Introduction to Partial Differential Equations - Week 2+3 - The Heat Equation

Antonio León Villares

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1 Deriving the Heat Equation

Let $u(t, \underline{x}), \underline{x} \in \mathbb{R}^n$. The ***heat equation*** *for u is:*

$$u_t - D\Delta u = f(t, \underline{x})$$

where:

- *$D > 0$:* ***diffusion coefficient***
- *Δ:* ***Laplacian operator****:*

$$\Delta = \sum_{i=1}^{n} \partial_i^2$$

1.1 Energy Through a Body

- **What 2 physical properties define heat transmission through a body?**
  - consider a body $\mathcal{B} \subset \mathbb{R}^n$
  - the 2 key factors affecting heat flow through $\mathcal{B}$ are:
    * the (mass) **density**: typically, as density increases, heat conductivity decreases

      $$\rho \sim [mass] \times [Volume]^{-1}$$

    * the **thermal energy per unit mass**: how much thermal energy is stored within the body, at a given time and position

      $$e(t, \underline{x}) \sim [energy] \times [mass]^{-1}$$

  - we also have to assume that the heat is supplied by some external source, at a given rate per unit mass:

    $$\mathcal{R}(t, \underline{x}) \sim [energy] \times [time]^{-1} \times [mass]^{-1}$$

- **How can we compute the total thermal energy within a body?**
  - let $E(t, V)$ denote the **energy** contained in some volume $V \subset \mathcal{B}$ of the body
  - for some infinitesimal volume $\delta x^1 \delta x^2 \dots \delta x^n$, we can assume that the thermal energy per unit mass remains constant; then, in this sub-volume, the thermal energy is:

    $$\rho e(t\underline{x}) \delta x^1 \delta x^2 \dots \delta x^n$$

  - hence, for the **whole** volume, we must have:

    $$E(t, V) = \int_V \rho e(t, \underline{x}) d^n \underline{x}$$

    (this is compressed notation for a set of n integrals, one for each of the variables x^i)

- **What is the rate of change of thermal energy within a body?**
  - we just differentiate $E(t, V)$
  - assuming E is "nice" and well-behaved, we can differentiate **within** the integral:

    $$\frac{d}{dt} E(t, V) = \frac{d}{dt} \int_V \rho e(t, \underline{x}) d\underline{x} = \int_V \rho \partial_t e(t, \underline{x}) d^n \underline{x}$$

1.2 Changes in Energy Through a Body

- **What 2 factors determine how heat flows through the body?**
  - we have that:
$$\int_V \rho \partial_t e(t, \underline{x}) d^n \underline{x}$$
    defines heat flow through $\mathcal{B}$
  - we now focus on the 2 factors which we (assume) affects this value:
    * the **rate** at which the **external source** supplies energy
    * the **distribution** and **flow** of heat **within** the body
- **How do we compute the the rate of energy transfer?**
  - in some infinitesimal volume, the total energy transfer rate per volume is:
$$\rho \mathcal{R}(t, \underline{x})$$
  - hence, the **total** rate of energy transfer by the external source is obtained by integrating over V:
$$\int_V \rho \mathcal{R}(t, \underline{x}) d^n \underline{x} \sim [energy] \times [time]^{-1}$$
- **How do we model the flow of heat through the body?**
  - let $\underline{q}$ be a **heat flux vector**, defining how heat flows through the body:
$$\underline{q} \sim [energy] \times [time]^{-1} \times [area]^{-1}$$
  - consider the surface ∂V of the volume; if $\underline{\hat{N}}$ is a **unit** outward vector to $d\sigma \subset \partial V$, then the flow of heat **in the direction of $\underline{\hat{N}}$ is:**
$$\underline{q} \cdot \underline{\hat{N}}$$
  - thus, the total heat going **in** to the body through the surface of V is:
$$-\int_{\partial V} \underline{q} \cdot \underline{\hat{N}} d\sigma \sim [energy] \times [time]^{-1}$$
  - if we apply the **Divergence Theorem**:
$$-\int_{\partial V} \underline{q} \cdot \underline{\hat{N}} d\sigma = -\int_V \nabla \cdot \underline{q} d^n \underline{x}$$
- **What relationship must heat flowing through a body satisfy?**
  - we assume that:

> *The rate of change of **total energy** in the sub-volume V is **equal** to the rate of heat energy flowing into V + the rate of heat energy **supplied** by the external source.*

  - we then have the following relationship:
$$\int_V \rho \partial_t e(t, \underline{x}) d^n \underline{x} = -\int_V \nabla \cdot \underline{q} d^n \underline{x} + \int_V \rho \mathcal{R}(t, \underline{x}) d^n \underline{x}$$
  - this should hold for **any** sub-volume V so:
$$\rho \partial_t e(t, \underline{x}) = -\nabla \cdot \underline{q} + \rho \mathcal{R}(t, \underline{x})$$

1.3 Fourier's Law

- **What is Fourier's law?**
  - to convert the above into a PDE, we need to assume a relationship between $\underline{q}$ (the heat flux) and $u(t, \underline{x})$ (the temperature)
  - this is **Fourier's Law**:
  $$\underline{q}(t, \underline{x}) = -\kappa \nabla u(t, \underline{x})$$
  - κ is the **thermal conductivity**
- **What is the physical interpretation of Fourier's Law?**
  - $\nabla u(t, \underline{x})$ is the **gradient vector**, which points in the direction of **greatest** temperature increase
  - thus, **Fourier's Law** states that the heat flux is in the direction of greatest temperature increase, which is perpendicular to level sets of $u(t, \underline{x})$ (that is, surfaces of **constant** temperature, $u(t, \underline{x}9 = c)$
  - heat also flows from **hot** to **cold** (hence the minus sign)
- **Is Fourier's Law a fundamental law of nature?**
  - no, it is just a simple, reasonable assumption of how heat flow works

1.4 The Heat Equation

- **How can we relate thermal energy distribution through the body $e(t, \underline{x})$ with the temperature of the body $u(t, \underline{x})$?**
  - we assume a simple model:
  $$e = c_v u$$
  - $c_v > 0$ is the **specific heat at constant volume**
  - again, this is just a reasonable assumption, not a law of nature

We have all the ingredients:

- $\rho \partial_t e(t, \underline{x}) = -\nabla \cdot \underline{q} + \rho \mathcal{R}(t, \underline{x})$
- $\underline{q}(t, \underline{x}) = -\kappa \nabla u(t, \underline{x})$
- $e(t, \underline{x}) = c_v u(t, \underline{x})$

Thus:

$$\begin{aligned}
& \rho \partial_t e(t, \underline{x}) = -\nabla \cdot \underline{q} + \rho \mathcal{R}(t, \underline{x}) \\
\implies & \rho c_v u_t(t, \underline{x}) = -\nabla \cdot (-\kappa \nabla u(t, \underline{x})) + \rho \mathcal{R}(t, \underline{x}) \\
\implies & u_t(t, \underline{x}) = \frac{\kappa}{\rho c_v} \Delta u(t, \underline{x}) + \frac{\mathcal{R}(t, \underline{x})}{c_v}
\end{aligned}$$

so:

$$D = \frac{\kappa}{\rho c_v} \qquad f(t, \underline{x}) = \frac{\mathcal{R}(t, \underline{x})}{c_v}$$

2 Well-Posedness and Types of Boundary Conditions

- **What is a well-posed problem?**
  - in ODEs, given an IVP, we were always guaranteed a **unique** solution (if it existed)
  - for PDEs, this isn't always the case
  - a PDE is **well-posed** if we provide some "data", such that we obtain a **unique** solution to the PDE, and the solution depends "continuously" on the data

For the following we consider the case $n = 1$: a 1 dimensional rod, with endpoints $x \in [0, L], t \in [0, T]$, with heat equation:

$$u_t - Du_{xx} = 0$$

2.1 Cauchy Boundary Conditions

- **What is a Cauchy boundary condition?**
  - a condition defining the value of $u(t, x)$ at $t = 0$ for all positions
  - for example:

$$u(0, x) = g(x), \qquad \forall x \in [0, L]$$

2.2 Dirichlet Boundary Conditions

- **What is a Dirichlet boundary condition?**
  - a condition defining the value of $u(t, x)$ at the endpoints for all times
  - for example:

$$u(t, 0) = h_0(t) \qquad u(t, L) = h_L(t), \qquad \forall t \in [0, T]$$

2.3 Neumann Boundary Conditions

- **What is a Neumann boundary condition?**
  - a condition defining the **inward rate of heat flow** at the boundaries (as described by **Fourier's Law**)
  - for example:

$$-u_x(t, 0) = h_0(t) \qquad u_x(t, L) = h_L(t), \qquad \forall t \in [0, T]$$

2.4 Robin Boundary Conditions

- **What is a Robin boundary condition?**
  - a condition defined by a **linear combination** of Dirichlet and Neumann conditions
  - for example:

$$-u_x(t, 0) + \alpha u(t, 0) = h_0(t) \qquad u_x(t, L) + \alpha u(t, L) = h_L(t), \qquad \forall t \in [0, T]$$

2.5 Mixed Boundary Conditions

- **What is a mixed boundary condition?**
  - above we have **homogeneous** boundary conditions, since they applied to a **homogeneous** PDE, and the conditions at the endpoints were of the same type
  - a **mixed boundary condition** uses different conditions at each endpoint (for example, Dirichlet when $x = 0$, and Neumann when $x = L$)

2.6 Examples of Well-Posed Problems

The following wre ***well-posed problems****, provided we make suitable assumptions for* g, h_0, h_L.

$$\begin{cases} u_t - Du_{xx} = 0, & t \in (0,T), x \in (0,L) \\ u(0,x) = g(x), & x \in [0,L] \qquad \textit{(Cauchy data)} \\ u(t,0) = h_0(t) \qquad u(t,L) = h_L(t), & \forall t \in [0,T] \qquad \textit{(Dirichlet data)} \end{cases}$$

$$\begin{cases} u_t - Du_{xx} = 0, & t \in (0,T), x \in (0,L) \\ u(0,x) = g(x), & x \in [0,L] \qquad \textit{(Cauchy data)} \\ -u_x(t,0) = h_0(t) \qquad u_x(t,L) = h_L(t), & \forall t \in [0,T] \qquad \textit{(Neumann data)} \end{cases}$$

$$\begin{cases} u_t - Du_{xx} = 0, & t \in (0,T), x \in (0,L) \\ u(0,x) = g(x), & x \in [0,L] \qquad \textit{(Cauchy data)} \\ -u_x(t,0) + \alpha u(t,0) = h_0(t) \quad u_x(t,L) + \alpha u(t,L) = h_L(t) & \textit{(Robin data)} \end{cases}$$

3 Recap: Fourier Series

3.1 The Inner Product for Functions

Let $u(x), v(x)$ *be non-zero function, and consider* $x \in [a,b]$. *The* ***inner product*** *of* u, v *is:*

$$\langle u, v \rangle = \int_a^b u(x)v(x)\, dx$$

We say 2 u, v *are* ***orthogonal*** *if:*

$$\langle u(x), v(x) \rangle = 0$$

3.1.1 Orthogonality of sine and cosine

Consider an interval $[-L, L]$. We have the following properties:

- $$\left\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right) \right\rangle = \begin{cases} 2L, & n = m = 0 \\ L, & n = m \neq 0 \\ 0, & n \neq m \end{cases}$$

- $$\left\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right) \right\rangle = \begin{cases} L, & n = m \\ 0, & n \neq m \end{cases}$$

- $$\left\langle \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right) \right\rangle = 0$$

In particular, the $2L$-periodic functions $\sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)$ are ***mutually orthogonal****.*

3.2 Euler-Fourier Coefficients

- **What is a Fourier Series?**
  - consider a $2L$ periodic function $f(x)$
  - its **Fourier Series** is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

  - a_n, b_n are known as **Fourier coefficients**
- **How do you compute the Fourier Coefficients?**
  - we use the **inner product**, and exploit orthogonality:

$$\left\langle f(x), \cos\left(\frac{n\pi x}{L}\right) \right\rangle = a_n L \implies a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx, \qquad n \geq 0$$

$$\left\langle f(x), \sin\left(\frac{n\pi x}{L}\right) \right\rangle = b_n L \implies b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx, \qquad n \geq 1$$

3.3 Parseval's Identity

Let:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

Then:

$$\|f\|^2 = \langle f, f \rangle = L \left(\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2 \right)$$

This can be regarded as an ***infinite dimensional Pythageorean theorem****, and leads to many beautiful identities.*

3.4 Convergence of Fourier Series

Here we use the interval $[0, 1]$*, with* $L = \frac{1}{2}$*. Any periodic function on this interval will be periodic on any other interval.*

Let $f \in L^2([0,1])$; that is:

$$\|f\|^2_{L^2([0,1])} = \int_0^1 |f(x)|^2 dx < \infty$$

then $f(x)$ can be Fourier Expanded as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(2n\pi x) + b_n \sin(2n\pi x)\right)$$

where:

$$a_n = 2\int_0^1 f(x)\cos(2n\pi x)\, dx, \qquad n \geq 0$$

$$b_n = 2\int_0^1 f(x)\sin(2n\pi x)\, dx, \qquad n \geq 0$$

This infinite sum ***converges in*** *L^2:*

$$\lim_{N\to\infty} \left\| f - \left(\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(2n\pi x) + b_n \sin(2n\pi x)\right)\right)\right\|_{L^2([0,1])} = 0$$

In fact, on any subinterval $[a,b] \subset (0,1)$, the Fourier Series ***converges uniformly****:*

$$\lim_{N\to\infty} \left\| f - \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(2n\pi x) + b_n \sin(2n\pi x)\right)\right\|_{C^0([a,b])}$$

where recall:

$$\|f\|_{C^0([a,b])} = \underset{x\in[a,b]}{sup} |f(x)|$$

4 Solving the Heat Equation

4.1 Separation of Variables

- **What is the method of separation of variables?**
  - a method for solving **some** types of PDEs
  - particularly useful for **linear** PDEs
- **How can the method be applied?**
  - we assume a solution:

    $$u(x,t) = v(t)w(x)$$

  - plugging this into the PDE, hope that v, w are solutions to solvable ODEs

4.2 Worked Example: Solution to the homogeneous Heat Equation

We consider the following **homogeneous** PDE, with Dirichlet conditions:

$$\begin{cases} u_t - u_{xx} = 0, & t \in [0, R], x \in [0, 1] \\ u(0, x) = x, & x \in [0, 1] \\ u(t, 0) = 0 & u(t, 1) = 0 \end{cases}$$

4.2.1 Separation of Variables

We apply separation of variables, by assuming:

$$u(t, x) = X(x)T(t)$$

Then:

$$u_t(t, x) = X(x)T'(t)$$

$$u_{xx}(t, x) = X''(x)T(t)$$

So the PDE becomes:

$$X(x)T'(t) - X''(x)T(t) = 0 \implies \frac{T'}{T} = \frac{X''}{X}$$

But notice, the LHS is just a function of t, whilst the RHS is just a function of x. In other words, if this relation is true $\forall t, x$, this can only be possible if $\exists \lambda \in \mathbb{R}$ such that:

$$\frac{T'}{T} = \lambda = \frac{X''}{X}$$

Thus, we have 2 ODEs:

$$T' = \lambda T$$

$$X'' = \lambda X$$

These are known as **eigenvalue problems**, whereby the λ is an **eigenvalue**, whilst the functions T, X are **eigenfunctions**.

4.2.2 Solving for T

This is a standard separable ODE:

$$\frac{dT}{dt} = \lambda T \implies \int \frac{1}{T} dT = \int \lambda dt$$

So:

$$ln|T| = \lambda t + C \implies T = Ae^{\lambda t}, \quad A = e^C$$

4.2.3 Solving for X

The characteristic polynomial is:

$$p(r) = r^2 - \lambda$$

$p(r) = 0$ has solutions dependent on the value of λ, so we check each case individually.

(1) $\underline{\lambda = 0}$

Then, the ODE reduces to:

$$X'' = 0 \implies X = Bx + C, \qquad B, C \in \mathbb{R}$$

Since $T \neq 0$, the boundary conditions:

$$u(t,0) = u(t,1) = 0$$

imply that:

$$X(0) = X(1) = 0$$

But notice, this would mean that:

$$X(0) = C = 0$$
$$X(1) = B + C = 0 \implies B = 0$$

so $X(x) = 0$ - a trivial solution.

(2) $\underline{\lambda > 0}$

Then:

$$p(r) = 0 \implies r = \pm\sqrt{\lambda}$$

Letting $\mu^2 = \lambda$, it follows that solutions are of the form:

$$X(x) = Be^{\mu t} + Ce^{-\mu t}$$

By the boundary conditions:

$$X(0) = B + C = 0 \implies B = -C$$
$$X(1) = Be^{\mu} + Ce^{-mu} \implies C(e^{-\mu} - e^{\mu}) = 0 \implies C = 0$$

Again, $B = C = 0$, so $X(x) = 0$ is the trivial solution.

(3) $\underline{\lambda < 0}$

Then, letting $-\mu^2 = \lambda$:

$$p(r) = 0 \implies r = \pm i\mu$$

So solutions are of the form:

$$X(x) = B\cos(\mu x) + C\sin(\mu x)$$

By the boundary conditions:

$$X(0) = B = 0$$
$$X(1) = C\sin^{\mu} = 0 \implies \mu = m\pi, \qquad m \in \mathbb{Z}^+$$

Hence, we have infinitely many solutions of the form:

$$X_m(x) = C\sin(\mu x) = C\sin(m\pi x)$$

and we require:

$$\lambda = -\mu^2 = -m^2\pi^2$$

4.2.4 A General Solution via Fourier Series

The infinitely many solutions to the heat equation take the form:

$$u_m(t,x) = D_m e^{-m^2\pi^2 t}\sin(m\pi x), \qquad D_m \in \mathbb{R}, m \in \mathbb{Z}^+$$

Here, the D_m must be such that we can satisfy the initial condition:

$$u(0,x) = x$$

To do this, we consider **the most general solution**, which will be a linear combination of these solutions:

$$u(t,x) = \sum_{m=1}^{\infty} a_m u_m(t,x) = \sum_{m=1}^{\infty} A_m e^{-m^2\pi^2 t}\sin(m\pi x), \qquad A_m = a_m D_m$$

Then:

$$u(0,x) = x \implies \sum_{m=1}^{\infty} A_m \sin(m\pi x) = x$$

But this is a Fourier Series, so the coefficients A_m are just the Fourier coefficients:

$$A_m = 2\int_0^1 x\sin(m\pi x)dx$$

We thus compute, by applying integration by parts with:

$$u = x \qquad du = dx$$
$$dv = \sin(m\pi x) \qquad v = -\frac{1}{m\pi}\cos(m\pi x)$$

so:

$$\begin{aligned} A_m &= 2\int_0^1 x\sin(m\pi x)dx \\ &= 2\left(\left[-\frac{x}{m\pi}\cos(m\pi x)\right]_0^1 + \frac{1}{m\pi}\int_0^1 \cos(m\pi x)dx\right) \\ &= 2\left(-\frac{1}{m\pi}\cos(m\pi) + \frac{1}{m\pi}\left[\int_0^1 \sin(m\pi x)\right]_0^1\right) \\ &= \frac{2\sin(m\pi) - \cos(m\pi)}{m\pi} \\ &= \frac{2}{m\pi}(-1)^{m+1} \end{aligned}$$

where we have used the fact that:

$$\sin(n\pi) = 0, \qquad \forall n \in \mathbb{Z}$$
$$\cos(n\pi) = (-1)^n, \qquad \forall n \in \mathbb{Z}$$

Thus, our general solution to the heat equation becomes:

$$u(t,x) = \sum_{m=1}^{\infty} (-1)^{m+1} e^{-m^2\pi^2 t}\frac{2}{m\pi}\sin(m\pi x)$$

Each term $u_m = (-1)^{m+1}e^{-m^2\pi^2 t}\frac{2}{m\pi}\sin(m\pi x)$ is known as a ***mode*** *of the solution. Notice that here both the modes and $u(t,x)$ rapidly (exponentially) decay to 0 as $t \to \infty$. However, this is due to the Dirichlet conditions; different, non-zero conditions could make the solution tend to some non-zero value.*

5 Understanding the Solution to the Heat Equation

We claim:

$$u(t,x) = \sum_{m=1}^{\infty} (-1)^{m+1} e^{-m^2\pi^2 t} \frac{2}{m\pi} \sin(m\pi x)$$

is a solution to the PDE. However, we need to verify the behaviour of this solution. In particular:

1. Does the sum even converge?
2. Does the sum solve the heat equation?
3. Does $u(t,x)$ satisfy the boundary conditions?
4. Do we have $\lim_{t\to 0^+} u(t,x) = u(0,x) = x$? We have that $u(x,1) = 0 \neq 1$, so this fails pointwise; the question is whether u behaves "properly" for all the other x.
5. Is this solution unique, given the initial/boundary conditions?

5.1 Convergence of the Infinite Sum

Intuitively, the presence of $e^{-m^2\pi^2 t}$ makes it clear that for any $t > 0$, the series expression for u will converge uniformly for $x \in [0,1]$.

We verify this formally.

Recall the Weierstrass M-Test:

> *Let E be a non-empty subset of $\mathbb{R}$. Let*
>
> $$f_k : E \to \mathbb{R}$$
>
> *and suppose that $\exists M_k \geq 0$ such that:*
>
> $$\sum_{k=0}^{\infty} M_k < \infty$$
>
> *If $|f_k(x)| \leq M_k$ for all $k \in \mathbb{N}$, and $x \in E$, then:*
>
> $$f = \sum_{k=0}^{\infty} f_k$$
>
> *converges* ***absolutely*** *and* ***uniformly*** *on E.*

Let:

$$u_m = (-1)^{m+1} e^{-m^2\pi^2 t} \frac{2}{m\pi} \sin(m\pi x)$$

Then, notice that:

$$\begin{aligned}
|u_m| &= |(-1)^{m+1} e^{-m^2\pi^2 t} \frac{2}{m\pi} \sin(m\pi x)| \\
&\leq \frac{2e^{-m^2\pi^2 t}}{m\pi} \\
&< e^{-m^2\pi^2 t} \qquad \textit{since } \frac{2}{m\pi} < \frac{1}{m} \leq 1 \\
&\leq e^{-m\pi^2 t} \\
&\leq (e^{-\pi^2 t})^m
\end{aligned}$$

But notice, $\forall t > 0$, we have $e^{-\pi^2 t} < 1$, so $\sum_{m=1}^{\infty} (e^{-\pi^2 t})^m$ is a geometric series, which converges. Thus, by the Weierstrass M-Test, it follows that:

$$u(t, x) = \sum_{m=1}^{\infty} (-1)^{m+1} e^{-m^2\pi^2 t} \frac{2}{m\pi} \sin(m\pi x)$$

converges **absolutely** and **uniformly**, for $t > 0$ as required.

However, this doesn't account for the case $t = 0$, in which we have:

$$u(0, x) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{2}{m\pi} \sin(m\pi x)$$

In fact, for this all that we can strive to show is that the series converges in the L^2 sense as $t \to 0$ (since there is a disconitnuity when $t = 0$):

$$\lim_{t\to 0} \|u(t, x) - x\|_{L^2([0,1])} = 0$$

5.2 An Infinite Sum as a Solution

Each mode in the sum solves the heat equation, by construction. We now verify that the sum does.

Since the series is uniformly convergent (and other easy to check details), we can differentiate $u(t, x)$ term-by-term (for both t and x). Hence:

$$u_t = \sum_{m=1}^{\infty} \partial_t \left[(-1)^{m+1} e^{-m^2\pi^2 t} \frac{2}{m\pi} \sin(m\pi x) \right] = 2 \sum_{m=1}^{\infty} (-1)^m m\pi e^{-m^2\pi^2 t} \sin(m\pi x)$$

$$u_{xx} = \sum_{m=1}^{\infty} \partial_x^2 \left[(-1)^{m+1} e^{-m^2\pi^2 t} \frac{2}{m\pi} \sin(m\pi x) \right] = 2 \sum_{m=1}^{\infty} (-1)^m m\pi e^{-m^2\pi^2 t} \sin(m\pi x)$$

so indeed:

$$u_t - u_{xx} = 0$$

5.3 Solution Satisfies Boundary Conditions

Since each of the u_m satisfy it, and they are 0 at the endpoints, it follows that u satisfies them aswell:

$$u(t, 0) = \sum_{m=1}^{\infty} (-1)^{m+1} e^{-m^2\pi^2 t} \frac{2}{m\pi} \sin(0) = 0$$

$$u(t, 1) = \sum_{m=1}^{\infty} (-1)^{m+1} e^{-m^2\pi^2 t} \frac{2}{m\pi} \sin(m\pi) = 0$$

5.4 Limit of Solution at the Start

5.5 Uniqueness of Solution

We have solved the following Dirichlet problem:

$$\begin{cases} u_t - u_{xx} = 0, & t \in (0,T], x \in [0,1] \\ u(0,x) = x, & x \in [0,1] \\ u(t,0) = 0 & u(t,1) = 0 \end{cases}$$

with a solution:

$$u(t,x) = \sum_{m=1}^{\infty} (-1)^{m+1} e^{-m^2\pi^2 t} \frac{2}{m\pi} \sin(m\pi x)$$

*We claim that $u(t,x)$ is a **unique** solution: if someone claims that $v(t,x)$ is another solution, then we can show that:*

$$u(t,x) = v(t,x)$$

5.5.1 Definition: Spacetime Cylinder

*Let $\Omega \subset \mathbb{R}^n$ be a **bounded** spatial **domain** (an **open, connected** subset of $\mathbb{R}^n$.). Let $T > 0$ be a time.*
*Then, the corresponding **spacetime cylinder** $Q_T \subset \mathbb{R}^{1+n}$ (first dimension corresponds to **time**, remaining dimensions are **space**) is:*

$$Q_T := (0,T) \times \Omega$$

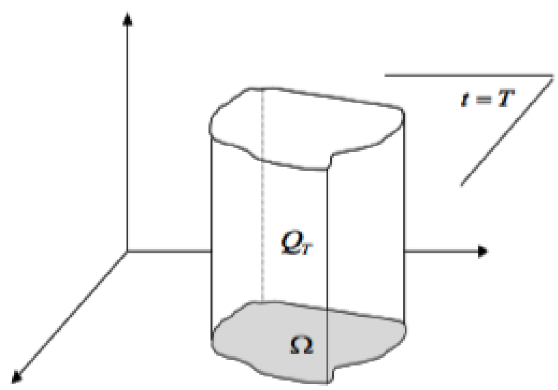


Fig. 2.3 The space-time cylinder Q_T

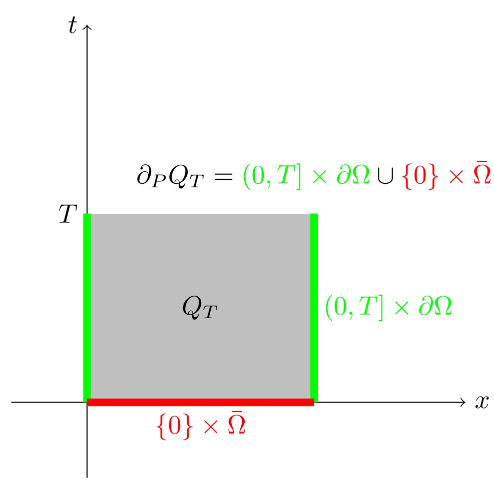
Figure 1: A spacetime cylinder in $\mathbb{R}^{1+2} = \mathbb{R}^3$. The domain Ω is a subset of $\mathbb{R}^2$. In the case of our heat equation, $\Omega \subset \mathbb{R}$, corresponding to the interval $[0, 1]$.

5.5.2 Definition: Parabolic Boundary

*The **parabolic boundary** $\partial_P Q_T$ of a **spacetime cylinder** Q_T is its bounding surface on some temporal range $(0, T]$:*

$$\partial_P Q_T := \{0\} \times \bar{\Omega} \cup (0, T] \times \partial\Omega$$

*Here $\bar{\Omega}$ denotes the closure of Ω (so the set $\Omega \cup \partial\Omega$). This is the union of our **spacetime cylinder's** bottom and sides.*



5.5.3 Theorem: Uniqueness of Heat Equation Solution on a Finite Interval

Solutions:

$$u \in C^{1,2}(\bar{Q}_T)$$

to the ***inhomogeneous heat equation****:*

$$\partial_t u - D\partial_x^2 u = f(t,x)$$

are ***unique*** *under* ***Dirichlet****,* ***Neumann****,* ***Robin*** *or* ***mixed*** *conditions. (Theorem 1.1)*

- *here u is a function which is continuously differentiable over $\bar{Q}_T$ up to order 1 in t, and up to order 2 in x*
- *technically, this theorem doesn't apply to the heat equation we have used up to now, due to the disconitnuity at $(0,1)$ (this means that strictly speaking $u(t,x) \notin C^{1,2}(\bar{Q}_T)$); addressing this issue is beyond the scope of the course.*

Proof. We show uniqueness for **Dirichlet** data, and when $D = 1$ (so $\Omega \subset \mathbb{R}$). The remaining cases are similar.

Assume there exist 2 solutions $u(t,x) \neq v(t,x)$ satisfying the Dirichlet problem. Define:

$$w = u - v$$

Then, since u, v satisfy the initial-boundary conditions, and linearity of PDEs make w a solution, it must be the case that it satisfies the following initial conditions:

$$\begin{cases} w_t - w_{xx} = 0, & t \in [0,T], x \in [0,L] \\ w(0,x) = 0, & x \in [0,L] \\ w(t,0) = 0 \quad w(t,L) = 0, & t \in [0,T] \end{cases}$$

Now, u is unique **if and only if** $w = 0$ for $(t,x) \in [0,T] \times [0,L]$.

We shall now use a trick called the **energy method**. Define:

$$E(t) := \int_0^L w^2(t,x)dx$$

Since $w^2 \geq 0$, it follows that $\forall t \in [0, T], E(t) \geq 0$.

Now, we can multiply the heat equation through by w to obtain:

$$w_t w - w_{xx} w = 0 \implies w_t w = w_{xx} w$$

But notice:

$$\frac{\partial}{\partial t}(w^2) = 2w_t w$$

So our modified heat equation becomes:

$$\frac{1}{2}\frac{\partial}{\partial t}(w^2) = w_{xx} w$$

We can integrate both sides with respect to x, on the interval $[0, L]$, noting that since w is well-behaved, we can take the partial derivative out of the integral:

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}(w^2) = w_{xx} w \\
\implies &\frac{1}{2}\int_0^L \frac{\partial}{\partial t}(w^2) dx = \int_0^L w_{xx} w dx \\
\implies &\frac{1}{2}\frac{d}{dt}\int_0^L w^2 dx = \int_0^L w_{xx} w dx \\
\implies &\frac{1}{2}\frac{dE}{dt} = \int_0^L w_{xx} w dx
\end{aligned}$$

Now, we can apply integration by parts on the RHS, via:

$$u = w \qquad du = w_x$$

$$dv = w_{xx} \qquad v = w_x$$

so:

$$\int_0^L w_{xx} w dx = [w w_x]_0^L - \int_0^L (w_x)^2 dx$$

But notice, by the initial conditions w is 0 at $x = 0, L$, so:

$$[w w_x]_0^L = 0$$

Hence:

$$\frac{dE}{dt} = -2\int_0^L (w_x)^2 dx$$

But $(w_x)^2 \geq 0$ for any x, so we must have that:

$$\frac{dE}{dt} \leq 0$$

so E must be a decreasing function.

However:

$$E(0) = \int_0^L w^2(t, 0) dx = 0$$

Hence, since $E(t) \geq 0$, and $E'(t) \leq 0$ and $E(0) = 0$ we must have that:

$$\forall t \in [0, T], \qquad E(t) = 0$$

But w^2 is **continuous** and **non-negative**, so this is only possible if $w^2 = 0 \implies w = 0$, as required.

Notice, in deriving this, we didn't require any knowledge about the form of the solution!

□

6 Workshop

1. **Use the method of separation of variables to solve the following initial-Neumann problem:**

$$\begin{cases} u_t - u_{xx} = 0, & (t,x) \in (0,\infty) \times (0,L) \\ u(0,x) = x, & x \in [0,L] \\ u_x(t,0) = u_x(t,L) = 0, & t \in (0,\infty) \end{cases}$$

Assume a solution of the form:

$$u(t,x) = T(t)X(x)$$

Then the PDE is equivalent to:

$$\lambda = \frac{T'}{T} = \frac{X''}{X}$$

where $\lambda \in \mathbb{R}$ is some constant.

We can solve for T:

$$T' = \lambda T \implies T = Ae^{\lambda t}$$

where $A \in \mathbb{R}$ is some constant.

We now solve for X:

$$X'' = \lambda X$$

This has characteristic polynomial:

$$P(r) = r^2 - \lambda$$

We split into 3 cases. Also notice that if:

$$u_x(t,0) = u_x(t,L) = 0 \implies T(t)X'(0) = T(t)X'(L) = 0$$

and this must hold for any t, so we must have that:

$$X'(0) = X'(L) = 0$$

(1) $\lambda = \mu^2$

If $\lambda > 0$, then the roots of $P(r)$ are:

$$r = \pm\mu$$

so that:

$$X(x) = Be^{\mu x} + Ce^{-\mu x}, \quad B, C \in \mathbb{R}$$
$$X'(x) = B\mu e^{\mu x} - C\mu e^{-\mu x}$$

Hence, applying the initial conditions:

$$X'(0) = 0 \implies \mu(B - C) = 0 \implies B = c$$

$$X'(L) = 0 \implies \mu B(e^{\mu L} - e^{-\mu L}) = 0$$

Since $e^{\mu L} - e^{-\mu L} \neq 0$, we must have that $B = 0$, so $X(x) = 0$ is the trivial solution.

(2) $\lambda = 0$

If $\lambda = 0$, then the roots of $P(r)$ are $r = 0$, and X is just linear:

$$X(x) = Bx + C$$

$$X'(x) = B$$

Hence, applying the initial conditions:

$$X'(0) = 0 \implies B = 0$$

Thus, X will just be a constant, so it's an uninteresting solution.

(3) $\lambda = -\mu^2$

If $\lambda < 0$, then the roots of $P(r)$ are:

$$r = \pm i\mu$$

so that:

$$X(x) = B\sin(\mu x) + C\cos(\mu x)$$

$$X'(x) = B\mu\cos(\mu x) - C\mu\sin(\mu x)$$

Hence, applying the initial conditions:

$$X'(0) = 0 \implies B\mu = 0 \implies B = 0$$

$$X'(L) = 0 \implies -C\mu\sin(\mu L) = 0 \implies \mu L = n\pi$$

Hence, if we define:

$$\mu_n = \frac{n\pi}{L}$$

We have found (infinitely-many) anon-trivial solution:

$$X_n(x) = C\cos(\mu_n x)$$

Putting all this together, the most general solution will be:

$$u(t, x) = \sum_{n=0}^{\infty} A_n e^{-\mu_n^2 t}\cos(\mu_n x)$$

If we apply the initial condition $u(0, x) = x$:

$$u(0, x) = x = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

In other words, A_n must be the Euler-Fourier Coefficients for x corresponding to cos. Hence, we know that:

$$\begin{aligned}
A_n &= \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2}{L}\left(-\frac{L}{n\pi}\left[x\sin\left(\frac{n\pi x}{L}\right)\right]_0^L - \frac{L}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx\right) \\
&= \frac{2}{L}\left(\frac{L^2}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{L}\right)\right]_0^L dx\right) \\
&= \frac{2L}{n^2\pi^2}\left((-1)^n - 1\right)
\end{aligned}$$

where we have used that:

$$\cos(n\pi) = (-1)^n$$

Hence:

$$u(t,x) = \frac{2L}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n - 1}{n^2}e^{-\frac{n^2\pi^2}{L^2}t}\cos\left(\frac{n\pi x}{L}\right)$$

2. **Use the method of separation of variables to solve the following mixed problem (just express the solution as a series, with coefficients independent x, t unspecified):**

$$\begin{cases}
u_t - Du_{xx} = 0, & (t,x) \in (0,\infty)\times(0,\pi) \\
u(0,x) = g(x), & x \in [0,\pi] \\
u_x(t,0) = 0, & t \in (0,\infty) \\
u_x(t,\pi) + u(t,\pi) = 0, & t \in (0,\infty)
\end{cases}$$

Working as above, we get that:

$$T' = \lambda D T \implies T = Ae^{\lambda D t}$$

For X, we split into 3 cases, and once again note that the initial conditions require that:

$$X'(0) = 0 \qquad X'(L) + X(L) = 0$$

---

(1) $\lambda = \mu^2$

If $\lambda > 0$, then:

$$X(x) = Be^{\mu x} + Ce^{-\mu x}, \quad B, C \in \mathbb{R}$$
$$X'(x) = B\mu e^{\mu x} - C\mu e^{-\mu x}$$

Hence, applying the initial conditions:

$$X'(0) = 0 \implies \mu(B - C) = 0 \implies B = c$$

$$X'(\pi)+X(\pi)=0 \implies B\mu e^{\mu\pi}-C\mu e^{-\mu\pi}+Be^{\mu\pi}+Ce^{-\mu\pi}=0 \implies B(e^{\mu\pi}(\mu+1)+e^{-\mu\pi}(-\mu+1))=0$$

Since $e^{\mu\pi}(\mu+1)+e^{-\mu\pi}(-\mu+1)\neq 0$, we must have that $B=0$, so $X(x)=0$ is the trivial solution.

(2) $\lambda = 0$

If $\lambda = 0$, then X is just linear:

$$X(x) = Bx + C$$
$$X'(x) = B$$

Hence, applying the initial conditions:

$$X'(0) = 0 \implies B = 0$$
$$X'(\pi) + X(\pi) \implies C = 0$$

Thus, X will be a trivial solution.

(3) $\lambda = -\mu^2$

If $\lambda < 0$, then:

$$X(x) = B\sin(\mu x) + C\cos(\mu x)$$
$$X'(x) = B\mu\cos(\mu x) - C\mu\sin(\mu x)$$

Hence, applying the initial conditions:

$$X'(0) = 0 \implies B\mu = 0 \implies B = 0$$
$$X'(\pi) + X(\pi) = 0 \implies -C\mu\sin(\mu\pi) + C\cos(\mu\pi) \implies C(1-\mu\tan(\mu\pi)) = 0$$

The equation:

$$\mu\tan(\mu\pi) = 0$$

has non-trivial solutions; let μ_n satisfy $\mu_n\tan(\mu_n\pi) = 0$. Then:

$$X(x) = C\cos(\mu_n x)$$

Thus, our most general solution is:

$$u(t,x) = \sum_{n=0}^{\infty} A_n e^{-\mu_n^2 Dt}\cos(\mu_n x)$$

3. **Consider the solution:**

$$u(t,x) = \sum_{m=1}^{\infty} (-1)^{m+1} e^{-m^2\pi^2 t}\frac{2}{m\pi}\sin(m\pi x)$$

to the initial boundary value heat equation problem:

$$\begin{cases} u_t - u_{xx} = 0, & (t,x) \in (0,\infty)\times(0,1) \\ u(0,x) = x, & x \in [0,1] \\ u(t,0) = u(t,1) = 0, & t \in (0,\infty) \end{cases}$$

Show that:

$$\lim_{t\to 0^+} \|u(t,x) - x\|_{L^2([0,1])} = 0$$

where the L^2 norm is taken over the x variable only. Feel free to make use of the "Some basic facts from Fourier analysis" theorem discussed in class.

Let:

$$A_m = (-1)^{m+1}\frac{2}{m\pi}$$

We have that the Fourier Series for x is:

$$f(x) = \sum_{m=1}^{\infty} A_m \sin(m\pi x)$$

In particular, we have that $f(x)$ converges to x in the L^2 sense:

$$\lim_{N\to\infty}\left\| x - \sum_{m=1}^{N} A_m \sin(m\pi x)\right\|_{L^2([0,1])} = 0$$

This means that:

$$\|u(t,x) - x\|_{L^2([0,1])} = \|u(t,x) - f(x)\|_{L^2([0,1])}$$

We thus compute:

$$\begin{aligned}\|u(t,x) - f(x)\|^2_{L^2([0,1])} &= \int_0^1 |u(t,x) - f(x)|^2 dx \\ &= \int_0^1 \left|\sum_{m=1}^{\infty} A_m e^{-m^2\pi^2 t}\sin(m\pi x) - \sum_{m=1}^{\infty} A_m \sin(m\pi x)\right|^2 dx \\ &= \int_0^1 \left|\sum_{m=1}^{\infty} A_m (e^{-m^2\pi^2 t} - 1)\sin(m\pi x)\right|^2 dx\end{aligned}$$

Now, let:

$$g(x) = \sum_{m=1}^{\infty} A_m (e^{-m^2\pi^2 t} - 1)\sin(m\pi x)$$

This will be a convergent series in L^2, since, $f(x) = \sum_{m=1}^{\infty} A_m (e^{-m^2\pi^2 t} - 1)\sin(m\pi x) \in L^2$ and $\forall t \geq 0, |A_m(e^{-m^2\pi^2 t} - 1)| \leq |A_m|$, so $\|g(x)\|^2_{L^2([0,1])} < \infty$, and thus, $g \in L^2$.

Hence, we can apply Parseval's Identity:

$$\begin{aligned}\|u(t,x) - f(x)\|^2_{L^2([0,1])} &= \|g(x)\|^2_{L^2([0,1])} \\ &= \frac{1}{2}\sum_{m=1}^{\infty} A_m^2 (e^{-m^2\pi^2 t} - 1)^2 \\ &= \sum_{m=1}^{\infty} \frac{2}{m^2\pi^2}(e^{-m^2\pi^2 t} - 1)^2\end{aligned}$$

We claim that this series is uniformly convergent. Since:

$$\frac{2}{\pi^2} \leq \frac{2}{3^2} \leq 1 \qquad \forall t \geq 0, (e^{-m^2\pi^2 t} - 1)^2 \leq 1$$

it follows that:

$$\sum_{m=1}^{\infty} \frac{2}{m^2\pi^2}(e^{-m^2\pi^2 t} - 1)^2 \leq \sum_{m=1}^{\infty} \frac{1}{m^2}$$

This is an absolutely convergent series, so by Weierstrass M-Test, it follows that:

$$\sum_{m=1}^{\infty} \frac{2}{m^2\pi^2}(e^{-m^2\pi^2 t} - 1)^2$$

is uniformly convergent.

Then, we can apply limits term by term to this series:

$$\lim_{t\to 0^+}\left(\sum_{m=1}^{\infty} \frac{2}{m^2\pi^2}(e^{-m^2\pi^2 t} - 1)^2\right) = \sum_{m=1}^{\infty} \frac{2}{m^2\pi^2}\left(\lim_{t\to 0^+}(e^{-m^2\pi^2 t}) - 1\right)^2 = \sum_{m=1}^{\infty} \frac{2}{m^2\pi^2}(1-1)^2 = 0$$

where we have used the continuity of squaring and the exponential.

Thus, it follows that:

$$\lim_{t\to 0^+} \|u(t,x) - f(x)\|^2_{L^2([0,1])} = \lim_{t\to 0^+}\left(\sum_{m=1}^{\infty} \frac{2}{m^2\pi^2}(e^{-m^2\pi^2 t} - 1)^2\right) = 0$$

so as required:

$$\lim_{t\to 0^+} \|u(t,x) - x\|_{L^2([0,1])} = 0$$

4. **Let $\ell > 0$ be a positive real number. Let:**

$$S = (0,\infty) \times (0,\ell)$$

and let:

$$u(t,x) \in C^{1,2}(\bar{S})$$

be the solution of the initial-boundary value problem:

$$\begin{cases} u_t - u_{xx} = 0, & (t,x) \in S \\ u(0,x) = \ell^{-2}x(l-x), & x \in [0,\ell] \\ u(t,0) = u(t,\ell) = 0, & t \in (0,\infty) \end{cases}$$

In this problem, you will us ethe energy method to show that the spatial L^2 norm of u decays exponentially *without actually having to solve the PDE.*

(a) **First, show that:**

$$\|u(0,x)\|_{L^2([0,\ell])} = \sqrt{\frac{\ell}{30}}$$

This is a straightforward computation:

$$\begin{aligned}
\|u(0,x)\|_{L^2([0,\ell])} &= \sqrt{\int_0^\ell \ell^{-4}x^2(\ell-x)^2dx} \\
&= \sqrt{\ell^{-4}\int_0^\ell x^2\ell^2 - 2\ell x^3 + x^4dx} \\
&= \sqrt{\ell^{-4}\left[\frac{x^3\ell^2}{3} - \frac{\ell x^4}{4} + \frac{x^4}{5}\right]_0^\ell} \\
&= \sqrt{\frac{\ell}{3} - \frac{\ell}{4} + \frac{\ell}{5}} \\
&= \sqrt{\frac{\ell}{30}}
\end{aligned}$$

(b) **Next, show that:**

$$\frac{d}{dt}\|u(t,x)\|^2_{L^2([0,\ell])} = -2\|\partial_x u(t,x)\|^2_{L^2([0,\ell])}$$

For this we use integration by parts:

$$\partial_x(u\partial_x u) = (\partial_x u)^2 + u\partial_x^2 u$$

alongside the fact that u solves the heat equation:

$$u_t - u_{xx} = 0 \implies u_t = u_{xx}$$

Then, using the fact that the integral is over x:

$$\begin{aligned}
\frac{d}{dt}\|u(t,x)\|^2_{L^2([0,\ell])} &= \int_0^\ell \frac{d}{dt}u^2dx \\
&= 2\int_0^\ell uu_tdx \\
&= 2\int_0^\ell uu_{xx}dx \\
&= 2\int_0^\ell (\partial_x(u\partial_x u) - (\partial_x u)^2)dx \\
&= [u\partial_x u]_0^\ell - 2\int_0^\ell (\partial_x u)^2)dx \\
&= -2\|\partial_x u(t,x)\|^2_{L^2([0,\ell])}
\end{aligned}$$

where we have used the initial condition $u(t,0) = u(t,\ell) = 0$.

(c) **Then, show that:**

$$|u(t,x)| \leq \sqrt{\ell}\|\partial_x u(t,x)\|_{L^2([0,\ell])}$$

Consider using the *Fundamental Theorem of Calculus*, alongside the *Cauchy-Schwarz Inequality.*

Since $u(t,0) = 0$, then:

$$u(t,x) = u(t,x) - u(t,0) = \int_0^x u_x(t,y)dy$$

Hence:

$$\begin{aligned}|u| &= |\int_0^x u_x(t,y)dy| \\ &\le \sqrt{\int_0^x |u_x|^2 dy}\sqrt{\int_0^x |1|^2 dy} \\ &\le \sqrt{\int_0^\ell |u_x|^2 dy}\sqrt{\int_0^\ell |1|^2 dy} \\ &= \sqrt{\ell}\|\partial_x u(t,x)\|_{L^2([0,\ell])}\end{aligned}$$

(d) **Thus, conclude that:**

$$\|u(t,x)\|^2_{L^2([0,\ell])} \le \ell^2 \|\partial_x u(t,x)\|^2_{L^2([0,\ell])}$$

and show that:

$$\frac{d}{dt}\left(\|u(t,x)\|^2_{L^2([0,\ell])}\right) \le -\frac{2}{\ell^2}\|u(t,x)\|^2_{L^2([0,\ell])}$$

If we square both sides and integrate over $x \in [0,\ell]$:

$$\begin{aligned}&|u| \le \sqrt{\ell}\|\partial_x u(t,x)\|_{L^2([0,\ell])} \\ \implies &\|u(t,x)\|^2_{L^2([0,\ell])} \le \ell\|\partial_x u(t,x)\|^2_{L^2([0,\ell])}\int_0^\ell 1dx \\ \implies &\|u(t,x)\|^2_{L^2([0,\ell])} \le \ell^2\|\partial_x u(t,x)\|^2_{L^2([0,\ell])}\end{aligned}$$

Hence, using the fact that:

$$\frac{d}{dt}\|u(t,x)\|^2_{L^2([0,\ell])} = -2\|\partial_x u(t,x)\|^2_{L^2([0,\ell])}$$

we have that:

$$\frac{d}{dt}\left(\|u(t,x)\|^2_{L^2([0,\ell])}\right) \le -\frac{2}{\ell^2}\|u(t,x)\|^2_{L^2([0,\ell])}$$

(e) **By integrating the differential inequality with respect to time, and using the initial condition $t = 0$, conclude that:**

$$\forall t \ge 0, \quad \|u(t,x)\|_{L^2([0,\ell])} \le \sqrt{\frac{\ell}{30}}e^{-t/\ell^2}$$

If we define:

$$E(t) = \|u(t,x)\|^2_{L^2([0,\ell])}$$

then the above gives us:

$$E'(t) \ge -\frac{2}{\ell^2}E(t)$$

so that:

$$E(t) \le Ae^{-t/\ell^2}$$

If we use the fact that $E(0) = \sqrt{\frac{\ell}{30}}$:

$$E(t) \le \sqrt{\frac{\ell}{30}}e^{-t/\ell^2}$$

as required.

5. **Let $\mu > 0$ be a positive constant and let $u(x,t)$ be a positive solution of class C^3 of the heat equation:**

$$u_t = \mu u_{xx}, \qquad x \in \mathbb{R}, t > 0$$

Show that:

$$v = -2\mu \frac{u_x}{u}$$

satisfies Burgers' equation:

$$v_t + v v_x = \mu v_{xx}, \qquad t > 0$$

We can apply the Chain Rule to determine the derivatives of v, v_t, v_{xx} in terms of u and its partial derivatives:

$$\begin{aligned} v_t &= \frac{\partial}{\partial t}\left(-2\mu \frac{u_x}{u}\right) \\ &= -2\mu \frac{u_{xt}u - u_t u_x}{u^2} \end{aligned}$$

$$\begin{aligned} v_x &= \frac{\partial}{\partial x}\left(-2\mu \frac{u_x}{u}\right) \\ &= -2\mu \frac{u_{xx}u - (u_x)^2}{u^2} \end{aligned}$$

Hence:

$$\begin{aligned} v_t + v v_x &= -2\mu \frac{u_{xt}u - u_t u_x}{u^2} + \left(-2\mu \frac{u_x}{u}\right)\left(-2\mu \frac{u_{xx}u - (u_x)^2}{u^2}\right) \\ &= -\frac{2\mu}{u^3}\left(u_{xt}u^2 - u_t u_x u - 2\mu(u_{xx}u_x u - (u_x)^3)\right) \end{aligned}$$

Using the fact that u satisfies the heat equation, $u_t = \mu u_{xx}$, so:

$$\begin{aligned} v_t + v v_x &= -\frac{2\mu}{u^3}\left(u_{xt}u^2 - u_t u_x u - 2\mu(u_{xx}u_x u - (u_x)^3)\right) \\ &= -\frac{2\mu}{u^3}\left(u_{xt}u^2 - \mu u_{xx}u_x u - 2\mu(u_{xx}u_x u - (u_x)^3)\right) \\ &= -\frac{2\mu}{u^3}\left(u_{xt}u^2 - 3\mu u_{xx}u_x u + 2\mu(u_x)^3)\right) \end{aligned}$$

On the other hand:

$$\begin{aligned} v_{xx} &= \frac{\partial}{\partial x}\left(-2\mu \frac{u_{xx}u - (u_x)^2}{u^2}\right) \\ &= -2\mu \frac{\frac{\partial}{\partial x}(u_{xx}u - (u_x)^2)u^2 - (u_{xx}u - (u_x)^2)\frac{\partial}{\partial x}(u^2)}{u^4} \\ &= -2\mu \frac{(u_{xxx}u + u_{xx}u_x - 2u_x u_{xx})u^2 - (u_{xx}u - (u_x)^2)(2uu_x)}{u^4} \\ &= -\frac{2\mu}{u^3}\left((u_{xxx}u - u_x u_{xx})u - (u_{xx}u - (u_x)^2)(2u_x)\right) \\ &= -\frac{2\mu}{u^3}\left(u_{xxx}u^2 - u_{xx}u_x u - 2u_{xx}u_x u + 2(u_x)^3\right) \\ &= -\frac{2\mu}{u^3}\left(u_{xxx}u^2 - 3u_{xx}u_x u + 2(u_x)^3\right) \end{aligned}$$

If we differentiate the heat equation with respect to x:

$$u_t = \mu u_{xx} \implies u_{tx} = \mu u_{xxx}$$

so:

$$\begin{aligned}
v_{xx} &= -\frac{2\mu}{u^3}\left(u_{xxx}u^2 - 3u_{xx}u_x u + 2(u_x)^3\right) \\
&= -\frac{2\mu}{u^3}\left(\frac{u_{tx}}{\mu}u^2 - 3u_{xx}u_x u + 2(u_x)^3\right) \\
\implies \mu v_{xx} &= -\frac{2\mu}{u^3}\left(u_{tx}u^2 - 3\mu u_{xx}u_x u + 2\mu(u_x)^3\right) \\
\implies \mu v_{xx} &= v_t + v v_x
\end{aligned}$$

Hence, if u satisfies the heat equation, then $v = -2\mu\frac{u_x}{u}$ satisfies Burgers' equation:

$$\mu v_{xx} = v_t + v v_x$$

as required.