Introduction to Partial Differential Equations - Week 4 - The Weak Maximum Principle & the Fundamental Solution to the Heat Equation

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Contents

1 The Weak Maximum Principle

1.1 Theorem: The Weak Maximum Principle

*Let $\Omega \subset \mathbb{R}^n$ be a **domain** (open and connected subset). Recall the **space-time cylinder**:*

$$Q_T = (0, T) \times \Omega$$

*and the **parabolic boundary**:*

$$\partial_p Q_T = \{0\} \times \bar{\Omega} \cup (0, T] \times \partial\Omega$$

*Let $w \in C^{1,2}(Q_T) \cap C(\bar{Q}_T)$ be a **solution** to the **heat equation**:*

$$w_t - D\Delta w = f$$

Then:

1. *if $f \leq 0$, $w(x, t)$ achieves its **maximum** in the region $\bar{Q}_T$ at one or more points of the **parabolic boundary** $\partial_p Q_T$. Then, w is a **subsolution**, and if w is strictly **negative** on $\partial_p Q_T$, then w is strictly **negative** on $\bar{Q}_T$.*

2. *if $f \geq 0$, $w(x, t)$ achieves its **minimum** in the region $\bar{Q}_T$ at one or more points of the **parabolic boundary** $\partial_p Q_T$. Then, w is a **supersolution**, and if w is strictly **positive** on $\partial_p Q_T$, then w is strictly **positive** on $\bar{Q}_T$.*

1.2 Intuition on the Weak Maximum Principle

The gist of the weak maximum principle is that the parabolic boundary is what defines where the maximum/minimum of the solution to the heat equation will be.

In one dimension, the heat equation is simply:

$$w_t - Dw_{xx} = f$$

Lets consider an even simpler scenario:

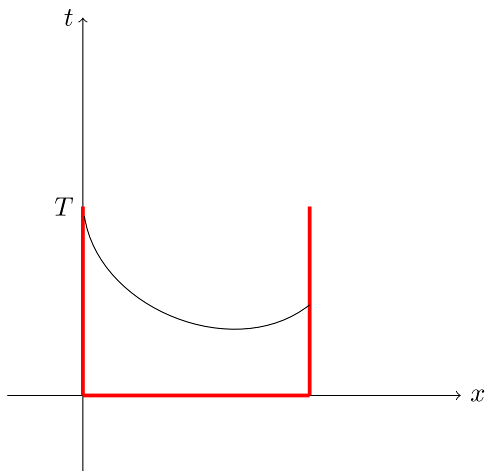
$$-w_{xx} = f$$

(1) $f \leq 0$

The heat equation becomes:

$$-w_{xx} \leq 0 \implies w_{xx} \geq 0$$

In other words, w will be **convex** (we will have $w \geq mx + b$, so w is a curve which is always dominate its tangent line):



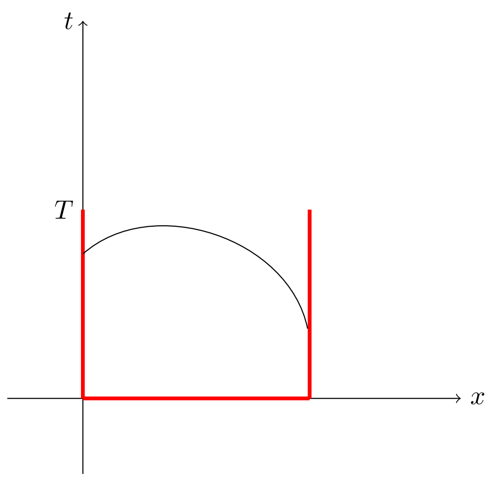
Hence, intuitively we see that the convex nature of the curve guarantees that at least one of the endpoints leads to a maximum (in this case on the left boundary).

(2) $f \geq 0$

The heat equation becomes:

$$-w_{xx} \geq 0 \implies w_{xx} \leq 0$$

In other words, w will be **concave** (we will have $w \leq mx + b$, so w is a curve which is always dominated by its tangent line):



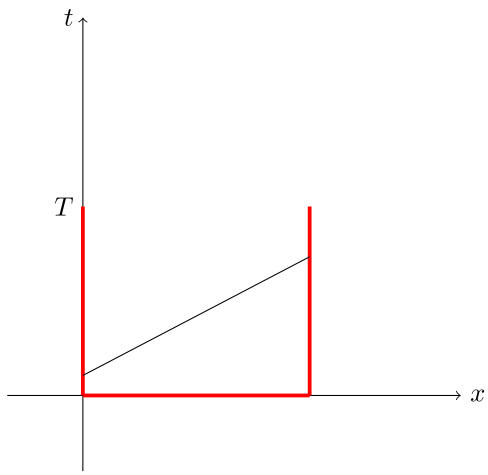
Hence, intuitively we see that the concave nature of the curve guarantees that at least one of the endpoints leads to a minimum (in this case on the right boundary).

(3) $f = 0$

The heat equation becomes:

$$-w_{xx} = 0 \implies w_{xx} = 0$$

In other words, w will be a straight line $w = mx + b$:



Hence, intuitively we see that w will be monotone, so it attains **both** a maximum and a minimum at the endpoints.

1.3 Proving the Weak Maximum Principle

Proof. We shall prove the Weak Maximum Principle when the spacetime dimension is 2 (so $x \in \mathbb{R}$). We will also only consider the case $f \leq 0$, since $f \geq 0$ is analogous.

Let $\varepsilon > 0$, and let w be a solution to the heat equation and define the function:

$$u = w - \varepsilon t \leq w \qquad t \in (0, T)$$

If we can derive information about the behaviour of u, we can then take the limit as $\varepsilon \to 0^+$ to obtain information about w.

We begin by plugging in u into the heat equation:

$$\begin{aligned} u_t - Du_{xx} &= (w_t - \varepsilon) - Dw_{xx} \\ &= (w_t - Dw_{xx}) - \varepsilon \\ &= f - \varepsilon \\ &< 0 \end{aligned}$$

since $f \leq 0$ and $\varepsilon > 0$.

We now seek to determine the maximum of u. For this, consider the spacetime cylinder confining u given by $Q_{T-\varepsilon}$, and assume that u attains a maximum in $\bar{Q}_{T-\varepsilon}$ at $(t_0, x_0) \in \partial_p Q_{T-\varepsilon}$.

We proceed by contradiction, and assume that $(t_0, x_0) \in Q_{T-\varepsilon}$. We can assume that $0 < t_0 \leq T - \varepsilon$, since if $t_0 = 0$, then the maximum is attained on $\{0\} \times \bar{\Omega}$, which is part of the parabolic boundary $\partial_p Q_{T-\varepsilon}$. Moreover, we must have that $x_0 \in \Omega$, since if $x_0 \in \bar{\Omega}$ then again it is part of the parabolic boundary

If (t_0, x_0) is a maximum, then we will have:

$$u_x(t_0, x_0) = 0$$

and:

$$u_t(t_0, x_0) \geq 0$$

with equality when $t_0 = T - \varepsilon$ (since $t = t_0$ is at the boundary of $Q_{T-\varepsilon}$, u can attain a maximum at (t_0, x_0) on $Q_{T-\varepsilon}$, but still be increasing for $t > T - \varepsilon$, so the gradient need not be 0, it can be positive).

Now, recall Taylor's Remainder Theorem, which allows us to approximate a function with a (truncated) Taylor series:

$$u(t_0, x) = u(t_0, x_0) + u_x(t_0, x_0)(x - x_0) + u_{xx}(t_0, x^*)\frac{(x - x_0)^2}{2}$$

where x^* is a point in the neighbourhood of x_0 (if this proof were for higher dimensions, then we would use Hessians, instead of derivatives).

But if we rearrange, noting the fact that $u_x(t_0, x_0) = 0$ we get:

$$u(t_0, x) - u(t_0, x_0) = u_{xx}(t_0, x^*)\frac{(x - x_0)^2}{2}$$

But since $u(t_0, x_0)$ is a maximum:

$$u(t_0, x_0) \geq u(t_0, x) \implies u_{xx}(t_0, x^*)\frac{(x - x_0)^2}{2} \leq 0$$

Since $\frac{(x-x_0)^2}{2} \geq 0$ for all x, this is only possible if $u_{xx}(t_0, x^*) \leq 0$. Thus, taking $x^* \to x_0$ implies:

$$u_{xx}(t_0, x_0) \leq 0$$

Thus, since $u_t(x_0, t_0) \geq 0$ and $u_{xx}(t_0, x_0) \leq 0$ it follows that:

$$u_t(x_0, t_0) - Du_{xx}(t_0, x_0) \geq 0$$

However, this is a direct contradiction with the fact that:

$$u_t(x_0, t_0) - Du_{xx}(t_0, x_0) = f - \varepsilon < 0$$

Thus, if $(t_0, x_0) \in Q_{T-\varepsilon}$, it can't be a maximum of u; thus, any maximum of u must lie in the parabolic boundary $\partial_p Q_{T-\varepsilon}$.

So now lets consider maxima of w. From definition, $u \leq w$. Moreover, by definition $\partial_p Q_{T-\varepsilon} \subset \partial_p Q_T$, so by definition any maximum in $\partial_p Q_{T-\varepsilon}$ must lie in $\partial_p Q_T$ too. Hence, we have the following inequality:

$$\max_{\bar{Q}_{T-\varepsilon}} u = \max_{\partial_p Q_{T-\varepsilon}} u \leq \max_{\partial_p Q_T} u \leq \max_{\partial_p Q_T} w$$

Now, we also have that:

$$w \leq u + \varepsilon T$$

since $u = w - \varepsilon t$ with $t \in (0, T)$. Hence, the maximum of w will be bounded above by the maximum of $u + \varepsilon T$:

$$\max_{\bar{Q}_{T-\varepsilon}} w \leq \max_{\bar{Q}_{T-\varepsilon}} (u + \varepsilon T) = \max_{\bar{Q}_{T-\varepsilon}} u + \varepsilon T \leq \max_{\partial_p Q_T} w + \varepsilon T$$

But now, since $\partial_p Q_T$ is a parabolic boundary, it is clear that:

$$\max_{\partial_p Q_T} w \leq \max_{\bar{Q_T}} w$$

However, using the uniform continuity of w, we can take the limit as $\varepsilon \to 0+$ such that:

$$\begin{aligned} & \max_{\bar{Q}_{T-\varepsilon}} w \leq \max_{\partial_p Q_T} w + \varepsilon T \\ \implies & \lim_{\varepsilon \to 0^+} \left(\max_{\bar{Q}_{T-\varepsilon}} w \right) \leq \lim_{\varepsilon \to 0^+} \left(\max_{\partial_p Q_T} w + \varepsilon T \right) \\ \implies & \max_{\bar{Q}_T} w \leq \max_{\partial_p Q_T} w \end{aligned}$$

In conclusion, we must thus have equality:

$$\max_{\bar{Q}_T} w = \max_{\partial_p Q_T} w$$

which is what we required.

□

1.4 Corollary: Comparison Principle and Stability

This corollary allows us to compare 2 different solutions to the heat equation, given that they have (possibly) different inhomogeneous terms.

Suppose v, w are solutions to the heat equations:

$$v_t - Dv_{xx} = f$$

$$w_t - Dw_{xx} = g$$

Then:

1. ***Comparison**: if $v \geq w$ on $\partial_p Q_T$ and $f \geq g$, then $v \geq w$ on **all** of Q_T.*
2. ***Stability**:*

$$\max_{\bar{Q}_T} |v - w| \leq \max_{\partial_p Q_T} |v - w| + T \max_{\bar{Q}_T} |f - g|$$

Proof. Define:

$$u = w - v$$

u also solves the heat equation, and by linearity:

$$u_t - Du_{xx} = g - f \leq 0$$

since $f \geq g$.

(1) **Comparison**

Now, the Weak Maximum Principle applies, which implies that u attains a maximum on $\partial_p Q_T$. But since $v \geq w$ on $\partial_p Q_T$, we have that $u \leq 0$ on the parabolic boundary, so it follows that $u \leq 0$ on all of Q_T, so $v \geq w$ always, as required.

(2) **Stability**

Define:

$$M = \max_{\bar{Q}_T} |f - g|$$

$$u = w - v - tM$$

Then:

$$u_t - Du_{xx} = (g - f) - M \leq 0$$

Hence, the Weak Maximum Principle applies to u, and:

$$\max_{\bar{Q}_T} u = \max_{\partial_P Q_T} u$$

Moreover:

$$\max_{\bar{Q}_T} (w - v) = \max_{\bar{Q}_T} (w - v - tM) + \max_{\bar{Q}_T} tM \leq \max_{\partial_P Q_T} u + T\max_{\bar{Q}_T} |f - g|$$

Here it is claimed that:

$$\max_{\partial_P Q_T} u \leq \max_{\partial_P Q_T} |w - v|$$

Thus, it follows that:

$$\max_{\bar{Q}_T} (w - v) \leq \max_{\partial_P Q_T} |w - v| + +T\max_{\bar{Q}_T} |f - g|$$

If we define $u = v - w$, we similarly get:

$$\max_{\bar{Q}_T} (v - w) \leq \max_{\partial_P Q_T} |w - v| + +T\max_{\bar{Q}_T} |f - g|$$

So we have that:

$$\max_{\bar{Q}_T} |w - v| \leq \max_{\partial_P Q_T} |w - v| + +T\max_{\bar{Q}_T} |f - g|$$

as required.

□

2 The Fundamental Solution to the Heat Equation

2.1 The Fundamental Solution to the Homogeneous Heat Equation

2.1.1 Definition: The Fundamental Solution to the Inhomogeneous Heat Equation

Consider the ***inhomogeneous heat equation****:*

$$u_t - D\Delta u = f$$

The ***fundamental solution*** *to this is:*

$$\Gamma_D(t, x) = \frac{1}{(4\pi Dt)^{n/2}} e^{-\frac{|x|^2}{4Dt}}, \qquad t > 0, x \in \mathbb{R}^n$$

where:

$$|x|^2 = \sum_{i=1}^{n} (x^i)^2$$

2.1.2 Lemma: The Fundamental Solution Solves the Homogeneous Heat Equation

*The **fundamental solution** $\Gamma_D(t,x)$ solves the **homogeneous** heat equation for all $x \in \mathbb{R}^n, t > 0$.*

2.1.3 Lemma: Properties of the Fundamental Solution

*The **fundamental solution** $\Gamma_D(t,x)$ satisfies the following:*

1. $$x \neq 0 \implies \lim_{t \to 0^+} \Gamma_D(t,x) = 0$$

2. $$x = 0 \implies \lim_{t \to 0^+} \Gamma_D(t,x) = \infty$$

3. $$\forall t > 0, \qquad \int_{\mathbb{R}^n} \Gamma_D(t,x) d^n x = 1$$

2.1.4 Definition: The Delta Distribution

*The **delta distribution (Dirac delta function)** δ centered at 0 acts on functions $\phi(x)$ via:*

$$\langle \delta, \phi \rangle = \phi(0)$$

More generally:

$$\langle \delta, \phi(x-y) \rangle = \phi(x)$$

which follows by defining:

$$\varphi(y) = \phi(x-y)$$

so:

$$\langle \delta, \phi(x-y) \rangle = \langle \delta, \varphi \rangle = \varphi(0) = \phi(x)$$

2.1.5 Lemma: The Fundamental Solution is a Delta Distribution at the Limit

Suppose that:

- $\phi(x)$ *is* ***continuous*** *on* $\mathbb{R}^n$
- $$\exists a, b \geq 0 \ : \ |\phi(x)| \leq a e^{b|x|^2}$$

  Then:

  $$\lim_{t \to 0+} \int_{\mathbb{R}^n} \Gamma_D(t,x)\phi(x) \ d^n x = \phi(0)$$

  That is, at $t = 0$*,* $\Gamma_D(t,x)$ *behaves like the delta distribution centered at 0:*

  $$\lim_{t \to 0^+} \Gamma_D(t,x) = \delta(x)$$

Proof. Since the integral of the fundamental solution is 1 for $t > 0$, we get the following identity:

$$\phi(0) = (\phi(x) + \phi(0) - \phi(x)) \int_{\mathbb{R}^n} \Gamma_D(t,x) d^n x = \int_{\mathbb{R}^n} \Gamma_D(t,x)\phi(x) d^n x + \int_{\mathbb{R}^n} \Gamma_D(t,x)(\phi(0) - \phi(x)) d^n x$$

Now, let $\varepsilon > 0$, and define a ball B of radius R centered at 0, such that:

$$\forall x \in B, \quad |\phi(0) - \phi(x)| \leq \varepsilon$$

(since ϕ is continuous, $\phi(x)$ can be made to be arbitrarily close to $\phi(0)$ by choosing an x arbitrarily close 0)

Now, if B^C denotes the complement of B, such that:

$$\mathbb{R}^n = B \cup B^C$$

then we can derive a bound for the second term in the equality above:

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} \Gamma_D(t,x)(\phi(0) - \phi(x)) d^n x \right| &= \left| \int_B \Gamma_D(t,x)(\phi(0) - \phi(x)) d^n x + \int_{B^C} \Gamma_D(t,x)(\phi(0) - \phi(x)) d^n x \right| \\
&\leq \int_B \Gamma_D(t,x)|\phi(0) - \phi(x)| d^n x + \int_{B^C} \Gamma_D(t,x)|\phi(0) - \phi(x)| d^n x \\
&\leq \int_B \Gamma_D(t,x)|\phi(0) - \phi(x)| d^n x + \int_{B^C} \Gamma_D(t,x)(|\phi(0)| + |\phi(x)| d^n x \\
&\leq \varepsilon \int_B \Gamma_D(t,x) d^n x + |\phi(0)| \int_{B^C} \Gamma_D(t,x) d^n x + \int_{B^C} \Gamma_D(t,x)|\phi(x)| d^n x
\end{aligned}$$

Using the estimate $\int_B \Gamma_D(t,x) d^n x \leq 1$, it thus follows that:

$$\left| \int_{\mathbb{R}^n} \Gamma_D(t,x)(\phi(0) - \phi(x)) d^n x \right| \leq \varepsilon + |\phi(0)| \int_{B^C} \Gamma_D(t,x) d^n x + \int_{B^C} \Gamma_D(t,x)|\phi(x)| d^n x$$

Hence, we have the bound:

$$\phi(0) = \int_{\mathbb{R}^n} \Gamma_D(t,x)\phi(x)d^n x + \int_{\mathbb{R}^n} \Gamma_D(t,x)(\phi(0) - \phi(x))d^n x$$
$$\implies \left|\phi(0) - \int_{\mathbb{R}^n} \Gamma_D(t,x)\phi(x)d^n x\right| \leq \varepsilon + |\phi(0)| \int_{B^C} \Gamma_D(t,x)d^n x + \int_{B^C} \Gamma_D(t,x)|\phi(x)|d^n x$$

Now, if we can show that as $t \to 0^+$:

$$|\phi(0)| \int_{B^C} \Gamma_D(t,x)d^n x + \int_{B^C} \Gamma_D(t,x)|\phi(x)|d^n x \to 0$$

then we will have proven the claim that:

$$\lim_{t\to 0+} \int_{\mathbb{R}^n} \Gamma_D(t,x)\phi(x)\ d^n x = \phi(0)$$

We show that this is true for the expression:

$$\int_{B^C} \Gamma_D(t,x)|\phi(x)|d^n x$$

since the working will be identical for the expression $|\phi(0)| \int_{B^C} \Gamma_D(t,x)d^n x$ (we can think of $\phi(0)$ as a constant function ϕ bounded with $b = 0$).

Now, we can employ the bound on ϕ:

$$\begin{aligned}|\phi(x)|\Gamma_D(t,x) &\leq ae^{b|x|^2} \frac{1}{(4\pi Dt)^{n/2}} e^{-\frac{|x|^2}{4Dt}} \\ &= \frac{a}{(4\pi Dt)^{n/2}} e^{-|x|^2\left(\frac{1}{4Dt} - b\right)}\end{aligned}$$

So now let:

$$|z|^2 = |x|^2 \left(\frac{1}{4Dt} - b\right)$$

Then:

$$\begin{aligned}&\frac{d}{dx^i}|z|^2 = \frac{d}{dx^i}\left(|x|^2\left(\frac{1}{4Dt} - b\right)\right) \\ \implies &\frac{d}{dx^i}\left(\sum_{j=1}^n (z^j)^2\right) = \frac{d}{dx^i}\left(\left(\frac{1}{4Dt} - b\right)\sum_{j=1}^n (x^j)^2\right) \\ \implies &2\frac{dz^i}{dx^i} = 2\left(\frac{1}{4Dt} - b\right)x^i \\ \implies &\frac{dz^i}{dx^i} = \left(\frac{1}{4Dt} - b\right)x^i\end{aligned}$$

Hence:

$$d^n z = \left(\frac{1}{4Dt} - b\right)^{n/2} d^n x$$

(This follows by the fact that we are going to have nested integrals over the varibables x^i, and for each we are multiplying the result of the integral by constant $\sqrt{\frac{1}{4Dt} - b}$ n times.)

So:

$$d^n x = \left(\frac{1}{\frac{1}{4Dt} - b}\right)^{n/2} d^n z$$

Notice, this change of variables will be allowed so long as:

$$\frac{1}{4Dt} - b > 0 \iff t < \frac{1}{4Db}$$

which is fine, since $t \to 0^+$, so t can always be made to be smaller than this quantity.

Going back to the integral, we no longer integrate over:

$$|x| \geq R$$

(since R is the radius of B, and we integrate over $x \in B^C$) but rather:

$$|z| \geq R\sqrt{\frac{1}{4Dt} - b}$$

Thus:

$$\begin{aligned}
\int_{B^C} \Gamma_D(t,x)|\phi(x)|d^n x &\leq \int_{B^C} \frac{a}{(4\pi Dt)^{n/2}} e^{-|x|^2\left(\frac{1}{4Dt}-b\right)} d^n x \\
&= \frac{a}{(4\pi Dt)^{n/2}} \int_{|z|\geq R\sqrt{\frac{1}{4Dt}-b}} e^{-|z|^2} \left(\frac{1}{\frac{1}{4Dt} - b}\right)^{n/2} d^n z \\
&= \frac{a}{\left(4\pi Dt\left(\frac{1}{4Dt} - b\right)\right)^{n/2}} \int_{|z|\geq R\sqrt{\frac{1}{4Dt}-b}} e^{-|z|^2} d^n z \\
&= \frac{a}{(\pi - 4b\pi Dt)^{n/2}} \int_{|z|\geq R\sqrt{\frac{1}{4Dt}-b}} e^{-|z|^2} d^n z
\end{aligned}$$

But now, notice that:

$$\lim_{t\to 0} R\sqrt{\frac{1}{4Dt} - b} = \infty$$

That is, the limits of integration become infinitesimally small; thus, even if $\frac{a}{(\pi-4b\pi Dt)^{n/2}} \to \infty$, the integral goes to 0 "quicker", so we deduce that:

$$\lim_{t\to 0^+} \int_{B^C} \Gamma_D(t,x)|\phi(x)|d^n x = 0$$

Hence, it follows that since:

$$\left|\phi(0) - \int_{\mathbb{R}^n} \Gamma_D(t,x)\phi(x)d^n x\right| \leq \varepsilon + |\phi(0)| \int_{B^C} \Gamma_D(t,x)d^n x + \int_{B^C} \Gamma_D(t,x)|\phi(x)|d^n x$$

then

$$\limsup_{t\to 0^+} \left|\phi(0) - \int_{\mathbb{R}^n} \Gamma_D(t,x)\phi(x)d^n x\right| = \varepsilon$$

so since ε is an arbitrary positive constant:

$$\lim_{t\to 0^+} \left|\phi(0) - \int_{\mathbb{R}^n} \Gamma_D(t,x)\phi(x)d^n x\right| = 0$$

as required.

□

2.2 The Convolution

2.2.1 Definition: The Convolution

Let f, g be functions on $\mathbb{R}^n$. Define the ***convolution*** *as the following function:*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)d^n y$$

One can interpret the convolution as an ***averaging****, whereby f is replaced by its average value, weighted by g at each point.*

2.2.2 Lemma: Properties of the Convolution

1. *The convolution is* ***commutative****:*

$$(f * g)(x) = (g * f)(x)$$

2. *The convolution is* ***associative****:*

$$f * (g * h) = (f * g) * h$$

3. *The* ***delta distribution*** *is the identity element:*

$$(f * \delta)(x) = f(x)$$

Proof. (1) **Commutative**

Define a new variable $z = x - y$. Then:

$$d^n z = -d^n y$$

$$\begin{aligned}
(f * g)(x) &= \int_{\mathbb{R}^n} f(y)g(x-y)d^n y \\
&= -\int_{z(-\infty)}^{z(\infty)} f(x-z)g(z)d^n z \\
&= \int_{z(\infty)}^{z(-\infty)} f(x-z)g(z)d^n z \\
&= \int_{\mathbb{R}^n} g(z)f(x-z)d^n z \\
&= (g * f)(x)
\end{aligned}$$

② **Associative**

This follows by using Fubini's Theorem, which allows us to exchange the order of integration. Check this StackExchange post for an explicit demonstration.

③ **Identity**

$$(f * \delta)(x) = \int_{\mathbb{R}^n} f(y)\delta(x-y)d^n y$$

Now, δ is characterised by "spiking" to infinity around its axis of symmetry, and being 0 elsewhere. In other words, for some $\varepsilon > 0$, we will have:

$$\int_{\mathbb{R}^n} f(y)\delta(x-y)d^n y = f(x)\int_{x-\varepsilon}^{x+\varepsilon} \delta(x-y)d^n y = f(x)$$

where we use the fact that δ integrates to 1, and that f will be "constant" for $y \in [x-\varepsilon, x+\varepsilon]$.

□

2.3 Solving the Global Cauchy Problem via the Fundamental Solution

2.3.1 Proposition: Differentiation Under the Integral

Let $I(a,b)$ be a function on $\mathbb{R} \times \mathbb{R}$, and let $b_0 \in \mathbb{R}$. Then if:

1. *$\forall b$ in a neighbourhood of b_0*

$$\int_{\mathbb{R}} |I(a,b)|da < \infty$$

2. *there exists a neighbourhood $\mathcal{N}$ of b_0 such that for **almost every** a $\partial_b I(a,b)$ exists for $b \in \mathcal{N}$ (that is, the derivative at b is undefined at countably many points)*

3. *there exists a function $U(a)$ (defined for almost every a) such that if $b \in \mathcal{N}$:*

$$|\partial_b I(a,b)| \leq U(a) \qquad \int_{\mathbb{R}} U(a)da < \infty$$

Then the function:

$$J(b) = \int_{\mathbb{R}} I(a,b)da$$

*is **differentiable** near b_0, and:*

$$\partial_b J(b) = \int_{\mathbb{R}} \partial_b I(a,b)da$$

The same applies if $I(a,b) \in \mathbb{R}^m \times \mathbb{R}^n$.

2.3.2 Theorem: The Fundamental Solution & Solving the Homogeneous Heat Equation with Global Cauchy Conditions

Let $g(x)$ be a continuous function in $\mathbb{R}^n$, such that:

$$\exists a, b > 0, \qquad |g(x)| < ae^{b|x|^2}$$

Then, there exists a solution $u(t,x)$ to the ***homogeneous heat equation****:*

$$u_t - D\Delta u = 0, \qquad t \in [0,T), x \in \mathbb{R}^n$$
$$u(0,x) = g(x), \qquad x \in \mathbb{R}^n$$

where:

$$T = \frac{1}{4Db}$$

In particular:

$$\begin{aligned} u(t,x) &= [g(\xi) * \Gamma_D(t,\xi)](x) \\ &= \int_{\mathbb{R}^n} g(y)\Gamma_D(t,x-y)d^n y \\ &= \frac{1}{(4\pi Dt)^{n/2}} \int_{\mathbb{R}^n} g(y) e^{-\frac{|x-y|^2}{4Dt}} d^n y \end{aligned}$$

where $u(t,x)$ is ***infinitely differentiable*** *on the set $(0,T) \times \mathbb{R}^n$. Moreover, for each* ***compact subinterval*** *$[0,T'] \subset [0,T)]$:*

$$\exists A, B > 0 : \forall (t,x) \in [0,T'] \times \mathbb{R}^n \quad |u(t,x)| \leq Ae^{B|x|^2}$$

and $u(t,x)$ is the ***unique*** *solution in the class of functions verifying this bound.*

The above demonstrates 2 interesting properties of $u(t,x)$:

1. *u is **smooth**, even if g is just **continuous***
2. *the solution:*

$$u(t,x) = \frac{1}{(4\pi Dt)^{n/2}} \int_{\mathbb{R}^n} g(y) e^{-\frac{|x-y|^2}{4Dt}} d^n y$$

   ***propagates at infinite speed**: at $t = 0$ we can think of the solution as being "concentrated" at $g(x)$; as soon as $t > 0$, u takes on non-zero values everywhere on $\mathbb{R}^n$ (thus, we have information on what happens at every point in the plane).*

Proof. This proof has 3 steps:

1. Verifying the bound on $u(t,x)$
2. Verifying that $u(t,x)$ solves the homogeneous heat equation
3. Verifying that $u(t,x)$ satisfies the initial condition $u(0,x) = g(x)$

We shall only do this for when $n = 1$ dimension, when $x \in \mathbb{R}$.

(1) **Bounded Solution**

Notice:

$$(|x| - |y|)^2 = x^2 - |2xy| + y^2$$

Since $(|x| - |y|)^2 \geq 0$ this implies:

$$x^2 + y^2 \geq |2xy|$$

Furthermore, setting $x = x\sqrt{\varepsilon^{-1}}, y = y\sqrt{\varepsilon}$ for $\varepsilon > 0$ we get:

$$|2xy| = \varepsilon^{-1}x^2 + \varepsilon y^2$$

Moreover:

$$|x-y|^2 = (x-y)^2 = x^2 - 2xy + y^2 \geq 0$$

Since this is always positive, we must have that:

$$(x-y)^2 \leq x^2 + |2xy| + y^2$$

(if $xy \leq 0$, then we have equality, and if $xy < 0$, then we are subtracting a positive amount, so the inequality holds). Hence, we can bound:

$$|x-y|^2 \leq x^2 + \varepsilon^{-1}x^2 + \varepsilon y^2 + y^2 = (1+\varepsilon^{-1})x^2 + (1+\varepsilon)y^2$$

Now, using the bound on g:

$$|g(x)| < ae^{b|x|^2}$$

we get that:

$$
\begin{aligned}
|g(x-y)| &< ae^{b|x-y|^2} \\
&\leq ae^{b((1+\varepsilon^{-1})x^2+(1+\varepsilon)y^2)} \\
&= ae^{b(1+\varepsilon^{-1})x^2}e^{b(1+\varepsilon)y^2}
\end{aligned}
$$

Then we can estimate u via:

$$
\begin{aligned}
|u(t,x)| &= |[g(\xi) * \Gamma_D(t,\xi)](x)| \\
&= |[\Gamma_D(t,\xi) * g(\xi)](x)| \\
&= \left| \int_{\mathbb{R}} \Gamma_D(t,y) g(x-y) dy \right| \\
&\leq \int_{\mathbb{R}} \Gamma_D(t,y)|g(x-y)|dy \\
&\leq \int_{\mathbb{R}} \Gamma_D(t,y) ae^{b(1+\varepsilon^{-1})x^2}e^{b(1+\varepsilon)y^2}dy \\
&= \frac{1}{(4\pi Dt)^{1/2}} ae^{b(1+\varepsilon^{-1})x^2} \int_{\mathbb{R}} e^{-\frac{y^2}{4Dt}} e^{b(1+\varepsilon)y^2}dy \\
&= \frac{1}{(4\pi Dt)^{1/2}} ae^{b(1+\varepsilon^{-1})x^2} \int_{\mathbb{R}} e^{-y^2\left[\frac{1}{4Dt}-b(1+\varepsilon)\right]}dy
\end{aligned}
$$

Now, define a new variable z via:

$$
z^2 = y^2\left[\frac{1}{4Dt} - b(1+\varepsilon)\right] = t^{-1}y^2\left[\frac{1}{4D} - bt(1+\varepsilon)\right]
$$

Then:

$$
2z\frac{dz}{dy} = 2t^{-1}y\left[\frac{1}{4D} - bt(1+\varepsilon)\right] \implies \frac{dz}{dy} = \frac{t^{-1}y\left[\frac{1}{4D} - bt(1+\varepsilon)\right]}{t^{-1/2}y\left[\frac{1}{4D} - bt(1+\varepsilon)\right]^{\frac{1}{2}}} = t^{-1/2}\left[\frac{1}{4D} - bt(1+\varepsilon)\right]^{\frac{1}{2}}
$$

so our estimate becomes:

$$
\begin{aligned}
|u(t,x)| &\leq \frac{1}{(4\pi Dt)^{1/2}} ae^{b(1+\varepsilon^{-1})x^2} \int_{\mathbb{R}} e^{-y^2\left[\frac{1}{4Dt}-b(1+\varepsilon)\right]}dy \\
&\leq \frac{1}{(4\pi Dt)^{1/2}} ae^{b(1+\varepsilon^{-1})x^2} \int_{\mathbb{R}} e^{-z^2} t^{1/2}\left[\frac{1}{4D} - bt(1+\varepsilon)\right]^{-\frac{1}{2}} dz \\
&= \frac{a}{(4\pi D)^{1/2}}\left[\frac{1}{4D} - bt(1+\varepsilon)\right]^{-\frac{1}{2}} e^{b(1+\varepsilon^{-1})x^2}\sqrt{\pi} \\
&= Ae^{Bx^2}
\end{aligned}
$$

as required.

Proving that u is the unique function to satisfy the bound is more challenging.

---

② u is a Homogeneous Solution

Let $\mathcal{L}$ be the operator for the heat equation:

$$\mathcal{L} = \partial_t - D\partial_x^2$$

Recall, u solves the heat equation if:

$$\mathcal{L}[u(t,x)] = 0, \qquad t > 0, x \in \mathbb{R}$$

Since $\Gamma_D(t,x)$ is a the fundamental solution, then:

$$\mathcal{L}[\Gamma_D(t,x)] = 0$$

By the proposition on differentiating under the integral, with $I = \Gamma_D(t,x)$ (we can use the fact that $\Gamma_D(t,x)$ contains a rapidly decaying exponential to prove the boundedness requirements) it follows that:

$$\mathcal{L}[u(t,x)] = \int_{\mathbb{R}} g(y)\mathcal{L}[\Gamma_D(t,x)]dy = 0$$

since $\mathcal{L}[\Gamma_D(t,x)] = 0$. Thus, u solves the heat equation.

Moreover, $u \in C^\infty\left((0, T = \frac{1}{4Db}) \times \mathbb{R}\right)$ is obtained by repeatedly differentiating with respect to t and x under the integral, since $\Gamma_D(t,x)$ is infinitely differentiable in both variables.

---

(3) u **Satisfies the Initial Conditions**

We now have to show that:

$$\lim_{t\to 0^+} u(t,x) = g(x)$$

This follows from the property of $\Gamma_D(t,x)$ of behaving like δ as $t \to 0^+$, and that δ behaves like the identity under convolution. Explicitly:

$$\lim_{t\to 0^+} u(t,x) = \lim_{t\to 0^+} [g(\xi) * \Gamma_D(t,\xi)](x) = [g * \delta](x) = g(x)$$

so $u(t,x)$ satisfies the initial condition.

□

2.3.3 Theorem: Duhamel's Principle

Let $g(x)$ and $T = \frac{1}{4Db}$ be defined as in the theorem above. Moreover, assume that:

$$f(t,x) \qquad \partial_i f(t,x) \qquad \partial_i \partial_j f(t,x), \qquad 1 \leq i,j \leq n$$

*are **continuous, bounded** function on $[0,T) \times \mathbb{R}^n$.*
*Then, there exists a **unique solution** $u(t,x)$ to the **inhomogeneous heat equation**:*

$$u_t - D\Delta u = f(t,x), \qquad t \in [0,T), x \in \mathbb{R}^n$$

$$u(0,x) = g(x), \qquad x \in \mathbb{R}^n$$

Furthermore, $u(t,x)$ is given by:

$$u(t,x) = [\Gamma_D(t,\xi) * g)](x) + \int_0^t [\Gamma_D(t-s,\nu) * f(s,\nu))](x)ds$$

and:

$$u \in C^0([0,T) \times \mathbb{R}) \cap C^{1,2}((0,T) \times \mathbb{R}^n)$$

2.4 Deriving the Fundamental Solution

2.4.1 Lemma: Solutions to the Heat Equation are Invariant Under Translations and Parabolic Dilations

*Let $u(t,x)$ be a solution to the **homogeneous heat equation**:*

$$u_t - D\Delta u = 0, \qquad (t,x) \in [0,\infty) \times \mathbb{R}^n$$

Let:

$$A, t_0 \in \mathbb{R} \qquad x_0 \in \mathbb{R}^n$$

Then:

$$u^*(t,x) = Au(t-t_0, x-x_0)$$

*is also a solution to the **homogeneous heat equation**.*
*Similarly, if $\lambda > 0$, the **amplified, parabolically scaled** function:*

$$u^*(t,x) = Au(\lambda^2 t, \lambda x)$$

is also a solution.

---

(1) **Amplified + Translated**

Let $u(x,t)$ be a solution tot he homogeneous heat equation, and consider:

$$u^*(t,x) = Au(t-t_0, x-x_0)$$

Then applying the chain rule:

$$u_t^* - \Delta u_{xx}^* = A(u_t(t-t_0, x-x_0) - \Delta u_{xx}(t-t_0, x-x_0)) = 0$$

(2) **Amplified + Parabolically Scaled**

Let $u(x,t)$ be a solution tot he homogeneous heat equation, and consider:

$$u^*(t,x) = Au(\lambda^2 t, \lambda x)$$

Then applying the chain rule:

$$u_t^* - \Delta u_{xx}^* = A(\lambda^2 u_t(\lambda^2 t, \lambda x) - \lambda^2 \Delta u_{xx}(\lambda^2 t, \lambda x)) = 0$$

2.4.2 Lemma: Total Thermal Energy is Constant for Solutions to the Heat Equation

Let:

$$u(t,x) \in C^{1,2}([0,\infty) \times \mathbb{R}^n)$$

be a solution to the heat equation:

$$u_t = \Delta u$$

Assume that for fixed t:

$$\lim_{|x|\to\infty} |x|^{n-1}|\nabla_x u(t,x)| = 0$$

uniformly *in x.*
Furthermore, that there exists a function:

$$f(x) \geq 0$$

such that:

$$|u_t| \leq f(x) \qquad \int_{\mathbb{R}^n} f(x) d^n x < \infty$$

Then, if we define the ***total thermal energy*** *as:*

$$\mathcal{T}(t) = \int_{\mathbb{R}^n} u(t,x) d^n x$$

then the ***total thermal energy*** *of u is* ***constant*** *in time:*

$$\mathcal{T}(t) = \mathcal{T}(0)$$

Proof. Let u be a solution to the homogeneous heat equation:

$$u_t = \Delta u$$

Moreover, let $B_R(0)$ be a ball of radius R centered at the origin.

By the hypotheses of the Lemma, we can differentiate the total thermal energy $\mathcal{T}$ under the integral (setting $I = u$). T:

$$\begin{aligned}
\frac{d}{dt}\mathcal{T}(t) &= \int_{\mathbb{R}^n} \frac{\partial}{\partial t} u(t,x) d^n x \\
&= \int_{\mathbb{R}^n} \frac{\partial}{\partial t} u(t,x) d^n x \\
&= \int_{\mathbb{R}^n} \Delta u d^n x \\
&= \lim_{R\to\infty} \int_{B_R(0)} \Delta u d^n x
\end{aligned}$$

Now, recall that:

$$\nabla \cdot (\nabla f) = \Delta f$$

and the **Divergence Theorem**:

> *Let $\Omega \subset \mathbb{R}^3$ be a* ***domain*** *(open, connected subset of $\mathbb{R}^n$). Denote the* ***boundary/surface*** *of Ω as $\partial\Omega$.*
> *Then:*
>
> $$\int_\Omega \nabla \cdot F(x,y,z) dx\, dy\, dz = \int_{\partial\Omega} \underline{F}(\sigma) \cdot \underline{\hat{N}}(\sigma) d\sigma$$
>
> *Here:*
>
> - $\underline{\hat{N}}(\sigma)$ *is the* ***unit outward normal vector*** *to the surface $\partial\Omega$*
> - *if $\partial\Omega \subset \mathbb{R}^3$ can be described as:*
>
> $$\partial\Omega = \{(x,y,z) \mid z = \phi(x,y)\}$$
>
>   *then:*
>
> $$d\sigma = \sqrt{1 + \|\nabla\phi(x,y)\|^2} dx\, dy = \sqrt{1 + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2} dx\, dy$$

Thus:

$$\begin{aligned}
\frac{d}{dt}\mathcal{T}(t) &= \lim_{R\to\infty} \int_{B_R(0)} \Delta u d^n x \\
&= \lim_{R\to\infty} \int_{\partial B_R(0)} \nabla u(t,\sigma) \cdot \underline{\hat{N}}(\sigma) d\sigma
\end{aligned}$$

We now apply a change of variables:

$$\sigma = R\omega \implies d\sigma = R^{n-1}d\omega$$

where ω denotes angular coordinates on the surface of the unit sphere (see this). Thus:

$$\frac{d}{dt}\mathcal{T}(t) = \lim_{R\to\infty}\int_{\partial B_R(0)} R^{n-1}\nabla u(t,R\omega)\cdot \underline{\hat{N}}(R\omega)d\omega$$

Lastly, by assumption:

$$\lim_{R\to\infty} R^{n-1}||\nabla_x u(t,R\omega)| = 0$$

uniformly, so we can bring the limit inside of the integral, thus:

$$\frac{d}{dt}\mathcal{T}(t) = 0$$

as required.

□

2.4.3 Derivation of the Fundamental Solution

We now derive the Fundamental Solution in 2 spacetime dimensions (so $x \in \mathbb{R}$).

Say we have:

$$u(t,x) = Au(D^2\lambda^2 t, \lambda x)$$

We want that the total thermal energy of u and u^* are equal. We compute:

$$\begin{aligned}\int_{\mathbb{R}^n} u^*(t,x)d^n x &= \int_{\mathbb{R}^n} u(D^2\lambda^2 t, \lambda x)d^n x \\ &= A\lambda^{-n}\int_{\mathbb{R}^n} u(D^2\lambda^2 t, z)d^n z\end{aligned}$$

Since we don't care about time in the integral, if we choose:

$$A = \lambda^n$$

we will have that:

$$u^*(t,x) = \lambda^n u^*(D^2\lambda^2 t, \lambda x)$$

has the same thermal energy as u.

We begin by introducing the **dimensionless** variable:

$$\zeta = \frac{x}{\sqrt{Dt}}$$

which is **invariant** under parabolic scaling:

$$t \mapsto \lambda^2 t \qquad x \mapsto \lambda x$$

We look for a fundamental solution of the form:

$$\Gamma_D(t,x) = \frac{1}{\sqrt{Dt}}V(\zeta)$$

We aim to find V, by exploiting the properties that a fundamental solution should have.

① **Integral of 1**

The fundamental solution satisfies:

$$\int_{\mathbb{R}^n} \Gamma_D(t,x) d^n x = 1$$

so:

$$\begin{aligned} 1 &= \int_{\mathbb{R}} \frac{1}{\sqrt{Dt}} V\left(\frac{x}{\sqrt{Dt}}\right) dx \\ &= \int_{\mathbb{R}} V(\zeta) d\zeta \end{aligned}$$

② **Satisfies Heat Equation**

If $\Gamma_D(t,x)$ is to satisfy the heat equation, then:

$$\Gamma_t - D\Gamma_{xx} = 0$$

If we compute these derivatives:

$$\begin{aligned} \Gamma_t &= \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{Dt}}\right) V(\zeta) + \frac{1}{\sqrt{Dt}} \frac{dV}{d\zeta} \frac{\partial \zeta}{\partial t} \\ &= -\frac{D}{2(Dt)^{3/2}} V(\zeta) - \frac{1}{\sqrt{Dt}} V'(\zeta) \frac{Dx}{2(Dt)^{3/2}} \\ &= -\frac{1}{\sqrt{D}t^{3/2}} \left(\frac{1}{2} V(\zeta) + \frac{1}{2} \zeta V'(\zeta)\right) \end{aligned}$$

$$\begin{aligned} \Gamma_x &= \frac{1}{\sqrt{Dt}} \frac{dV}{d\zeta} \frac{\partial \zeta}{\partial x} \\ &= \frac{1}{Dt} V'(\zeta) \end{aligned}$$

$$\begin{aligned} \Gamma_{xx} &= \frac{1}{Dt} \frac{d^2 V}{d\zeta^2} \frac{\partial \zeta}{\partial x} \\ &= \frac{1}{(Dt)^{3/2}} V''(\zeta) \end{aligned}$$

So:

$$\begin{aligned} & \Gamma_t - D\Gamma_{xx} = 0 \\ \iff & -\frac{1}{\sqrt{D}t^{3/2}} \left(\frac{1}{2} V(\zeta) + \frac{1}{2} \zeta V'(\zeta)\right) - \frac{D}{(Dt)^{3/2}} V''(\zeta) = 0 \\ \iff & -\frac{1}{\sqrt{D}t^{3/2}} \left(V''(\zeta) + \frac{1}{2} \zeta V'(\zeta) + \frac{1}{2} V(\zeta)\right) = 0 \\ \iff & V''(\zeta) + \frac{1}{2} \zeta V'(\zeta) + \frac{1}{2} V(\zeta) = 0 \end{aligned}$$

All this also tells us that if $V(\zeta)$ is a solution to the heat equation, so is $V(-\zeta)$ (it leads to the same equation). Thus, we might as well search for an even V. But if V is even, then.

$$V'(0) = 0$$

(V is symmetric about the y-axis, so it must have a critical point at the origin)

With this information, we can now deduce a suitable V. Notice, we can write the ODE above as:

$$\frac{d}{d\zeta}\left(V'(\zeta) + \frac{1}{2}\zeta V(\zeta)\right) = 0$$

which implies that $\exists k \in \mathbb{R}$ such that:

$$V'(\zeta) + \frac{1}{2}\zeta V(\zeta) = k$$

But if $\zeta = 0$, since $V'(0) = 0$, it follows that:

$$k = 0$$

so V satisfies the first-order ODE:

$$V'(\zeta) + \frac{1}{2}\zeta V(\zeta) = 0$$

Now notice that:

$$\frac{d}{d\zeta}\ln V(\zeta) = \frac{1}{V(\zeta)}V'(\zeta)$$

If we rearrange the ODE:

$$V'(\zeta) + \frac{1}{2}\zeta V(\zeta) = 0 \implies \frac{V'(\zeta)}{V(\zeta)} = -\frac{1}{2}\zeta$$

so we can rewrite it as:

$$\frac{d}{d\zeta}\ln V(\zeta) = -\frac{1}{2}\zeta$$

Integrating both sides with respect to ζ yields:

$$lnV(\zeta) = -\frac{1}{4}\zeta^2 + C \implies V(\zeta) = Ae^{-\frac{1}{4}\zeta^2}$$

To find A, we use the fact that V integrates to 1, which we found in ①:

$$\int_{\mathbb{R}} V(\zeta)d\zeta = 1$$
$$\implies \int_{\mathbb{R}} Ae^{-\frac{1}{4}\zeta^2}d\zeta = 1$$
$$\implies A\int_{\mathbb{R}} e^{-\frac{1}{4}\zeta^2}d\zeta = 1$$

Now, let:

$$u = \frac{1}{2}\zeta \implies \frac{du}{d\zeta} = \frac{1}{2}$$

so:

$$2A\int_{\mathbb{R}} e^{-u^2}du = 1 \implies 2A\sqrt{\pi} = 1 \implies A = \frac{1}{\sqrt{4\pi}}$$

Thus, we have that with $D = 1$:

$$V(\zeta) = \frac{1}{\sqrt{4\pi}}e^{-\frac{1}{4}\zeta^2} = \frac{1}{\sqrt{4\pi}}e^{-\frac{x^2}{4t}}$$

so as required:

$$\Gamma_D(t,x) = \frac{1}{\sqrt{t}}V(\zeta) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$$

3 Workshop

1. **Let:**

$$S = (0, \infty) \times (0, 1)$$

and let:

$$u \in C^{1,2}(\bar{S})$$

be the solution of the initial-boundary value problem:

$$\begin{cases} u_t - u_{xx} = 0, & (t,x) \in S \\ u(0,x) = x(1-x), \quad x \in [0,1] \\ u(t,0) = u(t,1) = 0, & t \in (0,\infty) \end{cases}$$

Show that:

$$\forall t \geq 0, x \in [0,1] \;:\; u(t,x) = u(t, 1-x)$$

We first show that $v(t,x) = u(t, 1-x)$ solves the same problem as u. Indeed it satisfies the PDE:

$$\begin{aligned} v_t - v_{xx} &= u_t(t, 1-x) - u_{xx}(t, 1-x) \\ &= u_t(t,y) - u_{xx}(t,y) \\ &= 0 \end{aligned}$$

Similarly, if $x \in [0,1]$:

$$v(0,x) = (1-x)(x - (1-x)) = x(1-x) = u(0,x)$$

and if $t \in (0, \infty)$:

$$v(t,0) = u(t,1) = 0 \qquad v(t,1) = u(t,0) = 0$$

Now, consider the parabolic boundary (we let $\Omega = (0,1)$):

$$\partial_p Q_T = \{0\} \times \bar{\Omega} \cup [0,T] \times \partial\Omega$$

By the Comparison Principle, if we can show that $u = v$ on $\partial_p Q_T$, and given that $0 = 0$, we will have that $u = v$ on all of $[0,T] \times [0,1]$. If we then take the limit as $T \to \infty$, we will get that $u(t,x) = u(t, 1-x)$ for any $t \geq 0$. But above we saw that u, v coincide when $x = 0, x = 1$ and when $t \in (0, T]$, so $u = v$ on $\partial_p Q_T$ as required.

2. **Let:**

$$S = (0, \infty) \times (0, 1)$$

and let:

$$u \in C^{1,2}(S) \cap C(\bar{S})$$

be the solution of the initial-boundary value problem:

$$\begin{cases} u_t - u_{xx} = 0, & (t,x) \in S \\ u(0,x) = x(1-x), & x \in [0,1] \\ u(t,0) = u(t,1) = k, & t \in (0,\infty), k \in \mathbb{R} \end{cases}$$

First, prove that $u(t,x) \geq 0$ for $(t,x) \in \bar{S}$. then, find all $\alpha > 0, \beta > 0$ such that on S:

$$u(t,x) \leq w(t,x) = \alpha x(1-x)e^{-\beta t}$$

Finally, show that:

$$\lim_{t \to \infty} u(t,x) = 0$$

and that the convergence is uniform for $x \in [0,1]$.

By the Weak Maximum Principle, u attains its maximum and minimum on the parabolic boundary. Notice, since u is continuous on all of $\bar{S}$:

$$\lim_{t\to 0} u(t,0) = u(0,0) = 0 \qquad \lim_{t\to 0} u(t,1) = u(0,1) = 0$$

Since $u(t,0), u(t,1)$ are constant, it thus follows that:

$$u(t,0) = u(t,1) = 0$$

Moreover:

$$u(0,x) = x(1-x) \geq 0$$

Hence, it follows that on the parabolic boundary:

$$u(t,x) \geq 0$$

so by the weak maximum principle $u(t,x) \geq =$ for $(t,x) \in \bar{S}$.

Now, consider:

$$w(t,x) = \alpha x(1-x)e^{-\beta t}$$

It will be useful to see the conditions under which w solves the heat equation, as we can then apply the maximum principle. We thus compute:

$$\begin{aligned} w_t - w_{xx} &= -\beta(\alpha x(1-x)e^{-\beta t}) - \frac{\partial}{\partial x}(\alpha e^{-\beta t}((1-x)-x)) \\ &= \alpha e^{-\beta t}\left[-\beta(x(1-x)) - \frac{\partial}{\partial x}(1-2x)\right] \\ &= \alpha e^{-\beta t}\left[2 - \beta(x(1-x))\right] \end{aligned}$$

If we can show that $w_t - w_{xx} \geq 0$, then we can apply the Comparison Principle/Maximum Principle. For this we require that:

$$2 - \beta \max\{x(1-x)\} \geq 0 \implies 2 - \frac{\beta}{4} \geq 0$$

so:

$$0 < \beta \leq 8$$

Moreover, to ensure that $w \geq u$, on the boundary with $t = 0$:

$$x(1-x) \leq \alpha x(1-x) \implies \alpha \geq 1$$

Then, by the comparison principle, we must have that $u \leq w$ on all S.

Moreover, $u \to 0$ uniformly, since:

$$\sup u \leq \sup w \leq \alpha e^{-\beta t} \to 0$$

3. **Show that the fundamental solution:**

$$\Gamma_D(t,x) = \frac{1}{(4\pi D t)^{n/2}} e^{-\frac{\|x\|^2}{4Dt}}, \qquad t > 0, \underline{x} \in \mathbb{R}^n$$

is a solution to the homogeneous heat equation:

$$u_t - D\Delta u = f$$

when $\underline{x} \in \mathbb{R}^n, t > 0$

This is a simple albeit boring computation:

$$\begin{aligned}
\Gamma_t &= \frac{\partial}{\partial t}\left((4\pi D t)^{-n/2}\right) e^{-\frac{\|\underline{x}\|^2}{4Dt}} + \frac{1}{(4\pi D t)^{n/2}} \frac{\partial}{\partial t}\left(e^{-\frac{\|\underline{x}\|^2}{4Dt}}\right) \\
&= -\frac{n}{2}\left((4\pi D t)^{-n/2-1}\right)(4\pi D) e^{-\frac{\|\underline{x}\|^2}{4Dt}} + \frac{1}{(4\pi D t)^{n/2}}\left(e^{-\frac{\|\underline{x}\|^2}{4Dt}}\right)\left(\frac{\|\underline{x}\|^2}{4Dt^2}\right) \\
&= \frac{1}{(4\pi D t)^{n/2}} e^{-\frac{\|\underline{x}\|^2}{4Dt}}\left[-\frac{n}{2t} + \frac{\|\underline{x}\|^2}{4Dt^2}\right] \\
&= \Gamma_D(t, \underline{x})\left[-\frac{n}{2t} + \frac{\|\underline{x}\|^2}{4Dt^2}\right]
\end{aligned}$$

For the derivative with respect to x^i:

$$\begin{aligned}
\Gamma_{x^i} &= \frac{1}{(4\pi D t)^{n/2}} \frac{\partial}{\partial x^i}\left(e^{-\frac{\|\underline{x}\|^2}{4Dt}}\right) \\
&= \Gamma_D(t, \underline{x})\left[-\frac{2x^i}{4Dt}\right] \\
\implies \Gamma_{x^i x^i} &= \frac{\partial}{\partial x^i}\left(\Gamma_D(t, \underline{x})\right)\left[-\frac{2x^i}{4Dt}\right] + \Gamma_D(t, \underline{x}) \frac{\partial}{\partial x^i}\left(-\frac{2x^i}{4Dt}\right) \\
&= \Gamma_D(t, \underline{x})\left[\frac{2x^i}{4Dt}\right]^2 - \frac{2}{4Dt}\Gamma_D(t, \underline{x}) \\
&= \Gamma_D(t, \underline{x})\left[\frac{(x^i)^2}{4D^2t^2} - \frac{1}{2Dt}\right] \\
\implies D\Delta\Gamma &= \Gamma_D(t, \underline{x})\left[\frac{\|\underline{x}\|}{4Dt^2} - \frac{n}{2t}\right]
\end{aligned}$$

so as required $\Gamma_t = D\Delta\Gamma$

4. **Find an explicit formula for the solution of the global Cauchy problem:**

$$\begin{cases} u_t = Du_{xx} + bu_x + cu, & x \in \mathbb{R}, t > 0 \\ u(0, x) = g(x), & x \in \mathbb{R} \end{cases}$$

where D, b, c **are constant coefficients. Show that if** $c < 0$ **and** g **is bounded:**

$$\lim_{t \to \infty} u(t, x) = 0$$

You might want to pick h, k **such that:**

$$v(x, t) = u(x, t)e^{hx+kt}$$

solves the heat equation:

$$v_t = Dv_{xx}$$

By the hint, define:

$$v(x, t) = u(x, t)e^{hx+kt}$$

We computes its partial derivatives:

$$v_t = kue^{hx+kt} + u_t e^{hx+kt} = e^{hx+kt}(ku + u_t)$$

$$v_x = hue^{hx+kt} + u_x e^{hx+kt} = e^{hx+kt}(hu + u_x)$$

$$v_{xx} = he^{hx+kt}(hu + u_x) + e^{hx+kt}(hu_x + u_{xx}) = e^{hx+kt}(h^2 u + 2hu_x + u_{xx})$$

In particular, if we want $v_t = Dv_{xx}$, we require that:

$$\begin{aligned} &ku + u_t = D(h^2 u + 2hu_x + u_{xx}) \\ \implies \quad &u_t = Du_{xx} + (2Dh)u_x + (Dh^2 - k)u \end{aligned}$$

But u solves the PDE:

$$u_t = Du_{xx} + bu_x + cu$$

so in particular:

$$b = 2Dh \implies h = \frac{b}{2D}$$

$$c = Dh^2 - k \implies k = Dh^2 - c = \frac{b^2 - 4Dc}{4D}$$

Now, v solves the heat equation, and:

$$v(0,x) = g(x)\exp\left(\frac{bx}{2D}\right) \implies |v(0,x)| \leq \sup_{x\in\mathbb{R}} |g| \exp\left(\frac{bx}{2D}\right)$$

Hence, since g is bounded, it follows that by the representation Theorem, we can write:

$$v(t,x) = g(x)\exp\left(\frac{bx}{2D}\right) * \Gamma_D(t,x) = \int_{\mathbb{R}} g(y)\exp\left(\frac{by}{2D}\right)\Gamma(x-y,t)dy$$

But then, since $v = ue^{hx+kt}$:

$$u(t,x) = \exp\left(-\left[\frac{bx}{2D} + \frac{(b^2-4Dc)t}{4D}\right]\right)\int_{\mathbb{R}} g(y)\exp\left(\frac{by}{2D}\right)\Gamma(x-y,t)dy$$

Now, provided that $c < 0$, then:

$$\exp\left(-\left[\frac{bx}{2D} + \frac{(b^2-4Dc)t}{4D}\right]\right) \to 0$$

(since $b^2 - 4Dc$ will be positive).

Using the exponential bound for $\int_{\mathbb{R}} g(y)\exp\left(\frac{by}{2D}\right)\Gamma(x-y,t)dy$ as a solution to the heat equation, we can then see that as required $u(t,x) \to 0$.

5. **In this problem you will consider PDEs on the set $(t,x) \in [0,\infty) \times \mathbb{R}^n$. You may assume that all of the functions involved are sufficiently differentiable. Let $\mathcal{L}$ be a linear differential operator of the form:**

$$\mathcal{L} = \partial_t - \Delta$$

Suppose that we want to solve the inhomogenous problem:

$$\begin{cases} \mathcal{L}u = f(t,\underline{x}), & \underline{x} \in \mathbb{R}^n, t > 0 \\ u(0,\underline{x}) = 0, & \underline{x} \in \mathbb{R}^n \end{cases}$$

Show that the inhomogenous problem is oslved by:

$$v(t,x) = \int_{s=0}^{s=t} v_s(t-x,x)dx$$

where each v_s solves the following homogeneous IVP:

$$\begin{cases} \mathcal{L}v_s = 0, & \underline{x} \in \mathbb{R}^n, t > 0 \\ v_s(0,\underline{x}) = f(t,x), & \underline{x} \in \mathbb{R}^n \end{cases}$$

v is known as *Duhamel's Principle.*

6. **Let u be a $C^2(\Omega) \cap C(\bar{\Omega})$ solution to the elliptic equation:**

$$a(x,y)u_{xx} + b(x,y)u_{yy} = 0$$

in a bounded domain $\Omega \subset \mathbb{R}^2$. Here, a, b are continuous, positive functions in Ω. Prove that:

$$\max_{\Omega} u = \max_{\partial\Omega} u$$

***Hint*: Consider the auxiliary function:**

$$w(x,y) = u(x,y) + \varepsilon((x-x_0)^2 + (y-y_0)^2)$$

near a possible maximum points $(x_0, y_0) \in \Omega$, where $\varepsilon > 0$. Show that:

$$u(x,y) \le \max_{\partial\Omega} u + \varepsilon C_0, \qquad (x,y) \in \Omega$$

where $C_0 > 0$ is independent of ε.

Let

$$w(x,y) = u(x,y) + \varepsilon((x-x_0)^2 + (y-y_0)^2)$$

. Define:

$$C(x,y) = ((x-x_0)^2 + (y-y_0)^2)$$

We now compute:

$$\begin{aligned} a(x,y)w_{xx} + b(x,y)w_{yy} &= a(x,y)\frac{\partial^2}{\partial x^2}(u(x,y) + \varepsilon((x-x_0)^2 + (y-y_0)^2)) \\ &+ b(x,y)\frac{\partial^2}{\partial y^2}(u(x,y) + \varepsilon((x-x_0)^2 + (y-y_0)^2)) \\ &= a(x,y)u_{xx} + 2a(x,y)\varepsilon + b(x,y)u_{yy} + 2b(x,y)\varepsilon \\ &= 2\varepsilon(a(x,y) + b(x,y)) \end{aligned}$$

But notice $\varepsilon, a(x,y), b(x,y) > 0$, so:

$$a(x,y)w_{xx} + b(x,y)w_{yy} > 0$$

Now, assume $\exists \underline{x}_M = (x_M, y_M) \in \Omega$, such that $\underline{x}_M$ is a maximum of w. Then, consider the Hessian:

$$H = \begin{pmatrix} w_{xx} & w_{xy} \\ w_{yx} & w_{yy} \end{pmatrix}$$

Since $\underline{x}_M$ is a **maximum**, it follows by the Second Derivative Test that at $\underline{x}_M$:

$$|H(x_M, y_M)| = w_{xx}(x_M, y_M)w_{yy}(x_M, y_M) - (w_{xy}(x_M, y_M))^2 > 0$$

and:

$$w_{xx}(x_M, y_M) < 0 \qquad w_{yy}(x_M, y_M) < 0$$

However, this would imply that:

$$a(x_m, y_m)w_{xx}(x_M, y_M) + b(x_m, y_m)w_{yy}(x_M, y_M) < 0$$

since $a, b > 0$. This contradicts the fact that $a(x, y)w_{xx} + b(x, y)w_{yy} > 0$, so no such (x_M, y_M) must exist.

Hence, if w attains a maximum, it must be within $\partial\Omega$. Thus:

$$w(x, y) \leq \max_{\Omega} w = \max_{\Omega} (u + \varepsilon C(x, y)) \leq \left(\max_{\Omega} u\right) + \varepsilon C_0 \leq \left(\max_{\partial\Omega} u\right) + \varepsilon C_0$$

where we have defined:

$$C_0 = \max_{\Omega} ((x - x_0)^2 + (y - y_0)^2)$$

Here we have used the Ω is a **bounded** domain to ensure that we have a bound on the ε term.

Moreover, we have that on Ω:

$$w(x, y) = u(x, y) + \varepsilon((x - x_0)^2 + (y - y_0)^2) \implies w(x, y) \geq u(x, y)$$

with equality at (x_0, y_0). Thus, it follows that:

$$\max_{\Omega} u \leq \max_{\Omega} w \leq \max_{\partial\Omega} u + \varepsilon C_0$$

But since ε is an arbitrary positive constant, and $C_0 > 0$ is independent of ε, in particular it must be the case that as $\varepsilon \to 0$ we obtain the desired equality:

$$\max_{\Omega} u = \max_{\partial\Omega} u$$