Introduction to Partial Differential Equations - Week 7 & 8 - Green Functions, Harnack's Inequality and Liouville's Theorem

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Contents

We have worked towards solving Poisson's Equation:

$$\Delta u = f$$

over all of $\mathbb{R}^3$*. We now consider how to solve the PDE given some boundary conditions over some domain* $\Omega \subset \mathbb{R}^3$*. For this, we develop* ***Green Functions****.*

> *We consider the* **boundary value** *Poisson PDE:*
>
> $$\Delta u(\underline{x}) = f(\underline{x}), \qquad \underline{x} \in \Omega \subset \mathbb{R}^n$$
>
> $$u(\underline{x}) = g(\underline{x}, \qquad \underline{x} \in \partial\Omega$$
>
> *If* $g \in C(\partial\Omega)$*, then this PDE has a* ***unique solution****:*
>
> $$u \in C^2(\Omega) \cap C(\bar{\Omega})$$

1 Green Functions

1.1 Definition: Green Function

> *A* ***Green function*** *in* Ω *is a function on* $(\underline{x}, \underline{y}) \in \Omega \times \Omega$ *such that for each* ***fixed*** $\underline{x} \in \Omega$*:*
>
> $$\Delta_y G(\underline{x}, \underline{y}) = \delta_x(\underline{y}) = \delta(\underline{y} - \underline{x}), \qquad \underline{y} \in \Omega$$
>
> $$G(\underline{x}, \underline{\sigma}) = 0, \qquad \underline{\sigma} \in \partial\Omega$$

1.2 Proposition: Green Function for a Domain

> *Let* Φ *be the* ***fundamental solution*** *for* Δ *in* $\mathbb{R}^n$*:*
>
> $$\Phi(\underline{x}) = \begin{cases} \frac{1}{2\pi} \ln \|\underline{x}\|, & n = 2 \\ -\frac{1}{\omega_n \|\underline{x}\|^{n-2}}, & n \geq 3 \end{cases}$$
>
> *The* ***Green function*** $G(\underline{x}, \underline{y})$ *for a* ***domain*** Ω *is given by:*
>
> $$G(\underline{x}, \underline{y}) = \Phi(\underline{x} - \underline{y}) - \phi(\underline{x}, \underline{y})$$
>
> *such that for each* $\underline{x} \in \Omega$*,* $\phi(\underline{x}, \underline{y})$ *solves the* ***Dirichlet Problem****:*
>
> $$\Delta_y \phi(\underline{x}, \underline{y}) = 0, \qquad \underline{y} \in \Omega$$
>
> $$\phi(\underline{x}, \underline{\sigma}) = \Phi(\underline{x} - \underline{\sigma}), \qquad \underline{\sigma} \in \partial\Omega$$

Proof. We verify that this indeed satisfies the requirements for a Green function:

$$\Delta_y G(\underline{x}, \underline{y}) = \Delta\Phi(\underline{x} - \underline{y}) + \Delta_y \phi(\underline{x}, \underline{y}) = \delta(\underline{x} - \underline{y}) = \delta_x(\underline{y})$$

since ϕ is harmonic from definition, and we already showed that $\Delta\Phi = \delta$.

Moreover:

$$G(\underline{x}, \underline{\sigma}) = \Phi(\underline{x} - \underline{\sigma}) - \phi(\underline{x}, \underline{\sigma}) = \Phi(\underline{x} - \underline{\sigma}) - \Phi(\underline{x} - \underline{\sigma}) = 0$$

from definition of ϕ on $\partial\Omega$.

□

1.3 Proposition: Representation Formula for Solutions to Poisson PDE

This representation for solutions will be particularly useful for when we compute actual solution to the boundary value Poisson PDE.

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*Let Φ be the **fundamental solution** for Δ in $\mathbb{R}^n$:*

$$\Phi(\underline{x}) = \begin{cases} \frac{1}{2\pi} \ln \|\underline{x}\|, & n = 2 \\ -\frac{1}{\omega_n \|\underline{x}\|^{n-2}}, & n \geq 3 \end{cases}$$

*Let $\Omega \subset \mathbb{R}^n$ be a **domain**, and assume that:*

$$u \in C^2(\bar{\Omega})$$

Then, $\forall \underline{x} \in \Omega$ we have:

$$\begin{aligned} u(\underline{x}) = &\int_\Omega \Phi(\underline{x} - \underline{y}) \Delta_y u(\underline{y}) d^n \\ &- \underbrace{\int_{\partial\Omega} \Phi(\underline{x} - \underline{\sigma}) \left(\nabla u(\underline{\sigma}) \cdot \hat{\underline{N}}(\underline{\sigma}) \right) d\underline{\sigma}}_{\text{single layer potential}} \\ &+ \underbrace{\int_{\partial\Omega} u(\underline{\sigma}) \left(\nabla\Phi(\underline{x} - \underline{\sigma}) \cdot \hat{\underline{N}}(\underline{\sigma}) \right) d\underline{\sigma}}_{\text{double layer potential}} \end{aligned}$$

Proof. We prove this when $n = 3$, such that:

$$\Phi(\underline{x}) = -\frac{1}{\omega_3}\frac{1}{\|\underline{x}\|} = -\frac{1}{4\pi\|\underline{x}\|}$$

For this, we employ **Green's Identity**:

$$\int_\Omega v(\underline{x})\Delta w(\underline{x}) - w(\underline{x})\Delta v(\underline{x}) d^n x = \int_{\partial\Omega} v\left(\nabla w(\underline{\sigma}) \cdot \hat{\underline{N}}(\underline{\sigma})\right) - w\left(\nabla v(\underline{\sigma}) \cdot \hat{\underline{N}}(\underline{\sigma})\right) d\underline{\sigma}$$

Firstly, recall that we showed that $\Delta\Phi(\underline{x}) = 0$ whenever $\underline{x} \neq 0$. Thus:

$$\Delta\frac{1}{\|\underline{x} - \underline{y}\|} = 0$$

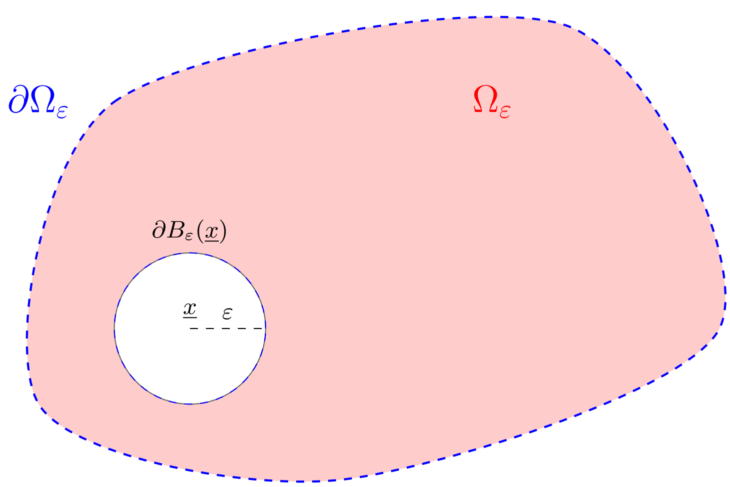
whenever $\underline{x} \neq \underline{y}$.

We begin by defining $B_\varepsilon(\underline{x})$, the ball of radius ε centered at $\underline{x}$. Then, consider:

$$\Omega_\varepsilon = \Omega \setminus B_\varepsilon(\underline{x})$$

such that:

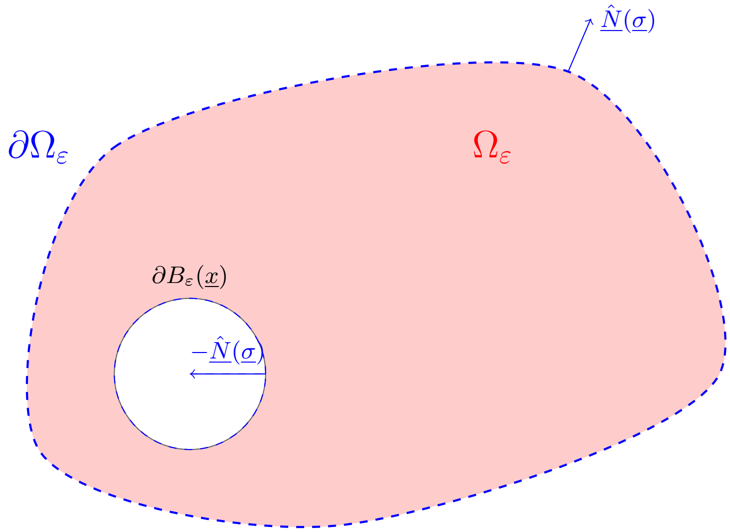
$$\partial\Omega_\varepsilon = \partial\Omega - \partial B_\varepsilon(\underline{x})$$



Then, applying Green's Identity:

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \frac{1}{\|\underline{x} - \underline{y}\|}\Delta u(\underline{y}) d^3 y \\
&= \int_{\Omega_\varepsilon} \frac{1}{\|\underline{x} - \underline{y}\|}\Delta u(\underline{y}) - u(\underline{y}) \underbrace{\Delta\frac{1}{\|\underline{x} - \underline{y}\|}}_{=0} d^3 y \\
&= \int_{\partial\Omega_\varepsilon} \frac{1}{\|\underline{x} - \underline{\sigma}\|}\left(\nabla u(\underline{\sigma}) \cdot \hat{\underline{N}}(\underline{\sigma})\right) - \nabla\frac{1}{\|\underline{x} - \underline{\sigma}\|}\left(u(\underline{\sigma}) \cdot \hat{\underline{N}}(\underline{\sigma})\right) d\underline{\sigma}
\end{aligned}$$

Now, $\partial\Omega_\varepsilon$ is composed of 2 different surfaces: the outward normal vector to $\partial\Omega$ faces **outwards**, whilst the outward normal vector to $\partial B_\varepsilon(\underline{x})$ will have to face **inwards** (for it to face **outwards** relative to $\partial\Omega_\varepsilon$):



Thus, we write the integral as:

$$\underbrace{\int_{\Omega_\varepsilon} \frac{1}{\|\underline{x}-\underline{y}\|}\Delta u(\underline{y})d^3y}_{L}$$

$$= \underbrace{\int_{\partial\Omega} \frac{1}{\|\underline{x}-\underline{\sigma}\|}\left(\nabla u(\underline{\sigma})\cdot\hat{\underline{N}}(\underline{\sigma})\right)d\underline{\sigma}}_{R_1}$$

$$- \underbrace{\int_{\partial\Omega} u(\underline{\sigma})\left(\nabla\frac{1}{\|\underline{x}-\underline{\sigma}\|}\cdot\hat{\underline{N}}(\underline{\sigma})\right)d\underline{\sigma}}_{R_2}$$

$$- \underbrace{\int_{\partial B_\varepsilon(\underline{x})} \frac{1}{\|\underline{x}-\underline{\sigma}\|}\left(\nabla u(\underline{\sigma})\cdot\hat{\underline{N}}(\underline{\sigma})\right)d\underline{\sigma}}_{R_3}$$

$$+ \underbrace{\int_{\partial B_\varepsilon(\underline{x})} u(\underline{\sigma})\left(\nabla\frac{1}{\|\underline{x}-\underline{\sigma}\|}\cdot\hat{\underline{N}}(\underline{\sigma})\right)d\underline{\sigma}}_{R_4}$$

We now show that as $\varepsilon \to 0^+$:

- $$L \to \int_{\Omega} \frac{1}{\|\underline{x}-\underline{y}\|}\Delta_y u(\underline{y})d^3y = -4\pi\int_{\Omega}\Phi(\underline{x}-\underline{y})\Delta_y u(\underline{y})d^3y$$

- $$R_1 \to \int_{\partial\Omega} \frac{1}{\|\underline{x}-\underline{\sigma}\|}\left(\nabla u(\underline{\sigma})\cdot\hat{\underline{N}}(\underline{\sigma})\right)d\underline{\sigma} \qquad \textit{(}4\pi\times\textit{ single layer potential)}$$

- $$R_2 \to -\int_{\partial\Omega} u(\underline{\sigma})\left(\nabla\frac{1}{\|\underline{x}-\underline{\sigma}\|}\cdot\hat{\underline{N}}(\underline{\sigma})\right)d\underline{\sigma} \qquad \textit{(}-4\pi\times\textit{ double layer potential)}$$

- $$R_3 \to 0$$

- $$R_4 \to -4\pi u(\underline{x})$$

from which the result follows by the fact that:

$$L = R_1 + R_2 + R_3 + R_4$$

since the factors $\frac{1}{4\pi}$ cancel out.

---

① L

Define:

$$M = \max_{y \in \bar{\Omega}} \Delta u(\underline{y})$$

Then:

$$\begin{aligned}
&\left| \int_{\Omega} \frac{1}{\|\underline{x} - \underline{y}\|} \Delta u(\underline{y}) d^3 y - L \right| \\
&= \left| \int_{\Omega} \frac{1}{\|\underline{x} - \underline{y}\|} \Delta u(\underline{y}) d^3 y - \int_{\Omega_\varepsilon} \frac{1}{\|\underline{x} - \underline{y}\|} \Delta u(\underline{y}) d^3 y \right| \\
&= \left| \int_{B_\varepsilon(\underline{x})} \frac{1}{\|\underline{x} - \underline{y}\|} \Delta u(\underline{y}) d^3 y \right| \\
&= \leq \int_{B_\varepsilon(\underline{x})} \frac{1}{\|\underline{x} - \underline{y}\|} |\Delta u(\underline{y})| d^3 y \\
&= \leq M \int_{B_\varepsilon(\underline{x})} \frac{1}{\|\underline{x} - \underline{y}\|} d^3 y
\end{aligned}$$

Thus, as $\varepsilon \to 0^+$, the ball over which we integrate becomes a point, so we can make this difference arbitrarily small. That is:

$$L \to \int_{\Omega} \frac{1}{\|\underline{x} - \underline{y}\|} \Delta u(\underline{y}) d^3 y$$

as expected.

② R_1

This one doesn't depend on ε, so the result is immediate.

③ R_2

This one doesn't depend on ε, so the result is immediate.

④ R_3

Define:

$$M' = \max_{y \in \bar{\Omega}} \|\nabla u(\underline{y})\|$$

Then:

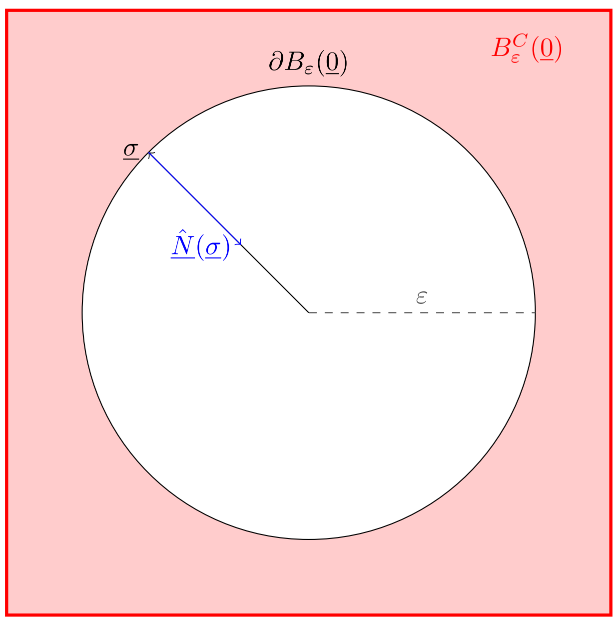
$$
\begin{aligned}
|R_3| &= \left| \int_{\partial B_\varepsilon(\underline{x})} \frac{1}{\|\underline{x}-\underline{\sigma}\|} \left(\nabla u(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma} \right| \\
&\leq \int_{\partial B_\varepsilon(\underline{x})} \frac{1}{\|\underline{x}-\underline{\sigma}\|} \|\nabla u(\underline{\sigma})\| d\underline{\sigma} \\
&\leq \int_{\partial B_\varepsilon(\underline{x})} \frac{1}{\varepsilon} M' d\underline{\sigma}, \quad \textit{(since we are on the surface of the ball)} \\
&= 4\pi\varepsilon^2 \times \frac{M'}{\varepsilon}
\end{aligned}
$$

so $|R_3| \to 0$ as $\varepsilon \to 0^+$ as required.

(5) R_4

We begin by recalling a result from last week:

$$
\nabla \frac{1}{\|\underline{\sigma}\|} \cdot \underline{\hat{N}} = -\frac{1}{\|\underline{\sigma}\|^2}
$$



Moreover, let:

$$
M'' = \max_{\underline{\sigma} \in \partial B_\varepsilon(\underline{x})} |u(\underline{x}) - u(\underline{\sigma})|
$$

Using this, we estimate:

$$\begin{aligned}
\left|\frac{1}{4\pi}R_4 - (-u(\underline{x}))\right| &= \frac{1}{4\pi}\left|u(\underline{x}) + \int_{\partial B_\varepsilon(\underline{x})} u(\underline{\sigma})\left(\nabla\frac{1}{\|\underline{x}-\underline{\sigma}\|}\cdot\underline{\hat{N}}(\underline{\sigma})\right)d\underline{\sigma}\right| \\
&= \frac{1}{4\pi}\left|\int_{\partial B_\varepsilon(\underline{x})}(u(\underline{x})-u(\underline{\sigma}))\frac{1}{\|\underline{x}-\underline{\sigma}\|^2}d\underline{\sigma}\right| \\
&\le \frac{1}{4\pi}\int_{\partial B_\varepsilon(\underline{x})}|u(\underline{x})-u(\underline{\sigma})|\frac{1}{\|\underline{x}-\underline{\sigma}\|^2}d\underline{\sigma} \\
&\le \frac{1}{4\pi}M''\int_{\partial B_\varepsilon(\underline{x})}\frac{1}{\|\underline{x}-\underline{\sigma}\|^2}d\underline{\sigma}
\end{aligned}$$

Consider spherical coordinates centered at $\underline{x}$:

$$(r,\theta,\phi)\in[0,\infty)\times[0,\pi)\times[0,2\pi)]$$

such that:

$$d\sigma = r^2\sin\theta d\theta d\phi$$

Then, we can write:

$$\int_{\partial B_\varepsilon(\underline{x})}\frac{1}{\|\underline{x}-\underline{\sigma}\|^2}d\underline{\sigma} = \int_0^{2\pi}\int_0^{\pi} d\theta d\phi = 4\pi$$

Thus:

$$|R_4 - (-u(\underline{x})| \le M''$$

But notice, M'' depends on ε, so the above difference becomes arbitarirly small, and:

$$R_4 \to -4\pi u(\underline{x})$$

as required.

Hence:

$$\underbrace{\int_{\Omega_\varepsilon} \frac{1}{\|\underline{x}-\underline{y}\|} \Delta u(\underline{y}) d^3 y}_{L}$$

$$= \underbrace{\int_{\partial\Omega} \frac{1}{\|\underline{x}-\underline{\sigma}\|} \left(\nabla u(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma})\right) d\underline{\sigma}}_{R_1}$$

$$- \underbrace{\int_{\partial\Omega} u(\underline{\sigma}) \left(\nabla \frac{1}{\|\underline{x}-\underline{\sigma}\|} \cdot \underline{\hat{N}}(\underline{\sigma})\right) d\underline{\sigma}}_{R_2}$$

$$- \underbrace{\int_{\partial B_\varepsilon(\underline{x})} \frac{1}{\|\underline{x}-\underline{\sigma}\|} \left(\nabla u(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma})\right) d\underline{\sigma}}_{R_3}$$

$$+ \underbrace{\int_{\partial B_\varepsilon(\underline{x})} u(\underline{\sigma}) \left(\nabla \frac{1}{\|\underline{x}-\underline{\sigma}\|} \cdot \underline{\hat{N}}(\underline{\sigma})\right) d\underline{\sigma}}_{R_4}$$

$$\implies \int_{\Omega} \frac{1}{\|\underline{x}-\underline{y}\|} \Delta u(\underline{y}) d^3 y = \int_{\partial\Omega} \frac{1}{\|\underline{x}-\underline{\sigma}\|} \left(\nabla u(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma})\right) d\underline{\sigma} - \int_{\partial\Omega} u(\underline{\sigma}) \left(\nabla \frac{1}{\|\underline{x}-\underline{\sigma}\|} \cdot \underline{\hat{N}}(\underline{\sigma})\right) d\underline{\sigma} - 4\pi u(\underline{x})$$

$$\implies u(\underline{x}) = \int_{\Omega} \Phi(\underline{x}-\underline{y}) \Delta_y u(\underline{y}) d^n y - \int_{\partial\Omega} \Phi(\underline{x}-\underline{\sigma}) \left(\nabla u(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma})\right) d\underline{\sigma} + \int_{\partial\Omega} u(\underline{\sigma}) \left(\nabla \Phi(\underline{x}-\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma})\right) d\underline{\sigma}$$

as required.

□

1.4 Proposition: Representation Formula for Solutions to Boundary Value Poisson PDE

*The above representation formula is **inconvenient**, in the sense that we require 3 pieces of information. Instead, we can use **Green Functions** to obtain a simpler representation.*

*Let Ω be a **domain** with a **smooth** boundary, and assume that:*

$$f \in C(\bar{\Omega}) \qquad g \in C(\partial\Omega)$$

*Then, **any** solution $u \in C^2(\bar{\Omega})$ (which will be **unique** in $C^2(\bar{\Omega})$) to the **boundary value Poisson problem** :*

$$\Delta u(\underline{x}) = f(\underline{x}), \qquad \underline{x} \in \Omega \subset \mathbb{R}^n$$

$$u(\underline{x}) = g(\underline{x}), \qquad \underline{x} \in \partial\Omega$$

can be represented as:

$$u(\underline{x}) = \int_{\Omega} f(\underline{y}) G(\underline{x}, \underline{y}) d^n y + \int_{\partial\Omega} g(\underline{\sigma}) \underbrace{\left(\nabla G(\underline{x}, \underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma})\right)}_{\text{Poisson kernel}} d\underline{\sigma}$$

*where $G(\underline{x}, \underline{y})$ is the **Green function** for Ω.*

*Here, having a **smooth** boundary isn't strictly necessary: for instance if Ω is a cube, or some regular shape, we can still have a representation formula. Smoothness is just convenient.*

Proof. Recall, a Green function on a domain Ω is of the form:

$$G(\underline{x}, \underline{y}) = \Phi(\underline{x} - \underline{y}) - \phi(\underline{x}, \underline{y})$$

where for fixed $\underline{x} \in \Omega$:

$$\Delta_y \phi(\underline{x}, \underline{y}) = 0, \qquad \underline{y} \in \Omega$$
$$\phi(\underline{x}, \underline{\sigma}) = \Phi(\underline{x} - \underline{\sigma}), \qquad \underline{\sigma} \in \partial\Omega$$

and:

$$G(\underline{x}, \underline{\sigma}) = 0, \qquad \underline{x} \in \Omega, \quad \underline{\sigma} \in \partial\Omega$$

Moreover, we have the representation formula for u as:

$$\begin{aligned} u(\underline{x}) &= \int_\Omega \Phi(\underline{x} - \underline{y}) \Delta_y u(\underline{y}) d^n y - \int_{\partial\Omega} \Phi(\underline{x} - \underline{\sigma}) \left(\nabla u(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma} + \int_{\partial\Omega} u(\underline{\sigma}) \left(\nabla \Phi(\underline{x} - \underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma} \\ &= \int_\Omega \Phi(\underline{x} - \underline{y}) f(\underline{y}) d^n y - \int_{\partial\Omega} \Phi(\underline{x} - \underline{\sigma}) \left(\nabla u(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma} + \int_{\partial\Omega} g(\underline{\sigma}) \left(\nabla \Phi(\underline{x} - \underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma} \end{aligned}$$

where we have used the fact that if u solves the boundary value Poisson porblem, then on Ω:

$$\Delta u = f$$

and on $\partial\Omega$:

$$u = g$$

Once again, recall **Green's Identity**:

$$\int_\Omega v(\underline{x}) \Delta w(\underline{x}) - w(\underline{x}) \Delta v(\underline{x}) d^n x = \int_{\partial\Omega} v \left(\nabla w(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) - w \left(\nabla v(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma}$$

If we let $u = w$ and $v = \phi$, we get that:

$$\int_\Omega \phi(\underline{x}, \underline{y}) \Delta_y u(\underline{y}) - u(\underline{y}) \Delta_y \phi(\underline{x}, \underline{y}) d^n x = \int_{\partial\Omega} \phi(\underline{x}, \underline{y}) \left(\nabla u(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) - u(\underline{\sigma}) \left(\nabla \Phi(\underline{x} - \underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma}$$

But notice:

- since u solves the boundary Poisson problem, on Ω:

  $$\Delta_y u = f$$

  and on $\partial\Omega$:

  $$u = g$$

- by construction, ϕ is such that on Ω:

$$\Delta_y \phi = 0$$

  and on $\partial\Omega$:

$$\phi(\underline{x}, \underline{\sigma}) = \Phi(\underline{x} - \underline{\sigma})$$

Hence:

$$\int_{\Omega} \phi(\underline{x}, \underline{y}) f(\underline{y}) = \int_{\partial\Omega} \Phi(\underline{x} - \underline{\sigma}) \left(\nabla u(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma} - \int_{\partial\Omega} g(\underline{\sigma}) \left(\nabla \phi(\underline{x} - \underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma}$$

Now, if we add the above to the representation for u:

$$\begin{aligned} u(\underline{x}) &= \int_{\Omega} \Phi(\underline{x} - \underline{y}) f(\underline{y}) d^n y - \int_{\partial\Omega} \Phi(\underline{x} - \underline{\sigma}) \left(\nabla u(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma} + \int_{\partial\Omega} g(\underline{\sigma}) \left(\nabla \Phi(\underline{x} - \underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma} \\ &+ \int_{\partial\Omega} \Phi(\underline{x} - \underline{\sigma}) \left(\nabla u(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma} - \int_{\partial\Omega} g(\underline{\sigma}) \left(\nabla \phi(\underline{x} - \underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma} - \int_{\Omega} \phi(\underline{x}, \underline{y}) f(\underline{y}) \\ &= \int_{\Omega} (\Phi(\underline{x} - \underline{y}) - \phi(\underline{x}, \underline{y}) f(\underline{y}) d^n y + \int_{\partial\Omega} g(\underline{\sigma}) \left(\nabla(\Phi(\underline{x} - \underline{\sigma}) - \phi(\underline{x}, \underline{\sigma})) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma} \\ &= \int_{\Omega} G(\underline{x}, \underline{y}) f(\underline{y}) d^n y + \int_{\partial\Omega} g(\underline{\sigma}) \left(\nabla G(\underline{x}, \underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma} \end{aligned}$$

as required.

□

2 Solving the Boundary Value Poisson PDE

2.1 Lemma: Green Function for a Ball Centered at the Origin

*Consider a ball $B_R(\underline{0}) \subset \mathbb{R}^n$ where $n \geq 3$. Then, the **Green Function** on $B_R(\underline{0})$ is:*

$$G(\underline{x}, y) = \begin{cases} \frac{1}{(n-2)\omega_n}\left[\|\underline{x}-\underline{y}\|^{2-n} - \left(\frac{\|\underline{x}\|}{R}\right)^{2-n}\left\|\frac{R^2\underline{x}}{\|\underline{x}\|^2} - \underline{y}\right\|^{2-n}\right], & \underline{x} \neq \underline{0} \\ \frac{1}{(n-2)\omega_n}\left[\|\underline{y}\|^{2-n} - R^{2-n}\right] & \underline{x} = \underline{0} \end{cases}$$

Furthermore, if $\underline{x} \in B_R(\underline{0})$ and $\underline{\sigma} \in \partial B_R(\underline{0})$ then:

$$\nabla G(\underline{x}, \underline{\sigma}) \cdot \hat{\underline{N}}(\underline{\sigma}) = \frac{R^2 - \|\underline{x}\|^{n-1}}{\omega_n R^{n-2}} \frac{1}{\|\underline{x}-\underline{\sigma}\|^n}$$

In particular, when $n = 3$:

$$G(\underline{x}, \underline{y}) = \begin{cases} -\frac{1}{4\pi\|\underline{x}-\underline{y}\|} + \frac{R}{4\pi\|\underline{x}\|\left\|\frac{R}{\|\underline{x}\|}\underline{x}-\underline{y}\right\|}, & \underline{x} \neq 0 \\ -\frac{1}{4\pi\|\underline{y}\|} + \frac{1}{4\pi R}, & \underline{x} = \underline{0} \end{cases}$$

$$\nabla G(\underline{x}, \underline{\sigma}) \cdot \hat{\underline{N}}(\underline{\sigma}) = \frac{R^2 - \|\underline{x}\|^2}{4\pi R} \frac{1}{\|\underline{x}-\underline{\sigma}\|^3}$$

Proof. Recall, a Green function over a domain Ω is given by:

$$G(\underline{x}, \underline{y}) = \Phi(\underline{x}-\underline{y}) - \phi(\underline{x}, \underline{y})$$

where Φ is the fundamental solution to $\nabla u = 0$, and for each $\underline{x} \in \Omega$, $\phi(\underline{x}, \underline{y})$ solves the **Dirichlet Problem**:

$$\Delta_y \phi(\underline{x}, \underline{y}) = 0, \qquad \underline{y} \in \Omega$$

$$\phi(\underline{x}, \underline{\sigma}) = \Phi(\underline{x}-\underline{\sigma}), \qquad \underline{\sigma} \in \partial\Omega$$

Lets operate with $n = 3$, and consider a ball $B_R(\underline{0})$ of radius R centered at the origin. To find G, we need to come up with a suitable $\phi(\underline{x}, \underline{y})$. One idea is to think of G as some sort of electric field. Φ represents the potential experienced by a point charge at some location in $B_R(\underline{0})$. Now, place an imaginary charge with charge q at some point $\underline{x}^* \in B_R^C(\underline{0})$. Then, we can think of G as:

$$G(\underline{x}, \underline{y}) = -\frac{1}{4\pi\|\underline{x}-\underline{y}\|} + \underbrace{\frac{q}{4\pi\|\underline{x}^*-\underline{y}\|}}_{-\phi(\underline{x}, \underline{y}?}$$

(here we use 4π are our area elemnet over the sphere)

But then, if G is a Green function, it will vanish on the boundary $\partial B_R(\underline{0})$:

$$G(\underline{x}, \underline{\sigma}) = 0, \qquad \underline{\sigma} \in \partial B_R(\underline{0}), \|\underline{\sigma}\| = R$$

and we can use this to determine $q, \underline{x}^*$.

Since G vanishes when $\underline{y} = \underline{\sigma}$, we will have that:

$$\frac{1}{4\pi\|\underline{x} - \underline{\sigma}\|} = \frac{q}{4\pi\|\underline{x}^* - \underline{\sigma}\|} \implies \|\underline{x}^* - \underline{\sigma}\|^2 = q^2\|\underline{x} - \underline{\sigma}\|^2$$

The strategy is to now put all the $\underline{\sigma}$ on the same side. Indeed:

$$\begin{aligned}
&\|\underline{x}^* - \underline{\sigma}\|^2 = q^2\|\underline{x} - \underline{\sigma}\|^2 \\
\implies &\langle \underline{x}^* - \underline{\sigma}, \underline{x}^* - \underline{\sigma} \rangle = q^2 \langle \underline{x} - \underline{\sigma}, \underline{x} - \underline{\sigma} \rangle \\
\implies &\langle \underline{x}^*, \underline{x}^* \rangle - 2\langle \underline{\sigma}, \underline{x}^* \rangle + \langle \underline{\sigma}, \underline{\sigma} \rangle = q^2 \langle \underline{x}, \underline{x} \rangle - 2q^2 \langle \underline{\sigma}, \underline{x} \rangle + q^2 \langle \underline{\sigma}, \underline{\sigma} \rangle \\
\implies &\langle \underline{x}^*, \underline{x}^* \rangle - q^2 \langle \underline{x}, \underline{x} \rangle + R^2 = 2\langle \underline{\sigma}, \underline{x}^* \rangle - 2q^2 \langle \underline{\sigma}, \underline{x} \rangle + q^2R^2 \\
\implies &\langle \underline{x}^*, \underline{x}^* \rangle - q^2 \langle \underline{x}, \underline{x} \rangle + (1 - q^2)R^2 = 2\left\langle \underline{\sigma}, \underline{x}^* - q^2\underline{x} \right\rangle
\end{aligned}$$

But notice, the RHS depends on $\underline{\sigma}$, whilst the LHS doesn't, and will be fixed. Since $\underline{x}, \underline{\sigma}$ are completely independent, equality holds **if and only if** both sides are equal to 0. In particular, by linearity of the dot product this implies that:

$$\underline{x}^* - q^2\underline{x} = 0 \implies \underline{x}^* = q^2\underline{x}$$

In turn, we then get a quadratic equation in q by considering the LHS:

$$\langle \underline{x}^*, \underline{x}^* \rangle - q^2 \langle \underline{x}, \underline{x} \rangle + (1 - q^2)R^2 = 0 \implies q^4\|\underline{x}\|^2 - q^2(\|\underline{x}\|^2 + R^2) + R^2 = 0$$

Using the quadratic formula:

$$\begin{aligned}
q^2 &= \frac{\|\underline{x}\|^2 + R^2 \pm \sqrt{(\|\underline{x}\|^2 + R^2)^2 - 4\|\underline{x}\|^2R^2}}{2\|\underline{x}\|^2} \\
&= \frac{\|\underline{x}\|^2 + R^2 \pm \sqrt{\|\underline{x}\|^4 - 2\|\underline{x}\|^2R^2 + R^4}}{2\|\underline{x}\|^2} \\
&= \frac{\|\underline{x}\|^2 + R^2 \pm \sqrt{(\|\underline{x}\|^2 - R^2)^2}}{2\|\underline{x}\|^2} \\
&= \frac{\|\underline{x}\|^2 + R^2 \pm (\|\underline{x}\|^2 - R^2)}{2\|\underline{x}\|^2}
\end{aligned}$$

Thus:

$$q^2 = \frac{2\|\underline{x}\|^2}{2\|\underline{x}\|^2} = 1 \implies q = 1$$

$$q^2 = \frac{2R^2}{2\|\underline{x}\|^2} = \frac{R^2}{\|\underline{x}\|^2} \implies q = \frac{R}{\|\underline{x}\|}$$

(notice, we enforce that $q > 0$)

If $q = 1$, we will get that $\phi(\underline{x}, \underline{y}) = \Phi(\underline{x} - \underline{y})$, which is uninteresting, since it corresponds to $G(\underline{x}, \underline{y}) = 0$.

Hence, we have that:

$$q = \frac{R}{\|\underline{x}\|} \qquad \underline{x}^* = \frac{R^2}{\|\underline{x}\|^2}\underline{x}$$

Notice, $\underline{x}$ and $\underline{x}^*$ are **collinear**, and as $x \to R$, $\underline{x}^* \to \underline{x}$.

Hence, we have that:

$$\phi(\underline{x}, \underline{y}) = -\frac{q}{4\pi\|\underline{x}^* - \underline{y}\|} = -\frac{1}{4\pi}\frac{R}{\|\underline{x}\|}\frac{1}{\left\|\frac{R^2}{\|\underline{x}\|^2}\underline{x} - \underline{y}\right\|}$$

Moreover, if we consider what happens as $\underline{x} \to \underline{0}$, and noting that $\frac{\underline{x}}{\|\underline{x}\|}$ will be a unit vector:

$$\phi(\underline{0}, \underline{y}) = -\lim_{\underline{x}\to\underline{0}} \frac{1}{4\pi}\frac{R}{\|\underline{x}\|}\frac{1}{\left\|\frac{R^2}{\|\underline{x}\|^2}\underline{x} - \underline{y}\right\|} = -\lim_{\underline{x}\to\underline{0}} \frac{R}{4\pi}\frac{1}{\left\|R^2\frac{\underline{x}}{\|\underline{x}\|} - \|\underline{x}\|\underline{y}\right\|} = -\frac{R}{4\pi R^2} = -\frac{1}{4\pi R}$$

This then gives us the desired Green Function for $n = 3$:

$$G(\underline{x}, \underline{y}) = \begin{cases} -\frac{1}{4\pi\|\underline{x} - \underline{y}\|} + \frac{R}{4\pi\|\underline{x}\|\left\|\frac{R^2}{\|\underline{x}\|^2}\underline{x} - \underline{y}\right\|}, & \underline{x} \neq \underline{0} \\ -\frac{1}{4\pi\|\underline{y}\|} + \frac{1}{4\pi R}, & \underline{x} = \underline{0} \end{cases}$$

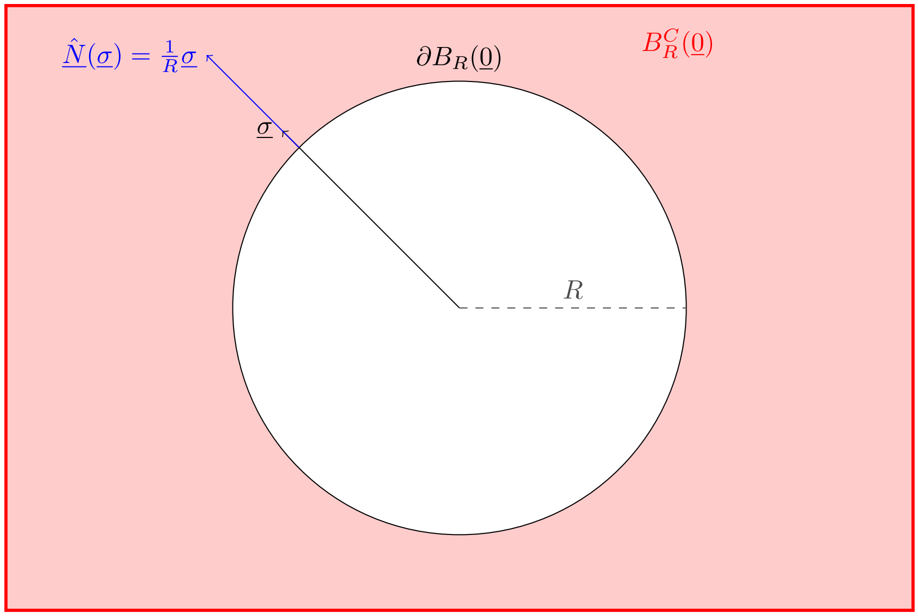
We now compute the gradient (with respect to $\underline{y}$, since with Green Functions we always think of $\underline{x}$ as being fixed):

$$\nabla_{\underline{y}} G(\underline{x}, \underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma})$$

With $n = 2$, and using $\underline{\sigma}$ as our coordinate on the **surface** of the unit ball, we'd have:

$$\underline{\hat{N}}(\underline{\sigma}) = \frac{1}{R}\underline{\sigma}$$

This follows geometrically from:



Hence, now we "just" need to compute $\nabla_{\underline{y}} G(\underline{\sigma})$. To this regard, we consider:

$$G(\underline{x}, \underline{y}) = -\frac{1}{4\pi\|\underline{x} - \underline{y}\|} + \frac{R}{4\pi\|\underline{x}\| \, \|\underline{x}^* - \underline{y}\|}$$

to reduce clutter.

Then:

$$\begin{aligned}
\frac{\partial}{\partial y^i}\left(\frac{1}{\|\underline{x} - \underline{y}\|}\right) &= \frac{\partial}{\partial y^i}\left(\left[\sum_{i=1}^{n}(x^i - y^i)^2\right]^{-1/2}\right) \\
&= -\frac{1}{2}\left[\sum_{j=1}^{n}(x^j - y^j)^2\right]^{-2/2} \frac{\partial}{\partial y^i}\left(\sum_{j=1}^{n}(x^j - y^j)^2\right) \\
&= -\frac{1}{2}\frac{1}{\|\underline{x} - \underline{y}\|^3} 2(x^i - y^i)\frac{\partial}{\partial y^i}\left(x^i - y^i\right) \\
&= \frac{x^i - y^i}{\|\underline{x} - \underline{y}\|^3}
\end{aligned}$$

Thus, it follows that:

$$\nabla_{\underline{y}} G(\underline{x}, \underline{\sigma}) = -\frac{\underline{x} - \underline{\sigma}}{4\pi\|\underline{x} - \underline{\sigma}\|^3} + \frac{R(\underline{x}^* - \underline{\sigma})}{4\pi\|\underline{x}\| \, \|\underline{x}^* - \underline{\sigma}\|^3}$$

Hence, and using the fact that when deriving G, we showed that:

$$\|\underline{x}^* - \underline{\sigma}\|^2 = q^2\|\underline{x} - \underline{\sigma}\|^2 = \frac{R^2}{\|\underline{x}\|^2}\|\underline{x} - \underline{\sigma}\|^2$$

we get that:

$$
\begin{aligned}
\nabla_{\underline{y}} G(\underline{x}, \underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) &= \left\langle -\frac{\underline{x} - \underline{\sigma}}{4\pi \|\underline{x} - \underline{\sigma}\|^3} + \frac{R(\underline{x}^* - \underline{\sigma})}{4\pi \|\underline{x}\| \, \|\underline{x}^* - \underline{\sigma}\|^3}, \frac{1}{R}\underline{\sigma} \right\rangle \\
&= \frac{1}{4\pi R} \left(-\frac{\langle \underline{\sigma}, \underline{x} - \underline{\sigma} \rangle}{\|\underline{x} - \underline{\sigma}\|^3} + \frac{R \langle \underline{\sigma}, \underline{x}^* - \underline{\sigma} \rangle}{\|\underline{x}\| \, \|\underline{x}^* - \underline{\sigma}\|^3} \right) \\
&= \frac{1}{4\pi R} \left(-\frac{\langle \underline{\sigma}, \underline{x} - \underline{\sigma} \rangle}{\|\underline{x} - \underline{\sigma}\|^3} + \frac{R \langle \underline{\sigma}, \underline{x}^* - \underline{\sigma} \rangle}{\|\underline{x}\| \frac{R^3}{\|\underline{x}\|^3} \|\underline{x} - \underline{\sigma}\|^3} \right) \\
&= \frac{1}{4\pi R} \left(-\frac{\langle \underline{\sigma}, \underline{x} - \underline{\sigma} \rangle}{\|\underline{x} - \underline{\sigma}\|^3} + \frac{\|\underline{x}\|^2 \langle \underline{\sigma}, \underline{x}^* - \underline{\sigma} \rangle}{R^2 \|\underline{x} - \underline{\sigma}\|^3} \right) \\
&= -\frac{1}{4\pi R^3 \|\underline{x} - \underline{\sigma}\|^3} \left(R^2 \langle \underline{\sigma}, \underline{x} - \underline{\sigma} \rangle - \|\underline{x}\|^2 \langle \underline{\sigma}, \underline{x}^* - \underline{\sigma} \rangle \right) \\
&= -\frac{1}{4\pi R^3 \|\underline{x} - \underline{\sigma}\|^3} \left(R^2 \langle \underline{\sigma}, \underline{x} - \underline{\sigma} \rangle - \|\underline{x}\|^2 \left\langle \underline{\sigma}, \frac{R^2}{\|\underline{x}\|^2} \underline{x} - \underline{\sigma} \right\rangle \right) \\
&= -\frac{1}{4\pi R \|\underline{x} - \underline{\sigma}\|^3} \left(\langle \underline{\sigma}, \underline{x} - \underline{\sigma} \rangle - \left\langle \underline{\sigma}, \underline{x} - \frac{\|\underline{x}\|^2}{R^2} \underline{\sigma} \right\rangle \right) \\
&= -\frac{1}{4\pi R \|\underline{x} - \underline{\sigma}\|^3} \left(\left\langle \underline{\sigma}, \frac{\|\underline{x}\|^2}{R^2} \underline{\sigma} - \underline{\sigma} \right\rangle \right) \\
&= -\frac{1}{4\pi R \|\underline{x} - \underline{\sigma}\|^3} \left(\|\underline{x}^2\| - R^2 \right) \\
&= \frac{R^2 - \|\underline{x}\|^2}{4\pi R} \frac{1}{\|\underline{x} - \underline{\sigma}\|^3}
\end{aligned}
$$

as required.

□

2.2 Lemma: Green Function for a Ball

*Consider a ball $B_R(\underline{p}) \subset \mathbb{R}^n$ where $n \geq 3$. Then, the **Green Function** on $B_R(\underline{p})$ is:*

$$G(\underline{x},\underline{y}) = \frac{1}{(n-2)\omega_n}\left[\|\underline{x}-\underline{y}\|^{2-n} - \left(\frac{\|\underline{x}-\underline{p}\|}{R}\right)^{2-n}\left\|\frac{R^2}{\|\underline{x}-\underline{p}\|^2}(\underline{x}-\underline{p})-(\underline{y}-\underline{p})\right\|^{2-n}\right]$$

when $\underline{x} \neq \underline{0}$ and:

$$G(\underline{0},\underline{y}) = \frac{1}{(n-2)\omega_n}\left[\|\underline{y}-\underline{p}\|^{2-n} - R^{2-n}\right]$$

when $\underline{x} = \underline{0}$. Furthermore, if $\underline{x} \in B_R(\underline{p})$ and $\underline{\sigma} \in \partial B_R(\underline{p})$ then:

$$\nabla G(\underline{x},\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) = \frac{R^2 - \|\underline{x}-\underline{p}\|^{n-1}}{\omega_n R^{n-2}} \frac{1}{\|\underline{x}-\underline{\sigma}\|^n}$$

In particular, when $n = 3$:

$$G(\underline{x},\underline{y}) = \begin{cases} -\dfrac{1}{4\pi\|\underline{x}-\underline{y}\|} + \dfrac{R}{4\pi\|\underline{x}-\underline{p}\|\left\|\frac{R}{\|\underline{x}-\underline{p}\|}(\underline{x}-\underline{p})-(\underline{y}-\underline{p})\right\|}, & \underline{x} \neq 0 \\ -\dfrac{1}{4\pi\|\underline{y}-\underline{p}\|} + \dfrac{1}{4\pi R}, & \underline{x} = \underline{0} \end{cases}$$

$$\nabla G(\underline{x},\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) = \frac{R^2 - \|\underline{x}-\underline{p}\|^2}{4\pi R} \frac{1}{\|\underline{x}-\underline{\sigma}\|^3}$$

This is immediate by using $\underline{x} \mapsto \underline{x}-\underline{p}$ and $\underline{y} \mapsto \underline{y}-\underline{p}$.

2.3 Theorem: Poisson's Formula

Let $B_R(\underline{p}) \subset \mathbb{R}^n$ be a ball of radius R centered at $\underline{p} \in \mathbb{R}^n$ and let $\underline{x} \in \mathbb{R}^n$. Let:

$$g \in C(\partial B_R(\underline{p}))$$

Then, the ***unique solution****:*

$$u \in C^2(B_R(\underline{p})) \cap C(\bar{B}_R(\underline{p}))$$

of the PDE:

$$\begin{cases} \Delta u(\underline{x}) = 0, & \underline{x} \in B_R(\underline{p}) \\ u(\underline{x}) = g(\underline{x}), & \underline{x} \in \partial B_R(\underline{p}) \end{cases}$$

can be represented using the ***Poisson formula****:*

$$u(\underline{x}) = \frac{R^2 - \|\underline{x} - \underline{p}\|^2}{\omega_n R} \int_{\partial B_R(\underline{p})} \frac{g(\underline{\sigma})}{\|\underline{x} - \underline{\sigma}\|^n} d\underline{\sigma}$$

In particular, when $n = 3$:

$$u(\underline{x}) = \frac{R^2 - \|\underline{x} - \underline{p}\|^2}{4\pi R} \int_{\partial B_R(\underline{p})} \frac{g(\underline{\sigma})}{\|\underline{x} - \underline{\sigma}\|^3} d\underline{\sigma}$$

Proof. This follows immediately by using the **Representation Formula** for solutions to the boundary value Poisson problem:

$$u(\underline{x}) = \int_\Omega f(\underline{y}) G(\underline{x}, \underline{y}) d^n y + \int_{\partial\Omega} g(\underline{\sigma}) \underbrace{(\nabla G(\underline{x}, \underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}))}_{Poisson\ kernel} d\underline{\sigma}$$

alongside the Green function we just derived (we work for $n = 3$):

$$G(\underline{x}, \underline{y}) = -\frac{1}{4\pi\|\underline{x} - \underline{y}\|} + \frac{R}{4\pi\|\underline{x}\| \left\| \frac{R^2}{\|\underline{x}\|^2}\underline{x} - \underline{y} \right\|}$$

and the normal gradient (known as Poisson kernel):

$$\nabla G(\underline{x}, \underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) = \frac{R^2 - \|\underline{x} - \underline{p}\|^2}{4\pi R} \frac{1}{\|\underline{x} - \underline{\sigma}\|^3}$$

Hence, plugging it all in, and using the fact that on Ω, $\Delta u = f = 0$ so:

$$u(\underline{x}) = \int_{\partial\Omega} g(\underline{\sigma}) \frac{R^2 - \|\underline{x} - \underline{p}\|^2}{4\pi R} \frac{1}{\|\underline{x} - \underline{\sigma}\|^3} d\underline{\sigma} = \frac{R^2 - \|\underline{x} - \underline{p}\|^2}{4\pi R} \int_{\partial B_R(\underline{p})} \frac{g(\underline{\sigma})}{\|\underline{x} - \underline{\sigma}\|^3} d\underline{\sigma}$$

□

3 Harnack's Inequality

3.1 Theorem: Harnack's Inequality

Let $B_R(\underline{0}) \subset \mathbb{R}^n$ be the ball of radius R centered at the origin, and let:

$$u \in C^2(B_R(\underline{p})) \cap C(\bar{B}_R(\underline{p}))$$

*be the **unique solution** of the PDE:*

$$\begin{cases} \Delta u(\underline{x}) = 0, & \underline{x} \in B_R(\underline{p}) \\ u(\underline{x}) = g(\underline{x}), & \underline{x} \in \partial B_R(\underline{p}) \end{cases}$$

*Assume that u is **non-negative** on $\bar{B}_R(\underline{0})$. Then, for any $\underline{x} \in B_R(\underline{0})$, we have that:*

$$\frac{R^{n-2}(R - \|\underline{x}\|)}{(R + \|\underline{x}\|)^{n-1}} u(\underline{0}) \leq u(\underline{x}) \leq \frac{R^{n-2}(R + \|\underline{x}\|)}{(R - \|\underline{x}\|)^{n-1}} u(\underline{0})$$

Proof. We prove this for $n = 3$.

By **Poisson's Formula**, we have that:

$$u(\underline{x}) = \frac{R^2 - \|\underline{x}\|^2}{4\pi R} \int_{\partial B_R(\underline{0})} \frac{g(\underline{\sigma})}{\|\underline{x} - \underline{\sigma}\|^3} d\underline{\sigma}$$

On the surface of our ball ($\underline{\sigma} \in \partial B_R(\underline{0})$) we will have that, by the Triangle Inequality:

$$\|\underline{x} - \underline{\sigma}\| \leq \|\underline{x}\| + \|\underline{\sigma}\|$$

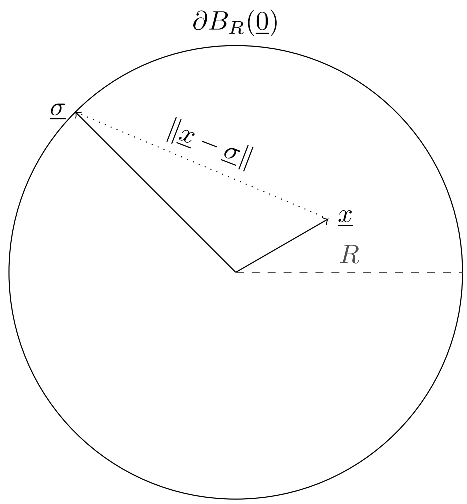
and by the Reverse Triangle Inequality:

$$\|\underline{x} - \underline{\sigma}\| \geq |\|\underline{x}\| - \|\underline{\sigma}\||$$

so

$$R - \|\underline{x}\| \leq \|\underline{x} - \underline{\sigma}\| \leq \|\underline{x}\| + R$$

where we have used that $\|\underline{\sigma}\| = R$, and that $R \geq \|\underline{x}\|$ always.



Hence, if we use $R - \|\underline{x}\| \leq \|\underline{x} - \underline{\sigma}\|$ alongside the fact that g is non-negative by assumption, we get that:

$$\begin{aligned}
u(\underline{x}) &\leq \frac{R^2 - \|\underline{x}\|^2}{4\pi R} \int_{\partial B_R(\underline{0})} \frac{g(\underline{\sigma})}{(R - \|\underline{x}\|)^3} d\underline{\sigma} \\
&= \frac{(R + \|\underline{x}\|)(R - \|\underline{x}\|)}{4\pi R} \int_{\partial B_R(\underline{0})} \frac{g(\underline{\sigma})}{(R - \|\underline{x}\|)^3} d\underline{\sigma} \\
&= \frac{R + \|\underline{x}\|}{4\pi R(R - \|\underline{x}\|)^2} \int_{\partial B_R(\underline{0})} g(\underline{\sigma}) d\underline{\sigma}
\end{aligned}$$

But now, since u solves the Poisson PDE, it is Harmonic, and so, by the **mean value property**:

$$u(\underline{0}) = \frac{1}{4\pi R^2} \int_{\partial B_R(\underline{0})} g(\underline{\sigma}) d\underline{\sigma}$$

Hence:

$$\begin{aligned}
u(\underline{x}) &\leq \frac{R + \|\underline{x}\|}{4\pi R(R - \|\underline{x}\|)^2} \int_{\partial B_R(\underline{0})} g(\underline{\sigma}) d\underline{\sigma} \\
&= \frac{R(R + \|\underline{x}\|)}{(R - \|\underline{x}\|)^2} \frac{1}{4\pi R^2} \int_{\partial B_R(\underline{0})} g(\underline{\sigma}) d\underline{\sigma} \\
&= \frac{R(R + \|\underline{x}\|)}{(R - \|\underline{x}\|)^2} u(\underline{0})
\end{aligned}$$

as required.

The other inequality follows identically by using $\|\underline{x} - \underline{\sigma}\| \leq \|\underline{x}\| + R$ instead.

□

3.2 Corollary: Liouville's Theorem

*As a corollary of **Harnack's Inequality**, we get **Liouville's Theorem**, which is what we need to prove that the solution to the Poisson equation is indeed unique.*

---

*Suppose that $u \in C^2(\mathbb{R}^n)$ is **harmonic** on $\mathbb{R}^n$.*
Assume that:

$$\exists M \in \mathbb{R} \ : \forall \underline{x} \in \mathbb{R}^n, \ u(\underline{x}) \geq M \text{ or } u(\underline{x}) \leq M$$

*Then, u is a **constant-valued function**.*

Proof. We begin with the case $u(\underline{x}) \geq M$. Define:

$$v = u + |M|$$

Clearly:

$$\Delta v = \Delta u + \Delta |M| = 0$$

so v is harmonic. Moreover, by adding $|M|$ we ensure that $v(\underline{x}) \geq 0$ for any x. Hence, Harnack's Inequality applies, from which we get:

$$\frac{R^{n-2}(R - \|\underline{x}\|)}{(R + \|\underline{x}\|)^{n-1}} v(\underline{0}) \leq v(\underline{x}) \leq \frac{R^{n-2}(R + \|\underline{x}\|)}{(R - \|\underline{x}\|)^{n-1}} v(\underline{0})$$

But then, as $R \to \infty$, and using the Squeeze Theorem, we conclude that:

$$v(\underline{x}) = v(\underline{0})$$

so v is constant, and thus, u must be constant.

If $u(\underline{x}) \leq M$, then use $w(\underline{x}) = -u(\underline{x}) + |M|$ instead of v, and argue in the same way.

□