Introduction to Partial Differential Equations - Week 5 & 6 - The Fundamental Solution to Laplace's/Poisson's Equations

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Contents

1 Laplace's and Poisson's Equations

1.1 Definition: The Laplacian Operator

*The **Laplacian** is a:*

- ***second-order***
- ***linear***
- ***constant-coefficient***

differential operator, defined as:

$$\nabla^2 = \Delta = \sum_{i=1}^{n} \partial_i^2$$

1.2 Definition: Laplace's Equation

*Let $\Omega \subset \mathbb{R}^n$ be a **domain** (open connected subset).*
*The **Laplace Equation** on Ω is the **homogeneous** PDE:*

$$\Delta u(\underline{x}) = 0, \qquad \underline{x} \in \Omega$$

1.3 Definition: Poisson's Equation

*Let $\Omega \subset \mathbb{R}^n$ be a **domain** (open connected subset).*
***Poisson's Equation** on Ω is the **inhomogeneous** PDE:*

$$\Delta u(\underline{x}) = f(\underline{x}), \qquad \underline{x} \in \Omega$$

- **What is a harmonic function?**
  - a function $u \in C^2(\Omega)$ satisfying **Laplace's Equation**:

$$\Delta u = 0$$

1.4 Example: Electromagnetism and Poisson's Equation

Maxwell's Equations define electromagnetic behaviour.
We consider:

- $\underline{E}$ *(electric field)*
- $\underline{B}$ *(magnetic induction)*
- $\underline{J}$ *(current density)*
- ρ *(charge density)*

Maxwell's Equations are:

$$\partial_t \underline{E} - \nabla \times \underline{B} = -\underline{J}$$

$$\nabla \cdot \underline{E} = \rho$$

$$\partial_t \underline{B} + \nabla \times \underline{E} = \underline{0}$$

$$\nabla \cdot \underline{B} = 0$$

We can think of **Laplace's Equation** as the heat equation, when u is time-independent (known as a **steady state solution**).

However, it is more interesting to think about it from the physical point of view of electromagnetism. Again, let's consider steady-state solutions, such that:

$$\partial_t \underline{E} = \partial_t \underline{B} = \underline{J} = \underline{0}$$

This tells us that:

$$\nabla \times \underline{E} = \underline{0}$$

Since Ω is a **domain**, in particular it is a connected set, so by Poincaré's Lemma, $\underline{E}$ must be a **conservative** vector field. That is, $\exists \phi$ such that:

$$\underline{E} = -\nabla \phi$$

We call ϕ the **electric potential**. This then tells us that:

$$\nabla \cdot \underline{E} = \rho \implies \nabla \left(-\nabla \cdot \phi \right) = -\Delta \phi = \rho$$

That is, the electric potential must be a solution to **Poisson's Equation**, with inhomogeneous term $-\rho$.

1.5 Example: Complex Analysis and Laplace's Equation

A complex function:

$$f(z) = u(z) + iv(z)$$

*is **differentiable** at:*

$$z_0 = z_0 + iy_0$$

***if and only if** u, v verify the **Cauchy-Riemann Equations** at z_0:*

$$u_x(x_0, y_0) = v_y(x_0, y_0) \qquad u_y(x_0, y_0) = -v_x(x_0, y_0)$$

If we differentiate each of the equations:

$$u_{xx}(x_0, y_0) = v_{yx}(x_0, y_0) \quad u_{yy}(x_0, y_0) = -v_{xy}(x_0, y_0)$$
$$u_{xy}(x_0, y_0) = v_{yy}(x_0, y_0) \quad u_{yx}(x_0, y_0) = -v_{xx}(x_0, y_0)$$

Then, assumign that $u, v \in C^2$ near z_0, we get that:

$$\Delta u = v_{yx} - v_{xy} = 0$$

$$\Delta v = -u_{yx} + u_{xy} = 0$$

That is, a **differentiable** complex function must be composed of **harmonic** real and imaginary parts!

2 Properties of Harmonic Functions

2.1 Boundary Conditions and Well-Posed Problems

- **Does Poisson's Equation require an initial condition?**
  - no, since it doesn't depend on **time**
  - we only need to prescribe **boundary** conditions
- **Which boundary conditions produce a well-posed problem?**
  - consider a **domain** $\Omega \subset \mathbb{R}^n$, with a **Lipschitz Boundary** (that is, $\partial\Omega$ is "locally regular", it can be described piecewise by regular functions)
  - the following produce **well-posed problems** foor $\Delta u = f$:
    1. **Dirichlet Data**:
    $$u(\underline{x}) = h(x), \qquad \forall \underline{x} \in \partial\Omega$$
    2. **Neumann Data**:
    $$\nabla u(\underline{x}) \cdot \underline{\hat{N}} = h(x), \qquad \forall \underline{x} \in \partial\Omega$$
    where $\underline{\hat{N}}$ denotes the unit outward normal vector to $\partial\Omega$
    3. **Robin-Type Data**:
    $$\nabla u(\underline{x}) \cdot \underline{\hat{N}} + \alpha u(\underline{x}) = h(x),\ \alpha > 0 \qquad \forall \underline{x} \in \partial\Omega$$
    4. **Mixed Conditions**: such as those arising by splitting $\partial\Omega$ into disjoint pieces:
    $$\partial\Omega = S_D \cup S_N$$
    and requiring that u satisfies some of the above conditions for $h(\underline{x})$ defined on S_D, and $g(\underline{x})$ defined on S_N
    5. **Conditions at Infinity**: if $\Omega = \mathbb{R}^n$ we can specify that $u(\underline{x})$ satisfies **asymptotic conditions** as $\|\underline{x}\| \to \infty$

2.2 Theorem: Uniqueness of Solutions to Poisson's Equation

Let $\Omega \subset R^n$ be a ***smooth, bounded*** *domain.*
Then, under ***Dirichlet, Robin*** *or* ***mixed boundary conditions****, there is* ***at most one*** *solution of regularity:*

$$u \in C^2(\Omega) \cap C^1(\bar{\Omega})$$

to the ***Poisson Equation****:*

$$\Delta u = f$$

In the case of ***Neumann*** *conditions, any 2 solution can differ by* ***at most*** *a constant.*

Proof. Consider 2 solutions u, v satisfying the Poisson Equation:

$$\Delta u = f \qquad \Delta v = f$$

Then, since this is a linear PDE $w = u - v$ is a solution to:

$$\Delta w = f - f = 0$$

Now, we apply the **Energy Method**. Define the energy to be:

$$E = \int_\Omega w^2 d^n x$$

Now, if we multiply the Lapalce Equation above by w, we obtain:

$$w \Delta w = 0$$

So integrating:

$$\int_\Omega w \Delta w d^n x = 0 \implies \int_\Omega w \Delta w + \|\nabla w\|^2 - \|\nabla w\|^2 d^n x = 0$$

But now notice that by the product rule, and using $\nabla \cdot \nabla w = \Delta w$:

$$\nabla \cdot (w \nabla w) = (\nabla w) \cdot (\nabla w) + w(\nabla \cdot \nabla w) = \|\nabla w\|^2 + w \Delta w$$

Hence, we can rewrite our integral as:

$$\int_\Omega \nabla \cdot (w \nabla w) d^n x - \int_\Omega \|\nabla w\|^2 d^n x = 0$$

It is now natural to apply the **Divergence Theorem**:

$$\int_{\partial\Omega} \underline{\hat{N}} \cdot (w \nabla w) d\underline{\sigma} - \int_\Omega \|\nabla w\|^2 d^n x = 0$$

where $\underline{\hat{N}}$ is the unit normal vector to the surface $\partial\Omega$

① **Dirichlet Data**

If u, v satisfy the Dirichlet Data, then it follows that:

$$\forall \underline{x} \in \partial\Omega, \quad u(\underline{x}) = v(\underline{x}) = g(\underline{x})$$

where g is some function, Thus, it follows that:

$$\forall \underline{x} \in \partial\Omega, \quad w(\underline{x}) = g - g = 0$$

Hence, the first surface integral vanishes, and we have:

$$\int_\Omega \|\nabla w\|^2 d^n x = 0$$

The fact that $\|\nabla w\|^2$ is continuous and non-negative implies (by results from Analysis - we did this as a homework) that:

$$\|\nabla w\|^2 = 0 \iff \nabla w = \underline{0}$$

in Ω. In other words, w will be constant on $\bar{\Omega}$. As we saw above, $w = 0$ on $\partial\Omega$, so $w = 0$ on all of $\bar{\Omega}$. Finally, this then implies that $u = v$, as required.

② **Robin Data**

If u, v satisfy the Robin Data, then it follows that:

$$\forall \underline{x} \in \partial\Omega, \quad \nabla u(\underline{x}) \cdot \underline{\hat{N}} + \alpha u(\underline{x}) = h(x), \;\; \alpha > 0$$
$$\forall \underline{x} \in \partial\Omega, \quad \nabla v(\underline{x}) \cdot \underline{\hat{N}} + \alpha v(\underline{x}) = h(x), \;\; \alpha > 0$$

So it follows that:

$$\forall \underline{x} \in \partial\Omega, \quad \nabla w(\underline{x}) \cdot \underline{\hat{N}} + \alpha w(\underline{x}) = h - h = 0, \;\; \alpha > 0$$

In other words, on the surface $\partial\Omega$ we have:

$$\nabla w(\underline{x}) \cdot \underline{\hat{N}} + \alpha w(\underline{x}) = 0 \implies w \nabla w(\underline{x}) \cdot \underline{\hat{N}} = -\alpha w^2(\underline{x})$$

Hence, our integral becomes:

$$\int_{\partial\Omega} -\alpha w^2(\underline{x}) d\underline{\sigma} - \int_\Omega \|\nabla w\|^2 d^n x = 0$$

But the fact that $\alpha w^2(\underline{x}), \|\nabla w\|^2 \geq 0$ imply as before that in particular:

$$\nabla w = 0$$

and the result follows.

③ **Neumann Data**

If u, v satisfy the Neumann Data, then it follows that:

$$\forall \underline{x} \in \partial\Omega, \quad \nabla u(\underline{x}) \cdot \underline{\hat{N}} = \nabla v(\underline{x}) \cdot \underline{\hat{N}} = g(\underline{x})$$

so we must have:

$$\forall \underline{x} \in \partial\Omega, \quad \nabla w(\underline{x}) \cdot \underline{\hat{N}} = g - g = 0$$

and our integral becomes:

$$\int_\Omega \|\nabla w\|^2 d^n x = 0$$

which again implies that:

$$\nabla w = 0$$

so w must be constant. However, since this time we only know that on the surface $w(\underline{x}) \cdot \underline{\hat{N}} = 0$, this is all we can say. Hence, if u, v satisfy Neumann conditions, any 2 solutions will differ at most by a constant, as required.

□

2.3 Theorem: Mean Value Properties of Harmonic Functions

Let $u(x)$ be harmonic in the domain $\Omega \subset \mathbb{R}^n$, and let:

$$B_R(\underline{x}) \subset \Omega$$

be a ball of radius R centered at $\underline{x} \in \mathbb{R}^n$. Then the following mean value formulae hold:

$$u(\underline{x}) = \frac{n}{\omega_n R^n} \int_{B_R(\underline{x})} u(\underline{y}) d^n y$$

$$u(\underline{x}) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(\underline{x})} u(\underline{\sigma}) d\underline{\sigma}$$

*where ω_n is the **surface area** of the unit ball centered at $\underline{0} \in \mathbb{R}^n$.*

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1. *The **surface area** of $B_1(\underline{0})$ in $\mathbb{R}^n$ is given by:*

$$\omega_n = \begin{cases} 2, & n = 0 \\ 2\pi, & n = 1 \\ \frac{2\pi}{n-1}\omega_{n-2}, & n > 1 \end{cases}$$

2. *Alternatively, it can be defined in terms of the **volume** V_n of the **unit sphere** in $\mathbb{R}^n$:*

$$\omega_n = (n+1)V_{n+1}$$

where we can define:

$$V_n = \begin{cases} 1, & n = 0 \\ 2\pi, & n = 1 \\ \frac{2\pi}{n}V_{n-2}, & n > 1 \end{cases}$$

3. *More generally:*

$$|B_R(\underline{x})| = \frac{\omega_n R^n}{n}$$

where $|B_R(\underline{x})|$ is the volume of $B_R(\underline{x})$. Similarly:

$$|\partial B_R(\underline{x})| = \omega_n R^{n-1}$$

where $|\partial B_R(\underline{x})|$ is the surface area of $B_R(\underline{x})$.

Proof. We consider the case $n = 2$; similar reasoning will give the higher dimensional cases. Moreover, we shall prove the claim when the ball is centered at the origin.

Define a ball $B_r(\underline{x})$ in $\mathbb{R}^n$ of radius r centered at $\underline{x}$. Consider the function

$$g(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(\underline{x})} u(\underline{\sigma}) d\underline{\sigma}$$

where $\sigma = \underline{x} + r\underline{\omega}$, and $\underline{\omega}$ is an angular coordinate in the surface of the unit ball in $\mathbb{R}^n$.

Now, since u is continuous, we can apply the **Mean Value Theorem** (Integral Version), which tells us that:

$$\lim_{r \to 0^+} g(r) = \lim_{r \to 0^+} \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(\underline{x})} u(\underline{\sigma}) d\underline{\sigma} = u(\underline{x})$$

The one-dimensional intuition is that if we have:

$$\frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} g(y) dy$$

By the Mean Value Theorem, we in fact have that $\exists c^* \in [x - \varepsilon, x + \varepsilon]$ *such that:*

$$\frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} g(y) dy = g(c^*)$$

But then, as $\varepsilon \to 0$*, the only possibility for* c^* *is* $c^* = x$*, so:*

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} g(y) dy = g(x)$$

This is analogous to the case above, albeit with $r \to 0^+$.

Hence, if we can show that $g'(r) = 0$, then g is constant, and since $\lim_{r \to 0^+} g(r) = u(\underline{x})$, we will have that $g(r) = u(\underline{x})$ for any r, which gives us the second mean value formula.

To this end, we compute $g'(r)$. By using the change of variables $\underline{\sigma} = \underline{x} + r\underline{\omega}$, which gives:

$$d\underline{\sigma} = r^{n-1} d\underline{\omega}$$

we have that:

$$g'(r) = \frac{\partial}{\partial r} \left(\frac{1}{\omega_n} \int_{\partial B_1(\underline{x})} u(\underline{x} + r\underline{\omega}) d\underline{\omega} \right) = \frac{1}{\omega_n} \int_{\partial B_1(\underline{x})} \partial_r u(\underline{x} + r\underline{\omega}) d\underline{\omega}$$

This allows us to integrate over the unit ball centered at $\underline{x}$. Moreover, notice that $\partial_r u(\underline{\sigma})$ will be the gradient vector of u dotted with $\underline{\omega}$ (since a partial derivative is nothing but a directional derivative in the direction of one of the axes, and $\underline{\omega}$ always points in the direction of the radial variable r). Hence:

$$g'(r) = \frac{1}{\omega_n} \int_{\partial B_1(\underline{x})} (\nabla u(\underline{x} + r\underline{\omega}) \cdot \underline{\hat{N}}) d\underline{\omega}$$

since $\underline{\omega}$ is a unit normal vector to the ball by construction. But now, the **Divergence Theorem** applies, and so we can write:

$$g'(r) = \frac{1}{\omega_n} \int_{B_1(\underline{x})} \Delta u(\underline{x} + r\underline{\omega}) d\underline{\omega}$$

However, u is Harmonic, so $\Delta u = 0$, and as required:

$$g'(r) = 0$$

so as required:

$$u(\underline{x}) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(\underline{x})} u(\underline{\sigma}) d\underline{\sigma}$$

Now we consider the case where we integrate over the ball. We have that:

$$u(\underline{x})\omega_n r^{n-1} = \int_{\partial B_r(\underline{x})} u(\underline{\sigma}) d\underline{\sigma}$$

But if we integrate with respect to r:

$$u(\underline{x})\frac{\omega_n r^n}{n} = \int_{B_r(\underline{x})} u(\underline{\sigma}) d\underline{\sigma} \implies u(\underline{x}) = \frac{n}{\omega_n r^n} \int_{B_r(\underline{x})} u(\underline{\sigma}) d\underline{\sigma}$$

as required.

□

2.3.1 Intuition About Mean Value Theorems

*The **mean value theorems** tell us that the the value of **harmonic functions** at a point is defined by **mean** value of the function for all points of in a sphere (or its surface) which is centered at the point.*

2.4 Theorem: Strong Maximum Principle

*Let $\Omega \subset \mathbb{R}^n$ be a **domain**, and assume that $u \in C(\Omega)$ satisfies the **mean value property**:*

$$u(\underline{x}) = \frac{n}{w_n R^n} \int_{B_R(\underline{x})} u(\underline{y}) d^n y$$

Then:

- *if $\underline{p} \in \Omega$ is an extremum of u, then u is constant on Ω*
- *otherwise, if Ω is **bounded** and $u \in C(\bar{\Omega})$ is **not** constant, then we must have that:*

$$\forall \underline{x} \in \Omega, \qquad u(x) < \max_{y \in \partial\Omega} u(y) \qquad u(x) > \min_{y \in \partial\Omega} u(y)$$

Proof. We argue for the minimum case when $n = 2$.

Assume that $\exists \underline{p} \in \Omega$ such that $u(\underline{p}) = m$ is a **minimum**. Define $B(\underline{p})$ as **any** ball centered at $\underline{p}$. Moreover, consider a smaller ball, $B_r(\underline{z}) \subset B(\underline{p})$. Since m is a minimum, in particular we have that:

$$u(\underline{z}) \geq m$$

Now, since u satisfies the mean value property, we have:

$$\begin{aligned}
m &= u(\underline{p}) \\
&= \frac{1}{|B(p)|} \int_{B(p)} u(\underline{y}) d^2 y \\
&= \frac{1}{|B(p)|} \left[\int_{B_r(\underline{z})} u(\underline{y}) d^2 y + \int_{B(\underline{p}) \setminus B_r(\underline{z})} u(\underline{y}) d^2 y \right] \\
&= \frac{1}{|B(p)|} \left[|B_r(\underline{z})| u(\underline{z}) + \int_{B(\underline{p}) \setminus B_r(\underline{z})} u(\underline{y}) d^2 y \right], \qquad \textit{since by MVP } u(z) = \frac{1}{|B_r(\underline{z})|} \int_{B_r(\underline{z})} u(\underline{y}) d^2 y \\
&\geq \frac{1}{|B(p)|} \left[|B_r(\underline{z})| u(\underline{z}) + \int_{B(\underline{p}) \setminus B_r(\underline{z})} m d^2 y \right] \\
&= \frac{1}{|B(p)|} \left[|B_r(\underline{z})| u(\underline{z}) + m(|B(p)| - |B_r(\underline{z})|) \right] \\
&= \frac{|B_r(\underline{z})|}{|B(p)|} u(\underline{z}) + m - \frac{|B_r(\underline{z})|}{|B(p)|} m
\end{aligned}$$

But this implies that:

$$\frac{|B_r(\underline{z})|}{|B(p)|} (u(\underline{z}) - m) \leq 0 \iff u(\underline{z}) \leq m$$

Hence, since by definition of m we must also have $u(\underline{z}) \geq m$, we conclude that:

$$u(\underline{z}) = m$$

However, $\underline{z}$ was arbitrary, so this holds $\forall x \in B(\underline{p})$. Moreover, since Ω is a domain, it is **open** and **connected**, so:

$$\forall \underline{x} \in \Omega, \qquad u(\underline{x}) = m$$

and so, u is constant if it attains a minimum on Ω.

$\square$

2.4.1 Corollary: Comparison Principle

Let $\Omega \subset \mathbb{R}^n$ be a ***bounded domain****, and let $f \in C(\partial\Omega)$. Then, the PDE:*

$$\begin{cases} \Delta u = 0, & \underline{x} \in \Omega \\ u(\underline{x}) = f(\underline{x}), & x \in \partial\Omega \end{cases}$$

has ***at most*** *one solution:*

$$u_f \in C^2(\Omega) \cap C(\bar{\Omega})$$

If u_f and u_g are the solutions to $f, g \in C(\partial\Omega)$, then:

$$\forall \underline{x} \in \partial\Omega, \quad f \geq g, f \neq g \implies \forall \underline{x} \in \Omega, \quad u_f > u_g$$

Proof. Define $w = u_f - u_g$. By linearity, and since $f \geq g$, w solves:

$$\begin{cases} \Delta w = 0, & \underline{x} \in \Omega \\ w(\underline{x}) = f - g \geq 0, & x \in \partial\Omega \end{cases}$$

Since w is harmonic, the Strong Maximum Principle applies, so either $w > 0$ is constant (a **positive constant**, since $f \neq g$) or w is non-constant, and so, attains a minimum on $\partial\Omega$:

$$\forall \underline{x} \in \Omega, \quad w(\underline{x}) > \min_{\underline{y} \in \partial\Omega} w(\underline{y}) = \min_{\underline{y} \in \partial\Omega} f(\underline{y}) - g(\underline{y}) \geq 0$$

Hence, no matter if w is a positive constant or non-constant on Ω, we have:

$$\forall \underline{x} \in \Omega, \quad w(\underline{x}) > 0 \implies u_f(\underline{x}) > u_g(\underline{x})$$

as required.

□

2.4.2 Corollary: Stability Estimate

Let $\Omega \subset \mathbb{R}^n$ be a ***bounded domain****, and let $f \in C(\partial\Omega)$. Then, the PDE:*

$$\begin{cases} \Delta u = 0, & \underline{x} \in \Omega \\ u(\underline{x}) = f(\underline{x}), & x \in \partial\Omega \end{cases}$$

has ***at most*** *one solution:*

$$u_f \in C^2(\Omega) \cap C(\bar{\Omega})$$

If u_f and u_g are the solutions to $f, g \in C(\partial\Omega)$, then:

$$\forall \underline{x} \in \Omega, \qquad |u_f(x) - u_g(x)| \leq \max_{\underline{y} \in \partial\Omega} |f(\underline{y}) - g(\underline{y})|$$

Proof. We perform the same argument as for the Comparison Principle, with $\pm w$, which gives us:

$$w(\underline{x}) > \min_{\underline{y} \in \partial\Omega} f(\underline{y}) - g(\underline{y}) > -\max_{\underline{y} \in \partial\Omega} |f(\underline{y}) - g(\underline{y})|$$

$$-w(\underline{x}) > \min_{\underline{y} \in \partial\Omega} - f(\underline{y}) + g(\underline{y}) > -\max_{\underline{y} \in \partial\Omega} |f(\underline{y}) - g(\underline{y})|$$

so

$$-w < \max_{\underline{y} \in \partial\Omega} |f(\underline{y}) - g(\underline{y})|$$

$$w < \max_{\underline{y} \in \partial\Omega} |f(\underline{y}) - g(\underline{y})|$$

so combining them:

$$|w| = |u_f - u_g| \leq \max_{\underline{y} \in \partial\Omega} |f(\underline{y}) - g(\underline{y})|$$

as required.

□

3 The Fundamental Solution to Poisson's Equation

The following illustrate the intuition and derivation for the Fundamental solution:

- by Li Chen
- by R.E Hunt

3.1 Intuition About the Fundamental Solution

We want to construct a fundamental solution which behaves like the ***Dirac-delta distribution****. As we saw with the* ***fundamental solution*** *to the* ***heat equation****, this gave us the property that through* ***convolution****, we could solve the inhomogeneous problem (in our case,* ***Poisson's Equation****). In particular, we shall see that the* ***fundamental solution*** Φ *satisfies:*

$$\Phi(\underline{x}) = \delta(\underline{x})$$

Moreover, we notice that unlike with the ***Heat Equation****,* ***Poisson's Equation*** *is* ***time independent****. Hence, instead of imposing initial conditions, we should impose some form of "decay" condition, such as:*

$$\lim_{\|\underline{x}\| \to \infty} |u(\underline{x})| = 0$$

This idea can be physically motivated by considering gravity or electromagnetism: away from a body, the ***potential*** *of the body (i.e gravitational potential, electric potential), should be* ***negligible****.*

3.2 Definition: Fundamental Solution for Laplace's Equation

The ***fundamental solution*** Φ *corresponding to the* ***Laplacian operator*** Δ *is:*

$$\Phi(\underline{x}) = \begin{cases} \frac{1}{2\pi} \ln \|\underline{x}\|, & n = 2 \\ -\frac{1}{\omega_n \|\underline{x}\|^{n-2}}, & n \geq 3 \end{cases}$$

where:

- $\|\underline{x}\|$ *is the standard vector norm*
- ω_n *is the surface area of the unit sphere in* $\mathbb{R}^n$

3.3 Lemma: Fundamental Solution at $\underline{x} = 0$

If $\underline{x} \neq \underline{0}$*, then* $\Delta\Phi(\underline{x}) = 0$.

Proof. Consider the case $n = 3$:

$$\Phi(\underline{x}) = -\frac{1}{4\pi\|\underline{x}\|}$$

Since $\underline{x} \neq \underline{0}$, we exploit the **radial symmetry** of Φ and can define a change of variables:

$$r^2 = \|\underline{x}\|^2 = x^2 + y^2 + z^2$$

Then:

$$2r\frac{\partial r}{\partial x^i} = 2x \implies \left(\frac{\partial r}{\partial x^i}\right)^2 + r\frac{\partial^2 r}{\partial (x^i)^2} = 1$$

and:

$$\frac{\partial \Phi}{\partial x^i} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial x^i}$$

$$\frac{\partial^2 \Phi}{\partial (x^i)^2} = \frac{\partial^2 \Phi}{\partial r^2}\left(\frac{\partial x^i}{\partial r}\right)^2 + \frac{\partial \Phi}{\partial r}\frac{\partial^2 r}{\partial (x^i)^2} = \frac{\partial^2 \Phi}{\partial r^2}\left(\frac{x^i}{r}\right)^2 + \frac{\partial \Phi}{\partial r}\frac{1}{r}\left(1 - \left(\frac{x_i}{r}\right)^2\right)$$

Hence:

$$\begin{aligned}\Delta \Phi &= \sum_{i=1}^{3} \frac{\partial^2 \Phi}{\partial r^2}\left(\frac{x_i}{r}\right)^2 + \frac{\partial \Phi}{\partial r}\frac{1}{r}\left(1 - \left(\frac{x_i}{r}\right)^2\right) \\ &= \frac{\partial^2 \Phi}{\partial r^2} + \frac{\partial \Phi}{\partial r}\frac{1}{r}(3-1) \\ &= \frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r}\frac{\partial \Phi}{\partial r}\end{aligned}$$

Thus, the Laplace Equation can be expressed via:

$$\frac{\partial \Phi}{\partial r} = \frac{1}{4\pi r^2} \qquad \frac{\partial^2 \Phi}{\partial r^2} = -\frac{1}{2\pi r^3}$$

Hence:

$$\Delta \Phi = -\frac{1}{2\pi r^3} + \frac{2}{r}\frac{1}{4\pi r^2} = 0$$

as expected.

□

3.4 Theorem: Solution to Poisson's Equation

Let:

$$f(\underline{x}) \in C_0^\infty(\mathbb{R}^n)$$

Then:

- *if* $\boldsymbol{n \geq 3}$*:*

$$\Delta u(\underline{x}) = f(\underline{x})$$

*has a **unique, smooth** solution satisfying:*

$$\lim_{\|\underline{x}\|\to\infty} |u(\underline{x})| = 0$$

- *if* $\boldsymbol{n = 2}$*:*

$$\Delta u(\underline{x}) = f(\underline{x})$$

*has a **unique** solution provided that:*

$$\lim_{\|\underline{x}\|\to\infty} \frac{u(x)}{\|\underline{x}\|} = 0 \qquad \lim_{\|\underline{x}\|\to\infty} \|\nabla u(\underline{x})\| = 0$$

*In particular, these **unique solutions** are given by:*

$$u(\underline{x}) = (\phi * f)(\underline{x}) = \begin{cases} \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln\|\underline{y}\| f(\underline{x}-\underline{y}) d^2y, & n=2 \\ -\frac{1}{\omega_n}\int_{\mathbb{R}^n} \|\underline{y}\|^{2-n} f(\underline{x}-\underline{y}) d^ny, & n\geq 3 \end{cases}$$

Moreover, $\exists C_n > 0$ *such that we can **estimate** the **decay** of* $u(x)$ *as* $\|\underline{x}\| \to \infty$*:*

$$|u(x)| \leq \begin{cases} C_2 \ln\|\underline{x}\|, & n=2 \\ \frac{C_n}{\|\underline{x}\|^{n-2}}, & n\geq 3 \end{cases}$$

Proof. We shall prove this for the case $n = 3$. Moreover, we use Δ_x, Δ_y to specify the variable with respect to which we compute the Laplacian (since we will take convolution, sometimes we will have functions in terms of $\underline{x}$, and others in terms of $\underline{y}$, so using Δ_x, Δ_y adds clarity). In this regard, due to symmetry we can see that:

$$\Delta_x f(\underline{x}-\underline{y}) = \Delta_y f(\underline{x}-\underline{y})$$

Now, recall the Theorem on differentiation under the integral:

Let $I(a,b)$ be a function on $\mathbb{R}\times\mathbb{R}$, and let $b_0 \in \mathbb{R}$. Then if:

1. *$\forall b$ in a neighbourhood of b_0*

$$\int_{\mathbb{R}} |I(a,b)| da < \infty$$

2. *there exists a neighbourhood $\mathcal{N}$ of b_0 such that for* ***almost every*** *a $\partial_b I(a,b)$ exists for $b \in \mathcal{N}$ (that is, the derivative at b is undefined at countably many points)*

3. *there exists a function $U(a)$ (defined for almost every a) such that if $b \in \mathcal{N}$:*

$$|\partial_b I(a,b)| \leq U(a) \qquad \int_{\mathbb{R}} U(a) da < \infty$$

Then the function:

$$J(b) = \int_{\mathbb{R}} I(a,b) da$$

is ***differentiable*** *near b_0, and:*

$$\partial_b J(b) = \int_{\mathbb{R}} \partial_b I(a,b) da$$

The same applies if $I(a,b) \in \mathbb{R}^m \times \mathbb{R}^n$.

Our functions are well-behaved, and so, we can bring the Laplacian into the integral:

$$\begin{aligned} \Delta_x u(\underline{x}) &= -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{\|\underline{y}\|}\Delta_x f(\underline{x}-\underline{y}) d^3y \\ &= -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{\|\underline{y}\|}\Delta_y f(\underline{x}-\underline{y}) d^3y \end{aligned}$$

Now, $u(\underline{x})$ satisfies the Poisson Equation if we can show that:

$$\Delta_x u(\underline{x}) = f(\underline{x})$$

To do this, we shall split $\mathbb{R}^3$ into a sphere and its complement. In particular, let $\varepsilon > 0$ and define a ball $B_\varepsilon(\underline{0})$ centred at the origin. Then, we can rewrite the integral as:

$$\Delta_x u(\underline{x}) = \underbrace{-\frac{1}{4\pi}\int_{B_\varepsilon(\underline{0})} \frac{1}{\|\underline{y}\|}\Delta_y f(\underline{x}-\underline{y}) d^3y}_{I} + \underbrace{-\frac{1}{4\pi}\int_{B^C_\varepsilon(\underline{0})} \frac{1}{\|\underline{y}\|}\Delta_y f(\underline{x}-\underline{y}) d^3y}_{II}$$

We now claim the following:

1. I goes to 0 as $\varepsilon \to 0^+$
2. $|f(\underline{x}) - II| \to 0$ as $\varepsilon \to 0^+$

from which it follows that $\Delta_x u(\underline{x}) = f(\underline{x})$ as required.

Before anything, define the following constant:

$$M = \sup_{\underline{y} \in \mathbb{R}^3} \{|f(\underline{y})| + \|\nabla f(\underline{y})\| + |\Delta_y f(\underline{y})|\}$$

(1) $I \to 0$

We use **spherical coordinates** (r, ω) (here ω encodes all the angular information ϕ, θ). Then, we have that:

$$d^3y = r^2 dr d\omega$$

where $\omega \in \partial B_\varepsilon(\underline{0})$ is an angular coordinate denoting a position on the surface of $B_\varepsilon(\underline{0})$, and $d\omega = \sin(\theta) d\theta d\phi$.

Thus, we have that:

$$\begin{aligned}
|I| &\leq \frac{1}{4\pi} \int_{B_\varepsilon(\underline{0})} \left| \frac{1}{\|\underline{y}\|} \Delta_y f(\underline{x} - \underline{y}) \right| d^3y \\
&= \frac{1}{4\pi} \int_0^\varepsilon \int_{\partial B_\varepsilon(\underline{0})} \left| \frac{1}{r} \Delta_y f(\underline{x} - \underline{y}) \right| r^2 d\omega dr \\
&\leq \frac{1}{4\pi} \int_0^\varepsilon r \int_{\partial B_\varepsilon(\underline{0})} M d\omega dr \\
&= \frac{1}{4\pi} \int_0^\varepsilon M r (4\pi\varepsilon^2) dr \\
&= \frac{M\varepsilon^4}{2}
\end{aligned}$$

So clearly:

$$\lim_{\varepsilon \to 0^+} |I| \leq \lim_{\varepsilon \to 0^+} \frac{M\varepsilon^4}{2} = 0 \implies \lim_{\varepsilon \to 0^+} |I| = 0$$

by Squeeze Theorem.

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(2) $|f(\underline{x}) - II| \to 0$

We begin by recalling **Green's Identity**:

$$\int_\Omega v(\underline{x}) \Delta w(\underline{x}) - w(\underline{x}) \Delta v(\underline{x}) d^n x = \int_{\partial\Omega} v \left(\nabla w(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) - w \left(\nabla v(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) \right) d\underline{\sigma}$$

In our case, we integrate over the region $B_\varepsilon^C(\underline{0})$. Let:

$$v(\underline{y}) = -\frac{1}{\|\underline{y}\|}$$

$$w(\underline{y}) = f(\underline{x} - \underline{y})$$

Then:

$$\begin{aligned}
\int_{B_\varepsilon^C(\underline{0})} v(\underline{y}) \Delta w(\underline{y}) - w(\underline{y}) \Delta v(\underline{y}) d^3y &= \int_{B_\varepsilon^C(\underline{0})} -\frac{1}{\|\underline{y}\|} \Delta_y f(\underline{x} - \underline{y}) + f(\underline{x} + \underline{y}) \Delta_y \frac{1}{\|\underline{y}\|} d^3y \\
&= \int_{B_\varepsilon^C(\underline{0})} -\frac{1}{\|\underline{y}\|} \Delta_y f(\underline{x} - \underline{y}) d^3y \\
&= 4\pi II
\end{aligned}$$

where we have used the fact that, as we saw above with polar coordinates:

$$\Delta_y \frac{1}{\|\underline{y}\|} = -\frac{2}{r^3} + \frac{2}{r}\frac{1}{r^2} = 0$$

Hence, we can use Green's Identity with II. Notice, $\hat{\underline{N}}(\underline{\sigma})$ will be the **inward** facing normal vector to the surface of the sphere $B_\varepsilon(\underline{0})$ (since we are integrating over the complement). Because of this, we need to "flip" the sign of the normal vector. Thus:

$$4\pi II = \int_{\partial B_\varepsilon^C(\underline{0})} \frac{1}{\|\underline{\sigma}\|}\left(\nabla f(\underline{x}-\underline{\sigma}) \cdot \hat{\underline{N}}(\underline{\sigma})\right) - f(\underline{x}-\underline{\sigma})\left(\nabla \frac{1}{\|\underline{\sigma}\|} \cdot \hat{\underline{N}}(\underline{\sigma})\right) d\underline{\sigma}$$

We note the following:

- since we integrate over the surface of a sphere of radius ε, our surface coordinate $\underline{\sigma}$ must satisfy:

$$\|\underline{\sigma}\| = \varepsilon$$

- as we did above, we can do a spherical change of coordinates, such that:

$$d\underline{\sigma} = \varepsilon^2 d\omega$$

  where $\underline{\sigma} = \varepsilon\omega$

Now, the first integrand will disappear as $\varepsilon \to 0^+$, since:

$$\begin{aligned}\left(\nabla f(\underline{x}-\underline{\sigma}) \cdot \hat{\underline{N}}(\underline{\sigma})\right) &\leq \left|\nabla f(\underline{x}-\underline{\sigma}) \cdot \hat{\underline{N}}(\underline{\sigma})\right| \\ &\leq |\nabla f(\underline{x}-\underline{\sigma})| \\ &\leq \left|\sup_{\underline{\sigma}\in\partial B_\varepsilon^C(\underline{0})} \nabla f(\underline{x}-\underline{\sigma})\right| \\ &\leq M\end{aligned}$$

Moreover, we claim that:

$$\nabla \frac{1}{\|\underline{\sigma}\|} \cdot \hat{\underline{N}} = -\frac{1}{\|\underline{\sigma}\|^2}$$

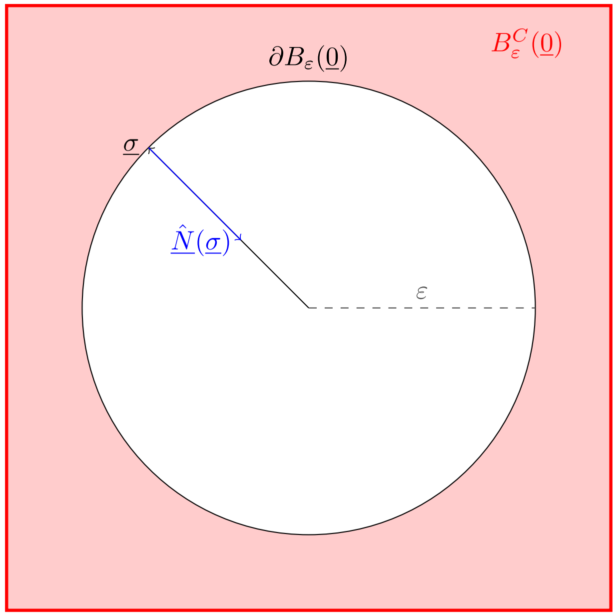
Indeed:

$$\frac{\partial}{\partial\sigma_i}\frac{1}{\|\underline{\sigma}\|} = \frac{\partial}{\partial\sigma_i}\left(\sum_i \sigma_i^2\right)^{-\frac{1}{2}} = -\frac{1}{2}\frac{1}{\left(\sum_i \sigma_i^2\right)^{\frac{3}{2}}} \times 2\sigma_i = -\frac{\sigma_i}{\|\underline{\sigma}\|^3}$$

so it follows that:

$$\nabla \frac{1}{\|\underline{\sigma}\|} = \frac{1}{\|\underline{\sigma}\|^3}\underline{\sigma}$$

Moreover, reasoning geometrically, the unit normal vector at $\underline{\sigma}$, given by $\hat{\underline{N}}(\underline{\sigma})$ is an inward facing vector perpendicular to the surface $\partial B_\varepsilon(\underline{0})$ (since it the normal vector is outward facing relative to the complement of the ball). But now, by definition, $\underline{\sigma}$ is a vector from the origin to the surface of the ball, so by definition, it incides perpendicularly on the surface.



In other words:

$$\hat{\underline{N}}(\underline{\sigma}) = \frac{1}{\|\underline{\sigma}\|}\underline{\sigma}$$

Thus, we have that:

$$\nabla\frac{1}{\|\underline{\sigma}\|}\cdot\hat{\underline{N}} = -\frac{1}{\|\underline{\sigma}\|^2} = \frac{1}{\|\underline{\sigma}\|^3}\underline{\sigma}\cdot\frac{1}{\|\underline{\sigma}\|}\underline{\sigma} = -\frac{1}{\|\underline{\sigma}\|^2} = -\frac{1}{\varepsilon^2}$$

Then, we can bound $4\pi II$ as:

$$\begin{aligned} 4\pi II &\leq \int_{\partial B_\varepsilon^C(\underline{0})}\left(\frac{M}{\varepsilon} + f(\underline{x}-\varepsilon\omega)\frac{1}{\varepsilon^2}\right)\varepsilon^2 d\omega \\ &= \int_{\partial B_\varepsilon^C(\underline{0})} M\varepsilon + f(\underline{x}-\varepsilon\omega)d\omega \end{aligned}$$

Now, f is continuous, so by the Mean Value Theorem for Integrals we have that $\exists\omega^*$ such that:

$$f(x-\varepsilon\omega^*) = \frac{1}{|\partial B_\varepsilon^C(\underline{0})|}\int_{B_\varepsilon^C(\underline{0})} f(\underline{x}-\varepsilon\omega)d\omega$$

Hence, and noting that $|\partial B_\varepsilon^C(\underline{0})| = 4\pi$ in $\mathbb{R}^3$, we obtain the bound:

$$4\pi II \leq 4\pi M\varepsilon + 4\pi f(x-\varepsilon\omega^*) \implies II \leq M\varepsilon + f(x-\varepsilon\omega^*)$$

Hence, as $\varepsilon \to 0^+$, we get that:

$$II \to f(x)$$

as required.

Hence, we have shown that $\Delta_x u(\underline{x}) = f(\underline{x})$, as required.

We still have to show our decay estimates for u as $\|\underline{x}\| \to \infty$. Assume that $f(\underline{x})$ vanishes outside of the ball $B_R(\underline{0})$. It suffices to estimate $|u(\underline{x})|$ when $\|\underline{x}\| > 2R$ (since we are going to consider $\|\underline{x}\| \to \infty$, it is sufficient to show a bound past some finite magnitude of $\underline{x}$).

Now, if $\|\underline{y}\| \leq R$ and $\|\underline{x}\| > 2R$, we have that:

$$\frac{1}{\|\underline{x} - \underline{y}\|} \leq \frac{1}{R} \leq \frac{2}{\|\underline{x}\|}$$

Thus, and recalling:

$$M = \sup_{\underline{y} \in \mathbb{R}^3} \{|f(\underline{y})| + \|\nabla f(\underline{y})\| + |\Delta_y f(\underline{y})|\}$$

we have that integrating over our ball $B_R(\underline{0})$

$$\begin{aligned}
|u(\underline{x})| &= \left| -\frac{1}{4\pi} \int_{\mathbb{R}^3} \|\underline{y}\|^{-1} |f(\underline{x} - \underline{y})| d^n y \right| \\
&\leq \frac{1}{4\pi} \int_{B_R(\underline{0})} \|\underline{y}\|^{-1} |f(\underline{x} - \underline{y})| d^n y, \quad \textit{(since } f \textit{ vanishes outside of } B_R(\underline{0})\textit{)} \\
&= \frac{1}{4\pi} \int_{B_R(\underline{0})} \|\underline{x} - \underline{y}\|^{-1} |f(\underline{y})| d^n y, \quad \textit{(by commutatitivity of convolution)} \\
&\leq \frac{1}{4\pi} \int_{B_R(\underline{0})} \frac{2}{\|\underline{x}\|} M d^n y \\
&\leq \frac{1}{4\pi} \int_{B_R(\underline{0})} \frac{2}{\|\underline{x}\|} M d^n y \\
&= |B_R(\underline{0})| \frac{1}{4\pi} \frac{2}{\|\underline{x}\|} M \\
&= \frac{4}{3} \pi R^3 \frac{1}{2\pi} \frac{1}{\|\underline{x}\|} M \\
&= \frac{2R^3 M}{3\|\underline{x}\|}
\end{aligned}$$

Hence, our bound is as required, and:

$$C_3 = \frac{2R^3 M}{3}$$

To prove uniqueness, we will rely on **Liouville's Theorem**, which shall be proven in the following weeks. □

4 Workshop

1. **Show that if $u \in C^\infty(\Omega)$ is harmonic in a domain Ω, also the derivatives of u of any order are harmonic in Ω.**

   It is sufficient to show that for any $x^i, i \in [1, n]$, we have that:

   $$v = u_{x^i}$$

   is harmonic (since then v is a harmonic function, and its derivatives will be harmonic, so any derivative of u will be harmonic)

We thus compute:

$$\Delta v = \sum_{j=1}^{n} v_{x^j x^j} = \sum_{j=1}^{n} u_{x^i x^j x^j} = \sum_{j=1}^{n} u_{x^j x^j x^i} = \frac{\partial}{\partial x^i}(\Delta u) = 0$$

so v is harmonic, and the result follows.

2. **We say that a function $u \in^C 2(\Omega), \Omega \subset \mathbb{R}^n$ is subharmonic in Ω if:**

$$\Delta u \geq 0$$

(if $\Delta u \leq 0$ then its superharmonic). Show that:

(a) **If u is subharmonic, then, for every $B_R(\underline{x}) \subset \Omega$:**

$$u(\underline{x}) \leq \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(\underline{x})} u(\underline{\sigma}) d\underline{\sigma}$$

(if u is superharmonic, the reverse inequality holds)

This follows immediately from the fact that harmonic functions obey the mean value property. Indeed, define a function:

$$g(r) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(\underline{x})} u(\underline{\sigma}) d\underline{\sigma}$$

where $\underline{\sigma} = \underline{x} + r\underline{\omega}$, and $\underline{\omega}$ represents an angular coordinate on the surface of a unit sphere. In particular, by applying the Mean Value Theorem, we see that:

$$\lim_{r \to 0^+} g(r) = u(\underline{x})$$

Moreover, we can now compute $g'(r)$. By using the change of variables $\underline{\sigma} = \underline{x} + r\underline{\omega}$, which gives:

$$d\underline{\sigma} = r^{n-1} d\underline{\omega}$$

we have that:

$$g'(r) = \frac{\partial}{\partial r}\left(\frac{1}{\omega_n} \int_{\partial B_1(\underline{x})} u(\underline{x} + r\underline{\omega}) d\underline{\omega}\right) = \frac{1}{\omega_n} \int_{\partial B_1(\underline{x})} \partial_r u(\underline{x} + r\underline{\omega}) d\underline{\omega}$$

This allows us to integrate over the unit ball centered at $\underline{x}$. Moreover, notice that $\partial_r u(\underline{\sigma})$ will be the gradient vector of u dotted with $\underline{\omega}$ (since a partial derivative is nothing but a directional derivative in the direction of one of the axes, and $\underline{\omega}$ always points in the direction of the radial variable r). Hence:

$$g'(r) = \frac{1}{\omega_n} \int_{\partial B_1(\underline{x})} (\nabla u(\underline{x} + r\underline{\omega}) \cdot \underline{\hat{N}}) d\underline{\omega}$$

since $\underline{\omega}$ is a unit normal vector to the ball by construction. But now, the **Divergence Theorem** applies, and so we can write:

$$g'(r) = \frac{1}{\omega_n} \int_{B_1(\underline{x})} \Delta u(\underline{x} + r\underline{\omega}) d\underline{\omega}$$

Since u is subharmonic, it thus follows that:

$$g'(r) \geq 0$$

But now, by the Fundamental Theorem of Calculus:

$$g(R) - \lim_{r \to 0^+} g(r) = \int_0^R g'(t) dt \geq 0$$

which implies that:

$$g(R) \geq u(\underline{x})$$

which is the result we were looking for.

(b) **If u is harmonic in Ω, then u^2 is subharmonic**

We have that:

$$\Delta u = \sum_{i=1}^{n} u_{x^i x^i} = 0$$

Then:

$$\frac{\partial u^2}{\partial x^i} = 2uu_{x^i} \implies \frac{\partial^2(u^2)}{\partial(x^i)^2} = 2\left[(u_{x^i})^2 + uu_{x^i x^i}\right]$$

Thus:

$$\Delta(u^2) = 2\sum_{i=1}^{n}\left[(u_{x^i})^2 + uu_{x^i x^i}\right] = 2\left[\sum_{i=1}^{n}(u_{x^i})^2 + u\sum_{i=1}^{n} u_{x^i x^i}\right] = 2\sum_{i=1}^{n}(u_{x^i})^2 \geq 0$$

so u^2 is subharmonic.

(c) **Let U be harmonic in Ω and $F : \mathbb{R} \to \mathbb{R}$ smooth. Under what conditions on F is $F(u)$ subharmonic?**

We compute:

$$\frac{\partial}{\partial x^i}(F(u)) = F'(u)u_{x^i}$$

$$\frac{\partial^2}{\partial (x^i)^2}(F(u)) = F''(u)(u_{x^i})^2 + F'(u)u_{x^i x^i}$$

Thus, and using the fact that u is harmonic:

$$\Delta(F(u)) = \sum_{i=1}^{n} F''(u)(u_{x^i})^2 = F''(u)\|\nabla u\|^2$$

since $\|\nabla u\|^2 \geq 0$, it follows that $F(u)$ is subharmonic **if and only if** $F''(u) \geq 0$.

3. **Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and $v \in C^2(\Omega) \cap C^2(\bar{\Omega})$ be a solution of the torsion problem:**

$$\begin{cases} v_{xx} + v_{yy} = -2 & \textbf{in } \Omega \\ v = 0 & \textbf{in } \partial\Omega \end{cases}$$

Show that $u = \|\nabla v\|^2$ attains its maximum on $\partial\Omega$.

We have that:

$$u = v_x^2 + v_y^2$$

such that:

$$u_x = 2(v_x v_{xx} + v_y v_{yx})$$

$$u_{xx} = 2(v_{xx}^2 + v_x v_{xxx} + v_{yx}^2 + v_y v_{yxx})$$

By symmetry we thus have:

$$u_{yy} = 2(v_{yy}^2 + v_y v_{yyy} + v_{yx}^2 + v_x v_{xyy})$$

Hence:

$$\Delta u = 2([v_{xx}^2 + v_{yy}^2 + 2v_{xy}^2] + v_x(v_{xxx} + v_{xyy}) + v_y(v_{yyy} + v_{yxx}))$$

But notice:

$$v_{xxx} + v_{xyy} = \frac{\partial}{\partial x}(v_{xx} + v_{yy}) = \frac{\partial}{\partial x}(-2) = 0$$

$$v_{yyy} + v_{yxx} = \frac{\partial}{\partial y}(v_{xx} + v_{yy}) = \frac{\partial}{\partial y}(-2) = 0$$

Hence:

$$\Delta u = 2(v_{xx}^2 + v_{yy}^2 + 2v_{xy}^2) \geq 0$$

Hence, the maximum principle applies, and since v is not constant, clearly u won't be constant. Thus, u attains its maximum on $\partial\Omega$.

4. **Let B_1 be the unit disc centered at $(0,0)$, and let U be a solution to:**

$$\begin{cases} \Delta u = y & \textbf{in } B_1 \\ u = 1 & \textbf{in } \partial B_1 \end{cases}$$

Find an explicit formula for u. Using knowledge from ODEs, it might make sense to seek for a polynomial, since the boundary condition is a polynomial.

Notice, on ∂B_1 we have that:

$$x^2 + y^2 = 1$$

Hence, we satisfy the boundary condition by using:

$$u(x,y) = F(x,y)(x^2 + y^2 - 1) + 1$$

Moreover, we know that:

$$\Delta y^3 = 6y$$

so we can guess that $F(x,y)$ should contain y. Indeed, if we use $F(x,y) = y$:

$$\begin{aligned} u_x &= 2xy \\ u_{xx} &= 2y \\ u_y &= x^2 + 3y^2 \\ u_{yy} &= 6y \end{aligned}$$

so:

$$\Delta u = 8y$$

Hence, we set:

$$F(x,y) = \frac{1}{8}y \implies u(x,y) = \frac{y(x^2 + y^2 - 1)}{8} + 1$$

5. **Let u be harmonic in $\mathbb{R}^n$, and let M be an orthogonal matrix of order N. Using the mean value property, show that $v(\underline{x}) = u(M\underline{x})$ is harmonic in $\mathbb{R}^n$.**

Since M is an orthogonal matrix, we have that $det(M) = 1$, so any transformation by M will be volume-preserving. Hence, $u(M\underline{x})$ will satisfy the mean value property, and so, $v(\underline{x})$ must be harmonic.

6. **Let u be harmonic in $\mathbb{R}^3$ such that:**

$$\int_{\mathbb{R}^3} |u(x)|^2 d^n x < \infty$$

Show that $u \equiv 0$.

Let $M \in \mathbb{R}$ such that:

$$\int_{\mathbb{R}^3} |u(x)|^2 d^n x = M$$

Moreover, we know that if $B_R(x)$ represents a ball of radius $R > 0$ in $\mathbb{R}^3$ centered at $x \in \mathbb{R}^3$

$$\lim_{R\to\infty} \int_{B_R(x)} |u(x)|^2 dx = \int_{\mathbb{R}^3} |u(x)|^2 d^n x = M$$

Furthermore, recall the Cauchy-Schwarz Inequality, given integrable functions $f, g \in L^2(\mathbb{R}^3)$:

$$\left| \int_{\mathbb{R}^3} f(x)g(x)d^3x \right|^2 \leq \left(\int_{\mathbb{R}^3} |f(x)|^2 d^3x \right) \left(\int_{\mathbb{R}^3} |g(x)|^2 d^3x \right)$$

Since u is harmonic, by the mean value property of harmonic functions, we have that if $x \in \Omega \subset \mathbb{R}^3$ and $B_R(x) \subset \Omega$, then:

$$u(x) = \frac{3}{4\pi R^3} \int_{B_R(x)} u(y) d^3y$$

where $R > 0$.

Thus, applying the Cauchy-Schwarz Inequality with $f(y) = |u(y)|, g(y) = 1$ (which are in L^2 by assumption):

$$\begin{aligned}
|u(x)| &= \left| \frac{3}{4\pi R^3} \int_{B_R(x)} u(y) d^3y \right| \\
&\leq \frac{3}{4\pi R^3} \int_{B_R(x)} |u(y)||1| d^3y \\
&\leq \frac{3}{4\pi R^3} \sqrt{\int_{B_R(x)} |u(y)|^2 d^3y} \sqrt{\int_{B_R(x)} 1 d^3y} \\
&\leq \frac{3}{4\pi R^3} \sqrt{\int_{\mathbb{R}^3} |u(y)|^2 d^3y} \sqrt{\frac{4\pi R^3}{3}} \\
&= \sqrt{\frac{3}{4\pi R^3}} M
\end{aligned}$$

Hence, since $|u(x)| \leq \sqrt{\frac{3}{4\pi R^3}} M$, as $R \to \infty$:

$$|u(x)| \to 0$$

since M is constant.

Hence, over $\mathbb{R}^3$ (which we can think of the ball $B_R(x)$ with $R \to \infty$), we will have:

$$|u(x)| = 0 \implies u(x) = 0$$

as required.