_r(\underline{x})} g(\underline{\sigma}) d\underline{\sigma} \right) \\ &= rG(r; \underline{x}) \end{aligned}$$

where we have used the fact that $u_t(0, \underline{x}) = g(\underline{x})$ when u solves the 1+3 global Cauchy problem.

Finally, we have that:

$$\lim_{r \to 0} U(t, r; \underline{x}) = u(t, \underline{x})$$

since u is continuous and:

$$\begin{aligned}
\lim_{r \to 0} U(t, r; \underline{x}) &= \lim_{r \to 0} \frac{1}{4\pi} \int_{\underline{\omega} \in \partial B_1(\underline{0})} u(t, \underline{x} + r\underline{\omega}) d\underline{\omega} \\
&= \frac{1}{4\pi} \int_{\underline{\omega} \in \partial B_1(\underline{0})} \lim_{r \to 0} (u(t, \underline{x} + r\underline{\omega})) d\underline{\omega} \\
&= \frac{1}{4\pi} \int_{\underline{\omega} \in \partial B_1(\underline{0})} u(t, \underline{x}) d\underline{\omega} \\
&= \frac{1}{4\pi} (4\pi u(t, \underline{x}))
\end{aligned}$$

this also tells us that:

$$\tilde{U}(t, 0; \underline{x}) = 0$$

by using:

$$\lim_{r \to 0^+} (rU(t, r; \underline{x}) = 0(u(t, \underline{x})) = 0$$

□

3.1.1 Corollary: Representation formula for $\tilde{U}$

Under the assumptions of the above Proposition, and for $0 \leq r \leq t$, *we have that:*

$$\tilde{U}(t, r; \underline{x}) = \frac{1}{2} \left(\tilde{F}(t + r; \underline{x}) - \tilde{F}(t - r; \underline{x}) \right) + \frac{1}{2} \int_{\rho = t - r}^{\rho = t + r} \tilde{G}(\rho; \underline{x}) d\rho$$

Proof. This follows immediately by the fact that $\tilde{U}$ satisfies the one-dimensional wave equation on the quarter plane. We just need to apply the Corollary to d'Alembert's formula.

□

3.2 Theorem: Kirchhoff's Formula

Kirchhoff's Formula provides us with a solution to the global Cauchy problem in 1+3 spacetime dimensions.

Assume that:

$$f \in C^3(\mathbb{R}^3) \qquad g \in C^2(\mathbb{R}^3)$$

Then, the ***unique*** *solution* $u(t,\underline{x})$ *to the* ***global Cauchy problem****:*

$$\begin{cases} -u_{tt}(t,\underline{x}) + \Delta u(t,\underline{x}) = 0, & (t,\underline{x}) \in [0,\infty) \times \mathbb{R}^3 \\ u(0,\underline{x}) = f(\underline{x}), & \underline{x} \in \mathbb{R}^3 \\ u_t(0,\underline{x}) = g(\underline{x}), & \underline{x} \in \mathbb{R}^3 \end{cases}$$

satisfies:

$$u \in C^2([0,\infty) \times \mathbb{R}^3)$$

and can be represented by:

$$\begin{aligned} u(t,\underline{x}) = \frac{1}{4\pi t^2} \int_{\partial B_t(\underline{x})} f(\underline{\sigma}) d\underline{\sigma} \\ + \frac{1}{4\pi t} \int_{\partial B_t(\underline{x})} \nabla f(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) d\underline{\sigma} \\ + \frac{1}{4\pi t} \int_{\partial B_t(\underline{x})} g(\underline{\sigma}) d\underline{\sigma} \end{aligned}$$

Proof. By the Proposition above we know that:

$$\begin{aligned} u(t,\underline{x}) &= \lim_{r\to 0^+} U(t,r;\underline{x}) \\ &= \lim_{r\to 0^+} \frac{\tilde{U}(t,r;\underline{x})}{r} \\ &= \lim_{r\to 0^+} \left[\frac{1}{2r} \left(\tilde{F}(t+r;\underline{x}) - \tilde{F}(t-r;\underline{x}) \right) + \frac{1}{2r} \int_{\rho=t-r}^{\rho=t+r} \tilde{G}(\rho;\underline{x}) d\rho \right] \\ &= \tilde{F}_t(t;\underline{x}) + \tilde{G}(t;\underline{x}) \end{aligned}$$

where we have used the definition of the partial derivative to obtain $\tilde{F}_t$, alongside the Mean Value Theorem to get $\tilde{G}$.

If we apply the definition of $\tilde{F}, \tilde{G}$, it thus follows that:

$$\begin{aligned}
u(t, \underline{x}) &= \partial_t \left(t \frac{1}{4\pi t^2} \int_{\partial B_t(\underline{x})} f(\underline{\sigma}) d\underline{\sigma} \right) + t \frac{1}{4\pi t^2} \int_{\partial B_t(\underline{x})} g(\underline{\sigma}) d\underline{\sigma} \\
&= \frac{1}{4\pi t^2} \int_{\partial B_t(\underline{x})} f(\underline{\sigma}) d\underline{\sigma} + t \partial_t \left(\frac{1}{4\pi t^2} \int_{\partial B_t(\underline{x})} f(\underline{\sigma}) d\underline{\sigma} \right) + \frac{1}{4\pi t} \int_{\partial B_t(\underline{x})} g(\underline{\sigma}) d\underline{\sigma} \\
&= \frac{1}{4\pi t^2} \int_{\partial B_t(\underline{x})} f(\underline{\sigma}) d\underline{\sigma} + t \partial_t \left(\frac{1}{4\pi t^2} \int_{\partial B_1(\underline{0})} f(\underline{x} + t\underline{\omega}) t^2 d\underline{\omega} \right) + \frac{1}{4\pi t} \int_{\partial B_t(\underline{x})} g(\underline{\sigma}) d\underline{\sigma} \\
&= \frac{1}{4\pi t^2} \int_{\partial B_t(\underline{x})} f(\underline{\sigma}) d\underline{\sigma} + \frac{t}{4\pi} \int_{\partial B_1(\underline{0})} \partial_t (f(\underline{x} + t\underline{\omega})) d\underline{\omega} + \frac{1}{4\pi t} \int_{\partial B_t(\underline{x})} g(\underline{\sigma}) d\underline{\sigma} \\
&= \frac{1}{4\pi t^2} \int_{\partial B_t(\underline{x})} f(\underline{\sigma}) d\underline{\sigma} + \frac{t}{4\pi} \int_{\partial B_1(\underline{0})} (\nabla f)(\underline{x} + t\underline{\omega}) \cdot \underline{\omega} \; d\underline{\omega} + \frac{1}{4\pi t} \int_{\partial B_t(\underline{x})} g(\underline{\sigma}) d\underline{\sigma} \\
&= \frac{1}{4\pi t^2} \int_{\partial B_t(\underline{x})} f(\underline{\sigma}) d\underline{\sigma} + \frac{t}{4\pi} \int_{\partial B_t(\underline{0})} \nabla f(\underline{\sigma}) \cdot N(\underline{\sigma}) t^2 \; d\underline{\sigma} + \frac{1}{4\pi t} \int_{\partial B_t(\underline{x})} g(\underline{\sigma}) d\underline{\sigma} \\
&= \frac{1}{4\pi t^2} \int_{\partial B_t(\underline{x})} f(\underline{\sigma}) d\underline{\sigma} + \frac{1}{4\pi t} \int_{\partial B_t(\underline{0})} \nabla f(\underline{\sigma}) \cdot N(\underline{\sigma}) \; d\underline{\sigma} + \frac{1}{4\pi t} \int_{\partial B_t(\underline{x})} g(\underline{\sigma}) d\underline{\sigma}
\end{aligned}$$

as required.

□

4 Workshop

1. **Let B_1 denote the solid open unit ball in $\mathbb{R}^3$ centered at the origin. Recall that the Green function $G(\underline{x}, \underline{y})$ for B_1 satisfies:**

$$\begin{aligned}
G(\underline{x}, \underline{y}) &= -\frac{1}{4\pi \|\underline{x} - \underline{y}\|} + \frac{1}{4\pi \|\underline{x}\| \left\| \frac{1}{\|\underline{x}\|^2} \underline{x} - \underline{y} \right\|}, \qquad \underline{x}, \underline{y} \in B_1, \underline{x} \neq \underline{0} \\
G(\underline{0}, \underline{y}) &= -\frac{1}{4\pi \|\underline{y}\|} + \frac{1}{\pi}, \qquad \underline{y} \in B_1 \\
\nabla G(\underline{x}, \underline{\sigma}) \cdot \hat{\underline{N}}(\underline{\sigma}) &= \frac{1 - \|\underline{x}\|^2}{4\pi} \frac{1}{\|\underline{x} - \underline{\sigma}\|^3}, \qquad \underline{x} \in B_1, \underline{\sigma} \in \partial B_1
\end{aligned}$$

   Show that $G(\underline{x}, \underline{y}) \leq 0$ for all $\underline{x}, \underline{y} \in B_1$.

2. **Let B_1 denote the solid open unit ball in $\mathbb{R}^3$. Let $f(\underline{x})$ be smooth on B_1, and let $g(\underline{\sigma})$ be smooth on ∂B_1, and let $u(\underline{x}$ be the unique smooth solution to:**

$$\begin{cases} \Delta u(\underline{x}) = f(\underline{x}), & \underline{x} \in B_1 \\ u(\underline{\sigma}) = g(\underline{\sigma}) & \underline{\sigma} \in \partial B_1 \end{cases}$$

   Recall that the solution $u(\underline{x})$ can be represented as:

$$u(\underline{x}) = \int_{B_1} f(\underline{y}) G(\underline{x}, \underline{y}) d\underline{y} + \int_{\partial B_1} g(\underline{\sigma}) \nabla G(\underline{x}, \underline{\sigma}) \cdot \hat{\underline{N}}(\underline{\sigma}) d\underline{\sigma}$$

   Show that:

   -
$$\int_{B_1} G(\underline{x}, \underline{y}) d\underline{y} = \frac{1}{6} \|\underline{x}\|^2 - \frac{1}{6}$$

- **Conclude that:**
$$\underline{x} \in B_1 \implies -\int_{B_1} G(\underline{x}, \underline{y}) d\underline{y} \leq \frac{1}{6}$$
- **Show that $\exists C > 0$, independent of f, g, such that:**
$$\max_{B_1} |u(\underline{x})| \leq C \left(\max_{B_1} |f(\underline{x})| + \max_{\partial B_1} |g(\underline{\sigma})| \right)$$

Let $u(\underline{x}) = \frac{1}{6}\|\underline{x}\|^2 - \frac{1}{6}$. Then:
$$\Delta u = 1 \qquad u(\underline{\sigma}) = 0, \quad \underline{\sigma} \in \partial B_1$$
Hence, u satisfies the PDE:
$$\begin{cases} \Delta u(\underline{x}) = 1, & \underline{x} \in B_1 \\ u(\underline{\sigma}) = 0 & \underline{\sigma} \in \partial B_1 \end{cases}$$
so by the representation formulae:
$$u(\underline{x}) = \int_{B_1} G(\underline{x}, \underline{y}) d\underline{y}$$
as required.

We have that:
$$\int_{B_1} G(\underline{x}, \underline{y}) d\underline{y} = \frac{1}{6}\|\underline{x}\|^2 - \frac{1}{6} \geq -\frac{1}{6} \implies -\int_{B_1} G(\underline{x}, \underline{y}) d\underline{y} \leq \frac{1}{6}$$

Using the representation formula once again, we have that:
$$\begin{aligned} |u(\underline{x})| &\leq \int_{B_1} |f(\underline{y})||G(\underline{x}, \underline{y})| d\underline{y} + \int_{\partial B_1} |g(\underline{\sigma})||\nabla G(\underline{x}, \underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma})| d\underline{\sigma} \\ &\leq \frac{1}{6} \max_{B_1} |f| + \max_{\partial B_1} |g(\underline{\sigma})| \frac{1 - \|\underline{x}\|^2}{4\pi} \int_{\partial B_1} \frac{1}{\|x - \underline{\sigma}\|^3} d\underline{\sigma} \\ &= \frac{1}{6} \left(\max_{B_1} |f(\underline{x})| + \max_{\partial B_1} |g(\underline{\sigma})| \right) \end{aligned}$$
where we have used **Poisson's Representation Formula**:
$$u(\underline{x}) = \frac{R^2 - \|\underline{x}\|^2}{4\pi R} \int_{\partial B_R(\underline{0})} \frac{g(\underline{\sigma})}{\|\underline{x} - \underline{\sigma}\|^3} d\underline{\sigma}$$
with $u = 1, R = 1$ to determine that:
$$\frac{1 - \|\underline{x}\|^2}{4\pi} \int_{\partial B_1} \frac{1}{\|x - \underline{\sigma}\|^3} d\underline{\sigma} = 1$$

3. **Let u be a harmonic function on $\mathbb{R}^3$, and assume that:**
$$\forall \underline{x} \in \mathbb{R}^3, \quad |u(\underline{x})| \leq \ln(|x| + 1)$$
Show that $u(\underline{x}) = 0$ for all $\underline{x}$

Recall Harnack's Inequality:

$$\frac{R^{n-2}(R-\|\underline{x}\|)}{(R+\|\underline{x}\|)^{n-1}}u(\underline{0}) \leq u(\underline{x}) \leq \frac{R^{n-2}(R+\|\underline{x}\|)}{(R-\|\underline{x}\|)^{n-1}}u(\underline{0})$$

For fixed R, define:

$$v(\underline{x}) = u(\underline{x}) + \ln(R+1)$$

Clearly, $v \geq 0$, so for each fixed fixed $\|\underline{x}\| \leq R$, we can apply Harnack's Inequality for v:

$$\frac{R^{n-2}(R-\|\underline{x}\|)}{(R+\|\underline{x}\|)^{n-1}}v(\underline{0}) \leq v(\underline{x}) \leq \frac{R^{n-2}(R+\|\underline{x}\|)}{(R-\|\underline{x}\|)^{n-1}}v(\underline{0})$$

We focus on the first inequality: the second inequality will proceed in a similar manner:

$$\frac{R^{n-2}(R-\|\underline{x}\|)}{(R+\|\underline{x}\|)^{n-1}}(u(\underline{0}) + \ln(R+1)) \leq u(\underline{x}) + \ln(R+1)$$

which implies that:

$$u(\underline{x}) \geq \frac{R^{n-2}(R-\|\underline{x}\|)}{(R+\|\underline{x}\|)^{n-1}}u(\underline{0}) + \left[\frac{R^{n-2}(R-\|\underline{x}\|)}{(R+\|\underline{x}\|)^{n-1}} - 1\right]\ln(R+1)$$

The right term vanishes as $R \to \infty$ (by L'Hôpital's or the following approximation):

$$\begin{aligned} &\lim_{R\to\infty}\left[\frac{R^{n-2}(R-\|\underline{x}\|)}{(R+\|\underline{x}\|)^{n-1}} - 1\right]\ln(R+1) \\ \leq &\lim_{R\to\infty}\left[\frac{R^{n-1}}{R^{n-1}} - 1\right](R+1) \\ =&0 \end{aligned}$$

Hence, we must have that:

$$u(\underline{x}) \geq \lim_{R\to\infty}\left[\frac{R^{n-2}(R-\|\underline{x}\|)}{(R+\|\underline{x}\|)^{n-1}}u(\underline{0}) + \left[\frac{R^{n-2}(R-\|\underline{x}\|)}{(R+\|\underline{x}\|)^{n-1}} - 1\right]\ln(R+1)\right] = u(\underline{0})$$

By the other inequality, we get that $u(\underline{x}) \leq u(\underline{0})$, so we must have that:

$$u(\underline{x}) = u(\underline{0})$$

for all $\underline{x}$. But now:

$$|u(\underline{0})| \leq \ln(1) = 0 \implies u(\underline{0}) = 0$$

so:

$$u(\underline{x}) = 0$$

as required.

4. **Consider the equation:**

$$L[u] = \Delta u(x) + k^2 u(x) = 0, \qquad x \in \mathbb{R}^3$$

called Helmoltz or reduced wave equation.

(a) **Show that the radial solutions:**

$$u = u(r), \qquad r = |x|$$

satisfying the outgoing Sommerfeld condition:

$$u_r + iku = \mathcal{O}\left(\frac{1}{r^2}\right), \qquad r \to \infty$$

are of the form:

$$\phi(r,k) = c\frac{e^{-ikr}}{r}, \qquad c \in \mathbb{C}$$

We have that in radial coordinates the Laplacian becomes:

$$\Delta u = u_{rr} + \frac{2}{r}u_r$$

Moreover:

$$\frac{\partial}{\partial r}(ru) = u + ru_r$$

$$\frac{\partial}{\partial r^2}(ru) = u_r + u_r + ru_{rr} = ru_{rr} + 2u_r$$

Thus, if u satisfies the Helmoltz equation:

$$L[u] = \Delta u(x) + k^2 u(x) = 0, \qquad x \in \mathbb{R}^3$$

$$\implies u_{rr} + \frac{2}{r}u_r + k^2 u = 0$$

$$\implies ru_r r + 2u_r + rk^2 u = 0$$

$$\implies \frac{\partial}{\partial r^2}(ru) + k^2(ru) = 0$$

If we define:

$$v(r) = ru(r)$$

then we have a second order ODE:

$$v'' + k^2 v = 0$$

with characteristic polynomial:

$$P(\eta) = \eta^2 + k^2$$

which has roots $\eta = \pm ik$. Thus, the solutions will be:

$$v(r) = Ae^{ikr} + Be^{-ikr}$$

which implies:

$$u(r) = \frac{A}{r}e^{ikr} + \frac{B}{r}e^{-ikr}$$

where $A, B \in \mathbb{C}$.

If we differentiate u with respect to r:

$$u_r = A\left(\frac{r(ik)e^{ikr} - e^{ikr}}{r^2}\right) + B\left(\frac{r(-ik)e^{-ikr} - e^{-ikr}}{r^2}\right)$$

Notice, if u is to satisfy the outgoing Sommerfeld condition, we must set $A = 0$, because then:

$$u_r = -(ik)u - B\frac{e^{-ikr}}{r^2} \implies u_r + iku = \mathcal{O}\left(\frac{1}{r^2}\right)$$

Hence, solution to the Helmoltz equation satisfying the outgoing Sommerfeld condition must be of the form:

$$u(r,k) = B\frac{e^{-ikr}}{r}$$

as required.

(b) **For f smooth and compactly supported in $\mathbb{R}^3$ define the potential:**

$$U(x) = c\int_{\mathbb{R}^3} f(y)\frac{e^{-ik\|x-y\|}}{\|x-y\|}dy$$

Show that setting $c = \frac{1}{4\pi}$ leads to:

$$L[U(x)] = -f(x)$$

We make the variable substitution:

$$\underline{z} = \underline{y} - \underline{x}$$

to obtain:

$$U = c\int_{\mathbb{R}^3} f(\underline{x}+\underline{z})\frac{e^{-ik\|\underline{z}\|}}{\|\underline{z}\|}d\underline{z}$$

Let $\varepsilon > 0$. We can then write the integral as:

$$LU = c\int_{B_\varepsilon(\underline{0})} L\left[f(\underline{x}+\underline{z})\frac{e^{-ik\|\underline{z}\|}}{\|\underline{z}\|}\right]d\underline{z} + c\int_{\mathbb{R}^3\setminus B_\varepsilon(\underline{0})} L\left[f(\underline{x}+\underline{z})\frac{e^{-ik\|\underline{z}\|}}{\|\underline{z}\|}\right]d\underline{z}$$

We now show that the first integral goes to 0 as $\varepsilon \to 0$. Recall by Green's Identity:

$$\int_\Omega u(\underline{x})\Delta v(\underline{x}) - v(\underline{x})\Delta u(\underline{x})d\underline{x} = \int_{\partial\Omega} u(\underline{\sigma})(\nabla v(\underline{\sigma})\cdot\underline{\hat{N}}(\underline{\sigma})) - v(\underline{\sigma})(\nabla u(\underline{\sigma})\cdot\underline{\hat{N}}(\underline{\sigma}))d\underline{\sigma}$$

Moreover, since the operator L is defined over $\underline{x}$:

$$c\int_{B_\varepsilon(\underline{0})} L\left[f(\underline{x}+\underline{z})\frac{e^{-ik\|\underline{z}\|}}{\|\underline{z}\|}\right]d\underline{z} = c\int_{B_\varepsilon(\underline{0})}(\Delta f(\underline{x}+\underline{z}) + k^2 f(\underline{x}+\underline{z}))\frac{e^{-ik\|\underline{z}\|}}{\|\underline{z}\|}d\underline{z}$$

If we use $v = f$, it follows that by Green's Identity:

$$\int_{B_\varepsilon(\underline{0})}\Delta f(\underline{x}+\underline{z})\frac{e^{-ik\|\underline{z}\|}}{\|\underline{z}\|}d\underline{z} = \int_{B_\varepsilon(\underline{0})} f(\underline{x}+\underline{z})\Delta\frac{e^{-ik\|\underline{z}\|}}{\|\underline{z}\|}d\underline{z} + \int_{\partial B_\varepsilon(\underline{0})}\frac{e^{-ik\|\underline{\sigma}\|}}{\|\underline{\sigma}\|}\left(\nabla f\cdot\underline{\hat{N}}(\underline{\sigma})\right) - f\left(\nabla\frac{e^{-ik\|\underline{\sigma}\|}}{\|\underline{\sigma}\|}\cdot\underline{\hat{N}}\right)d\underline{\sigma}$$

But notice, since $\frac{e^{-ik\|\underline{z}\|}}{\|\underline{z}\|}$ solves the Helmholtz equation, we have that:

$$\Delta\frac{e^{-ik\|\underline{z}\|}}{\|\underline{z}\|} = -k^2\frac{e^{-ik\|\underline{z}\|}}{\|\underline{z}\|}$$

Thus, we will get a cancellation, such that:

$$c\int_{B_\varepsilon(\underline{0})} L\left[f(\underline{x}+\underline{z})\frac{e^{-ik\|\underline{z}\|}}{\|\underline{z}\|}\right]d\underline{z} = c\int_{\partial B_\varepsilon(\underline{0})}\frac{e^{-ik\|\underline{\sigma}\|}}{\|\underline{\sigma}\|}\left(\nabla f(\underline{x}+\underline{\sigma})\cdot\underline{\hat{N}}(\underline{\sigma})\right) - f(\underline{x}+\underline{\sigma})\left(\nabla\frac{e^{-ik\|\underline{\sigma}\|}}{\|\underline{\sigma}\|}\cdot\underline{\hat{N}}\right)d\underline{\sigma}$$

Using the compact support of f (which in particular implies that it is bounded) alongside the fact that $|e^{i\theta}| = 1$:

$$\begin{aligned}
&\left|\int_{\partial B_\varepsilon(\underline{0})}\frac{e^{-ik\|\underline{\sigma}\|}}{\|\underline{\sigma}\|}\left(\nabla f(\underline{x}+\underline{\sigma})\cdot\underline{\hat{N}}(\underline{\sigma})\right)d\underline{\sigma}\right| \\
&\leq \int_{\partial B_\varepsilon(\underline{0})}\frac{1}{\|\underline{\sigma}\|}|\nabla f|d\underline{\sigma} \\
&= \frac{1}{\varepsilon}\sup_{\partial B_\varepsilon(\underline{0})}|\nabla f|4\pi\varepsilon^2 \\
&= \sup_{\partial B_\varepsilon(\underline{0})}|\nabla f|4\pi\varepsilon
\end{aligned}$$

so:

$$\lim_{\varepsilon \to 0} \left| \int_{\partial B_\varepsilon(\underline{0})} \frac{e^{-ik\|\underline{\sigma}\|}}{\|\underline{\sigma}\|} \left(\nabla f(\underline{x} + \underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma} \right| = 0$$

5. **Suppose that:**

$$u \in C^2((0,\infty) \times \mathbb{R}) \cap C^1([0,\infty) \times \mathbb{R})$$

is a solution to:

$$u_{tt} = u_{xx}$$

in $(0,\infty) \times \mathbb{R}$**. Let:**

$$E(t) = \frac{1}{2} \int_{\mathbb{R}} u_x^2(t,x) + u_t^2(t,x) dx$$

and suppose that:

$$E(0) < \infty$$

Prove that $E(t)$ **is constant.**

6. **Suppose that**

$$u \in C^2((0,\infty) \times \mathbb{R}) \cap C^1([0,\infty) \times \mathbb{R})$$

is a solution to:

$$\begin{cases} u_{tt} - u_{xx} = f(t,x), & (t,x) \in (0,\infty) \times \mathbb{R} \\ u(0,x) = \phi(x) \\ u_t(0,x) = \psi(x) \end{cases}$$

Assuming that f, ϕ, ψ **have compact support, prove that the solution** u **is unique.**

7. **Consider the initial boundary value problem:**

$$\begin{cases} u_{tt} + u_{xt} - 12u_{xx} = 0, & (t,x) \in (0,\infty) \times \mathbb{R} \\ u(0,x) = \phi(x) \\ u_t(0,x) = \psi(x) \end{cases}$$

where ϕ, ψ **have compact supports. Make a change of variables to redcue the PDE to canonical form:**

$$U_{\zeta\zeta} - U_{\eta\eta} = 0$$

and hence express u **in terms of** ϕ **and** ψ**.**