Introduction to Partial Differential Equations - Week 1 - PDEs & the Divergence Theorem

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September 2022

Contents

1 Partial Differential Equations

1.1 Definition: Differential Equations

Consider a function:

$$u = u(x^1, \dots, x^n)$$

*A **partial differential equation** in the unknown u is an equation involving u and its **partial derivatives**:*

$$F(u, u_{x^1}, \dots, u_{x^n}, u_{x^1x^1}, \dots, u_{x^{i_1}} \dots u_{x^{i_N}}, x^1, \dots, x^n) = 0$$

where:

$$i_1, \dots, i_N \in [1, n]$$

1.2 PDEs Notation

- **How do we express a partial derivative**
  - the following are equivalent:
    $$\frac{\partial u}{\partial x^i} = u_{x^i} = \partial_i u$$
  - for multiple derivatives:
    $$\frac{\partial^2 u}{\partial x^i \partial x^j} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} u = u_{x^i} u_{x^j} = \partial_i \partial_j u$$
- **What is the order of a PDE?**
  - the **largest** partial derivative appearing in the PDE

1.3 Types of PDEs

1.3.1 Homogeneous vs Inhomogeneous

- **What is a homogeneous PDE?**
  - a PDE of the form:
    $$F(u, u_{x^1}, \dots, u_{x^n}, u_{x^1x^1}, \dots, u_{x^{i_1}} \dots u_{x^{i_N}}) = 0$$
  - for example, a **second-order**, **homogeneous** PDE is:
    $$\partial^2_{x^i} u - 5\partial_{x^j} u = 0$$
- **What is an inhomogeneous PDE?**
  - a PDE of the form:
    $$F(u, u_{x^1}, \dots, u_{x^n}, u_{x^1x^1}, \dots, u_{x^{i_1}} \dots u_{x^{i_N}}) = f(x^1, \dots, x^n)$$
  - for example, a **second-order**, **inhomogeneous** PDE is
    $$\partial^2_{x^i} u - 5\partial_{x^j} u = 2x^1 - 5x^2$$

1.3.2 Constant vs Variable Coefficient

- **What is a constant coefficient PDE?**
  - a PDE where the coefficients multiplying the derivatives are constant
  - for example, a **second-order**, **inhomogeneous**, **constant coefficient** PDE is:

$$-\partial_t^2 u + 2\partial_x^2 u + u = t$$

- **What is a variable coefficient PDE?**
  - a PDE where the coefficients multiplying the derivatives are functions
  - for example, a **third-order**, **inhomogeneous**, **variable coefficient** PDE is:

$$\partial_t u + 2(1 + x^2)\partial_x^3 u + u = t$$

1.3.3 Linear vs Non-Linear

- **What is a linear PDE?**
  - a PDE where the derivatives are combined linearly
  - in other words, we can use a **linear differential operator**:

$$\mathcal{L}[au + bv] = a\mathcal{L}[u] + b\mathcal{L}[v]$$

    to express the PDE:

$$\mathcal{L}[u] = f(x^1, \dots, x^n)$$

  - for example, the operator:

$$\mathcal{L} = -\partial_t^2 + (t^2 - x^2)\partial_x^2 + 1$$

    defines a **second-order**, **linear** PDE

- **What is a non-linear PDE?**
  - a PDE where the derivatives are not related linearly
  - for example, the operator:

$$\mathcal{L}[u] = \partial_x^2 u + u^2 \partial_y^2 u(1 - \cos(u))\partial_t u$$

    defines a **second-order**, **non-linear** PDE

1.3.4 Physical Examples of PDEs

- **Wave Equation**

$$-\partial_t^2 u + \partial_x^2 u = 0$$

- **Heat Equation**

$$-\partial_t u + \partial_x^2 u = 0$$

- **Laplace's Equation**

$$\partial_x^2 u + \partial_y^2 u + \partial_z^2 u = 0$$

- **Poisson's Equation**

$$\partial_x^2 u + \partial_y^2 u + \partial_z^2 u = f(x, y, z)$$

- **Schrödinger's Equation**

$$\iota\partial_t u + \partial_x^2 u = 0$$

- **Transport Equation**

$$u_t + u_x = 0$$

- **Burger's Equation**

$$u_t + uu_x = 0$$

- **Maxwell's Equation in a Vacuum**

$$\partial_t \underline{E} - \nabla \times \underline{B} = 0$$

$$\partial_t \underline{B} + \nabla \times \underline{E} = 0$$

$$\nabla \cdot \underline{E} = 0$$

$$\nabla \cdot B = 0$$

where $\underline{E}$ and $\underline{B}$ are **vector-fields** (electric and magnetic fields respectively)

1.3.5 The Goals of Solving PDEs

1. *Does the PDE have any solutions?*
2. *What kind of "data" do we need to specify in order to solve the PDE?*
3. *Are the solutions corresponding to the given data unique?*
4. *What are the basic qualitative properties of the solution?*
5. *Does the solution deveop singularities? O fwhat nature?*
6. *What happens if we slightly vary the data? Does the solution then also vary only slightly?*
7. *What kinds of quantitative estimates can be derived for the solutions?*
8. *How can we define the "norm" of a solution, in a way that is useful for the problem at hand?*

1.4 The Principle of Superposition

Let $u_1, \dots, u_M$ be solutions to the linear PDE:

$$\mathcal{L}[u] = 0$$

If $c_1, \dots, c_M \in \mathbb{R}$, then:

$$\sum_{i=1}^{M} c_i u_i$$

is also a solution.

Proof. By linearity:

$$\mathcal{L}\left[\sum_{i=1}^{M} c_i u_i\right] = \sum_{i=1}^{M} c_i \mathcal{L}[u_i] = 0$$

□

> *Notice, this means that the solutions are **closed** under **addition** and **scalar multiplication**. Moreover, it is clear that $u = 0$ is always a solution to a homogeneous, linear PDE. It follows that the set of solutions to $\mathcal{L}[u] = 0$ is a **vector space**.*

1.5 Proposition: Relationship Between Inhomogeneous and Homogeneous Solutions to Linear PDEs

> *Let S_H be the set of **all** solutions to the **linear**, **homogeneous** PDE:*
>
> $$\mathcal{L}[u] = 0$$
>
> *Further, assume that u_p is a **particular** solution to the **linear**, **inhomogeneous** PDE:*
>
> $$\mathcal{L}[u] = f(x^1, \dots, x^n)$$
>
> *Then, the set of all solutions to the **inhomogeneous** PDE is given by:*
>
> $$S_I = \{u_p + u_h \mid u_h \in S_H\}$$

Proof. We have that:

$$\mathcal{L}[u_p] = f$$

Consider another solution w, such that:

$$\mathcal{L}[w] = f$$

Then, by linearity:

$$\mathcal{L}[w] - \mathcal{L}[u_p] = f - f = 0 \implies \mathcal{L}[w - u_p] = 0$$

But then, $w - u_p$ is a homogeneous solution, so:

$$w - u_p = u_h \in S_H$$

Then, we can express:

$$w = u_p + u_h$$

In other words, if w is a solution to the inhomogeneous problem, then $w \in S_I$

Now, consider an element $w \in S_I$. By definition, we can write:

$$w = u_p + u_h, \qquad u_h \in S_H$$

But then:

$$\mathcal{L}[w] = \mathcal{L}[u_p + u_h] = \mathcal{L}[u_p] + \mathcal{L}[u_h] = f + 0 = f$$

That is, if $w \in S_I$, then w is a solution to the inhomogeneous problem.

Thus, we have shown that all solutions to $\mathcal{L}[u] = f$ must be within S_I, as required.

□

1.6 Solving PDEs via Geometric Intuition

1.6.1 The Constant Coefficient Transport Equation

We consider the equation:

$$a\partial_x u(x,y) + b\partial_y u(x,y) = 0, \qquad a, b \in \mathbb{R}$$

The first thing to notice is that we can rewrite this as:

$$\nabla u \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

Recall, this is the definition of a **directional derivative**: it follows that the derivative of u in the direction (a,b) is 0. In other words, along lines defined by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$, u takes on **the same value**. Hence, it follows that u is **entirely** determined by the the line on which the x, y lie.

A line in the direction $\begin{pmatrix} a \\ b \end{pmatrix}$ has gradient $\frac{b}{a}$, and can be described by the relation:

$$bx - ay = c$$

Notice, varying c changes the y-intercept of the line, so changing c allows us to shift the line, thus covering all of $\mathbb{R}^2$. Hence, the position of a line depends solely on c, and the value of u depends on the position of the line.

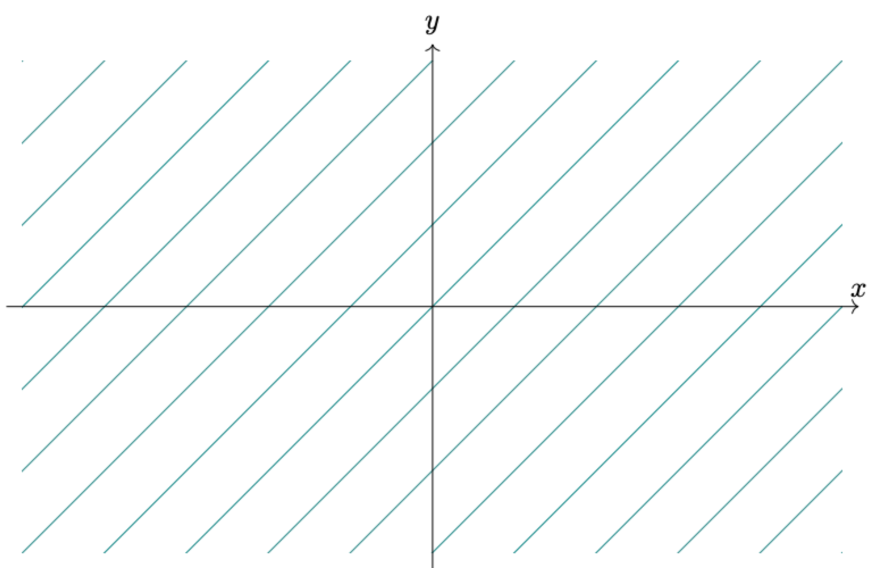

Figure 1: Varying c shifts the line $bx - ay = c$ about the $\mathbb{R}^2$ plane.

In other words, we can think of u as a function of c:

$$u(x,y) = f(c) = f(bx - ay)$$

where f is some function, to be decided given some "data" about how u behaves on some subset of its domain (a point won't be sufficient, since the value of u is determined by the value along lines).

For instance, if we know that:

$$u(x,0) = x^2$$

it follows that:

$$x^2 = f(bx) \implies x \mapsto ((b^{-1}x))^2$$

That is:

$$f(\psi) = b^{-2}\psi^2$$

So our solution to the transport equation becomes:

$$u(x, y) = f(bx - ay) = b^{-2}(bx - ay)^2 = (x - b^{-1}ay)^2$$

1.6.2 The Variable Coefficient Transport Equation

In a similar vein to before, we consider the PDE:

$$y\partial_x u + x\partial_y u = 0$$

Let $\underline{x} = \begin{pmatrix} y \\ x \end{pmatrix}$ Then, we have that:

$$\nabla u \cdot \underline{x} = 0$$

In other words, the directional derivative of u at a point (x, y) in the direction of $\underline{x}$ is 0, so u is constant on the curve $\mathcal{C}$ traced out by (x, y).

We can parametrise $\mathcal{C}$ via:

$$x \mapsto (x, y(x))$$

(we can think of y as a function of x, depending on the position on $\mathcal{C}$)

At a given point $P(x, y)$, the slope will be $\frac{x}{y}$, so:

$$\frac{dy}{dx} = \frac{x}{y}$$

This is a separable ODE, so we can solve for $y = y(x)$:

$$\frac{dy}{dx} = \frac{x}{y} \implies \int y dy = \int x dx$$
$$\implies \frac{y^2}{2} = \frac{x^2}{2} + c, \qquad c \in \mathbb{R}$$

Hence, u is constant on the hyperbolae $y^2 - x^2 = C$, so we can think of u as a function of C:

$$u(x, y) = f(C) = f(y^2 - x^2)$$

More generally, if we have a PDE:

$$a(x, y)\partial_x u + b(x, y)\partial_y u = 0$$

we can solve it, given that we can integrate:

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}$$

1.7 Analysis Recap: Norms

- **What is a norm?**
  - intuitively, allow us to "measure" the "size" of a function

1.7.1 C^k Norms

- **What is the C^k norm?**
  - consider a function f with domain $\Omega \subset \mathbb{R}$
  - for $k \in \mathbb{Z}, k \geq 0$ define the C^k norm of f on Ω by:

$$\|f\|_{C^k(\Omega)} := \sum_{a=0}^{k} \sup_{x\in\Omega} |f^{(a)}(x)|$$

  - here, $f^{(a)}(x)$ denotes the a^{th} order derivative of f
  - for instance:

$$\|\sin(x)\|_{C^7(\mathbb{R})} = 8$$

    since the derivative of $\sin(x)$ is $\cos(x)$, and the derivative of $\cos(x)$ is $-\sin(x)$, so over the reals, $|f^{(a)}(x)|$ attains a supremum of 1.

- **How does the C^k norm generalise for multidimensional domains?**
  - consider $\Omega \subset \mathbb{R}^n$
  - the C^k norm is obtained by summing over all partial derivatives of order $\leq k$
  - if $\Omega \subset \mathbb{R}^2$, we compute:

$$\begin{aligned}\|f\|_{C^2(\Omega)} := &\sup_{(x,y)\in\Omega} |f(x,y)| + \sup_{(x,y)\in\Omega} |\partial_x f(x,y)| + \sup_{(x,y)\in\Omega} |\partial_y f(x,y)| \\ &+ \sup_{(x,y)\in\Omega} |\partial_x^2 f(x,y)| + \sup_{(x,y)\in\Omega} |\partial_x \partial_y f(x,y)| + \sup_{(x,y)\in\Omega} |\partial_y^2 f(x,y)|\end{aligned}$$

- **Can we prioritise the weight of certain variables when computing the C^k norm?**
  - if $f = f(t,x)$, then $C^{i,j}$ denotes that we want to consider the first i partial derivatives with respect to t, and the j first partial derivatives with respect to x:

$$\|f\|_{C^{5,2}(\mathbb{R})} := \sum_{a=0}^{5} \sup_{(t,x)\in\mathbb{R}^2} |\partial_t^a f(t,x)| + \sum_{a=0}^{2} \sup_{(t,x)\in\mathbb{R}^2} |\partial_x^a f(t,x)|$$

1.7.2 L^p Norms

- **What is the L^p norm?**
  - let $1 \leq p < \infty, p \in \mathbb{R}$
  - consider a function f with domain $\Omega \in \mathbb{R}^n$
  - the L^p norm of f is:

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p d^n x\right)^{\frac{1}{p}}$$

1.7.3 Norm Properties

Both the C^k and L^p norms satisfy the following properties:

- ***Non-Negativity**: $\|f\| \geq 0$, with $\|f\| = 0$ **if and only if** $f(x) = 0$ almost everywhere (that is, it is non-zero on a set of measure 0)*
- ***Scaling**: $\|\lambda f\| = |\lambda|\|f\|$*
- ***Triangle Inequality**: $\|f + g\| \leq \|f\| + \|g\|$*

2 The Divergence Theorem

2.0.1 Vector Fields and Divergence

- **What is a vector field?**
  - a function mapping vectors to vectors:

$$\underline{F} : \Omega \to \mathbb{R}^n, \qquad \Omega \subset \mathbb{R}^n$$

$$\underline{F}(x^1, \dots, x^n) = \begin{pmatrix} F^1(x^1, \dots, x^n) \\ \vdots \\ F^n(x^1, \dots, x^n) \end{pmatrix}$$

  - here F^i are scalar fields:

$$F^i : \Omega \to \mathbb{R}$$

- **What is the divergence of a vector field?**
  - the divergence is:

$$\nabla \cdot \underline{F} := \sum_{i=1}^{n} \partial_i F^i$$

  - can think of it as a "dot product":

$$\begin{pmatrix} \frac{\partial}{\partial x^1} \\ \vdots \\ \frac{\partial}{\partial x^n} \end{pmatrix} \cdot \begin{pmatrix} F^1 \\ \vdots \\ F^n \end{pmatrix}$$

2.0.2 Theorem: The Divergence Theorem

The divergence theorem is a version of integration by parts for higher dimensions.

Let $\Omega \subset \mathbb{R}^3$ be a ***domain*** *(open, connected subset of $\mathbb{R}^n$). Denote the* ***boundary/surface*** *of Ω as $\partial\Omega$.*
Then:

$$\int_\Omega \nabla \cdot F(x,y,z)dx\ dy\ dz = \int_{\partial\Omega} \underline{F}(\sigma) \cdot \underline{\hat{N}}(\sigma)d\sigma$$

Here:

- $\underline{\hat{N}}(\sigma)$ *is the* ***unit outward normal vector*** *to the surface $\partial\Omega$*
- *if $\partial\Omega \subset \mathbb{R}^3$ can be described as:*

  $$\partial\Omega = \{(x,y,z) \mid z = \phi(x,y)\}$$

  then:

  $$d\sigma = \sqrt{1 + \|\nabla\phi(x,y)\|^2}dx\ dy = \sqrt{1 + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2}dx\ dy$$

3 Workshop

The following are useful formulae, either derived in this workshop (in some questions which we don't include, since we did them in Honours Analysis), or just in general.

- ***Divergence Theorem***

$$\int_{\Omega} \nabla \cdot \underline{F}(\underline{x}) d\underline{x} = \int_{\partial\Omega} \nabla F(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}) d\underline{\sigma}$$

- ***Laplacian****:*

$$\nabla \cdot \nabla F = \Delta F$$

- ***Cauchy-Schwarz Inequality****:*

$$| \langle \underline{v}, \underline{w} \rangle | \leq \|\underline{v}\| \|\underline{w}\|$$

- ***Cauchy-Schwarz Inequality for Integrals***

$$\left| \int_{\Omega} f(x) g(x) dx \right| \leq \sqrt{\int_{\Omega} |f(x)|^2 dx} \sqrt{\int_{\Omega} |g(x)|^2 dx}$$

1. **Find a solution to $\Delta u = 0$ in the whole of $\mathbb{R}^n$, such that:**

   (a) u **is a linear function**

   Any linear function is harmonic:

   $$u(\underline{x}) = x_1$$

   (b) u **is a quadratic polynomial**

   2 simple examples:

   $$u(\underline{x}) = x_1^2 - x_2^2$$
   $$u(\underline{x}) = x_1 x_2$$

   (c) u **is a cubic polynomial**

   Notice that:

   $$\Delta x_2^3 = 6x_2$$
   $$\Delta x_1^2 x_2 = 2x_2$$

   Hence:

   $$u(\underline{x}) = 3x_1^2 x_2 - x_2^3$$

   works.

2. **Let Ω be a domain with a smoooth boundary $\partial\Omega$. Let $u, v \in C^2(\bar{\Omega})$, where $\bar{\Omega}$ denotes the closure of Ω. Show that the *Green Identity* holds:**

$$\int_\Omega u(\underline{x})\Delta v(\underline{x}) - v(\underline{x})\Delta u(\underline{x})d\underline{x} = \int_{\partial\Omega} u(\underline{\sigma})(\nabla v(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma})) - v(\underline{\sigma})(\nabla u(\underline{\sigma}) \cdot \underline{\hat{N}}(\underline{\sigma}))d\underline{\sigma}$$

where $\underline{\hat{N}}(\underline{\sigma})$ is the outward unit-normal to $\partial\Omega$ at $\underline{\sigma}$

Define the following vector field:

$$\underline{F} = u\nabla v - v\nabla u$$

Then:

$$\nabla \cdot F = u\Delta v - v\Delta u$$

Hence, by applying the Divergence Theorem the result follows.

3. **Prove that if $\varepsilon \in (0, 0.5)$, then:**

$$f(x) = \sin(x)\frac{\ln(x^2+1)}{|x|^{1-\varepsilon}}$$

satisfies $f \in L^2(\mathbb{R})$:

$$\int_{\mathbb{R}} |f(x)|^2 dx < \infty$$

Note that f^2 will be even, so:

$$\int_{\mathbb{R}} |f|^2 dx = 2\int_0^\infty |f|^2 dx$$

Now, define δ, M as arbitrary constants, such that:

$$(0, \infty) = (0, \delta) \cup [\delta, M] \cup (M, \infty)$$

Recall from Honours Analysis that a function is in $L^2([a, b])$ if it is Lebesgue Integrable on $[a, b]$. Continuous functions are Lebesgue Integrable, and since f is continuous, so is f^2. Hence, $f^2 \in L^2([\delta, M])$. This means that to show that $f^2 \in L^2(\mathbb{R})$ it is sufficient to show that:

$$\int_0^\delta |f|^2 dx < \infty \qquad \int_M^\infty |f|^2 dx < \infty$$

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(1) **Integral on $(0, \delta)$**

Since δ can be arbitrarily small, we can enforce that $\delta < \frac{\pi}{2}$. Then, we know that:

$$0 \le x \le \delta < \frac{\pi}{2} \implies \sin(x) \le x$$

Thus:

$$\begin{aligned}
\int_0^\delta |f|^2 dx &= \int_0^\delta |\sin(x)|^2 \frac{\ln(x^2+1)}{|x|^{2-2\varepsilon}} dx \\
&\le \ln(\delta^2+1) \int_0^\delta \frac{|x|^2}{|x|^2/x^{2\varepsilon}} dx \\
&= \ln(\delta^2+1) \int_0^\delta x^{2\varepsilon} dx \\
&= \ln(\delta^2+1) \frac{\delta^{2\varepsilon+1}}{2\varepsilon+1} \\
&< \infty
\end{aligned}$$

where we have used the fact that $\ln(x)$ is an increasing function.

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(2) **Integral on** (M, ∞)

Since M can be made arbitrarily large, we can enforce that $\exists M$ such that:

$$x \geq M \implies \exists \beta \in \mathbb{R}^+ \,:\, \ln(x^2+1) \leq x^\beta$$

(we know that this is feasible, since $\ln(x^2+1) \leq x^2+1$ when $|x| \geq 0$)

> *One can also use L'Hôpital's Rule to verify taht this is sensible:*
>
> $$\lim_{x\to\infty} \frac{\ln(x^2+1)}{x^\beta} = \frac{2}{\beta} \lim_{x\to\infty} \frac{\frac{x^2}{x^2+1}}{x^\beta} = 0$$

Using this M, we have that:

$$\begin{aligned}
\int_M^\infty |f|^2 dx &= \int_M^\infty |\sin(x)|^2 \frac{\ln(x^2+1)}{|x|^{2-2\varepsilon}} dx \\
&\leq \int_M^\infty \frac{\ln(x^2+1)}{|x|^{2-2\varepsilon}} dx \\
&\leq \int_M^\infty \frac{|x|^{2\beta}}{|x|^{2-2\varepsilon}} dx \\
&\leq \int_M^\infty \frac{|x|^{2\beta}}{|x|^{2-2\varepsilon}} dx \\
&= \int_M^\infty |x|^{2\beta-2+2\varepsilon} dx \\
&= \frac{1}{2\beta+2\varepsilon-1} \left[\lim_{D\to\infty} \left(D^{2\beta+2\varepsilon-1}\right) - M^{2\beta+2\varepsilon-1} \right]
\end{aligned}$$

Hence, the integral converges provided that:

$$2\beta + 2\varepsilon - 1 < 0 \implies \beta < \frac{1-2\varepsilon}{2}$$

Hence, picking $\beta = \frac{1-2\varepsilon}{4}$, it follows that $f \in L^2$, as required.

4. **Let u be a harmonic function in a domain $\Omega \subset \mathbb{R}^n$. Let:**

$$w(\underline{x}) = \|\nabla u(\underline{x})\|^2 \qquad v(\underline{x}) = u^2(\underline{x})$$

Compute $\Delta w, \Delta v$.

We have that:

$$\Delta u = \sum_{i=1}^n (\partial^2_{x_i^2} u) = 0$$

Moreover:

$$w = \sum_{i=1}^{n} (\partial_{x_i} u)^2$$

Hence:

$$\partial_{x_j} w = 2\sum_{i=1}^{n} (\partial_{x_i} u)(\partial^2_{x_i x_j} u)$$

$$\begin{aligned}\partial^2_{x_j} w &= 2\sum_{i=1}^{n} (\partial^2_{x_i x_j} u)^2 + (\partial_{x_i} u)(\partial^3_{x_i x_j^2} u) \\ &= 2\sum_{i=1}^{n} (\partial^2_{x_i x_j} u)^2 + \partial_{x_i}(u\partial^2_{x_j^2} u)\end{aligned}$$

So it follows that:

$$\Delta w = 2\sum_{j=1}^{n}\sum_{i=1}^{n} (\partial^2_{x_i x_j} u)^2 + \partial_{x_i}(u\partial^2_{x_j^2} u)$$

(it's possible to simplify this, since when $i = j$, we will pick up some Laplacians of u which will cancel out. For instance, in 2 dimensions:

$$\Delta w = 2(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)$$

with the term:

$$2u_x \partial_x(\Delta u) + 2u_y(\partial_y(\Delta u)$$

having vanished)

Similarly:

$$\begin{aligned}&\partial_{x_i} v = 2u(\partial_{x_i} u) \\ \implies &\partial^2_{x_i^2} v = 2((\partial_{x_i} u)^2 + (\partial_{x_i} u)(\partial^2_{x_i^2} u)) \\ \implies &\Delta v = 2\sum_{i=1}^{n} (\partial_{x_i} u)^2 + (\partial_{x_i} u)(\partial^2_{x_i^2} u) \\ \implies &\Delta v = 2\sum_{i=1}^{n} (\partial_{x_i} u)^2 + u(\partial^2_{x_i^2} u) \\ \implies &\Delta v = 2\|\nabla u\|^2 = 2w\end{aligned}$$

5. **You have seen many explicit solutions to various linear ODE's in Honours Differential Equations. However, not all ODE's have such solutions. As an example, let us consider the following equation:**

$$y'' + a\sin(y) = 0, \qquad a > 0$$

This equation describes the oscillation of a simple pendulum, where the constant $a = \frac{g}{L}$ and g is the gravitational constant, and L the length of the string.

(a) **Under the initial conditions $y(x_0) = \alpha, y'(x_0) = \beta$ show that the solution y satisfies the first order ODE:**

$$y'(x) = \pm\sqrt{2a\cos(y) + \beta^2 - 2a\cos(\alpha)}$$

Differentiating the first order ODE with respect to x:

$$\begin{aligned}
& y'(x) = \pm\sqrt{2a\cos(y) + \beta^2 - 2a\cos(\alpha)} \\
\implies & y''(x) = \frac{d}{dx}\left(\pm(2a\cos(y) + \beta^2 - 2a\cos(\alpha))^{\frac{1}{2}}\right) \\
\implies & y''(x) = \frac{1}{2y'(x)} \times \frac{d}{dx}(2a\cos(y) + \beta^2 - 2a\cos(\alpha)) \\
\implies & y''(x) = \frac{1}{2y'(x)} \times (-2a\sin(y)y'(x)) \\
\implies & y''(x) = -a\sin(y) \\
\implies & y'' + a\sin(y) = 0
\end{aligned}$$

Thus, if y satisfies the pendulum equation, it must satisfy the first order ODE.

(b) **Hence, show that we have the implicit relation:**

$$x - x_0 = \int_{\alpha}^{y} \frac{dt}{\sqrt{2a\cos(t) + \beta^2 - 2a\cos(\alpha)}}$$

This integral cannot be computed explicitly; in fact, it is called an *elliptic integral*, and it is one of the special functions in analysis.

Let $s = x, y = t$. We have:

$$\frac{1}{\sqrt{2a\cos(t) + \beta^2 - 2a\cos(\alpha)}}\frac{dt}{ds} = 1$$

(we take the positive square root since we describe a simple pendulum) We integrate both sides with respect to s, where $s \in [x_0, x]$:

$$\begin{aligned}
& \int_{s=x_0}^{x=x} 1ds = \int_{s=x_0}^{s=x} \frac{1}{\sqrt{2a\cos(t) + \beta^2 - 2a\cos(\alpha)}}\frac{dt}{ds}ds \\
\implies & [s]_{x_0}^{x} = \int_{t=y(x_0)}^{t=y(x)} \frac{dt}{\sqrt{2a\cos(t) + \beta^2 - 2a\cos(\alpha)}} \\
\implies & x - x_0 = \int_{\alpha}^{y} \frac{dt}{\sqrt{2a\cos(t) + \beta^2 - 2a\cos(\alpha)}}
\end{aligned}$$

as required.

(c) **Finally, do a clever substitution to write the implicit relation in the following equivalent form:**

$$\sqrt{a}(x - x_0) = \int_{\frac{\sin(\alpha/2)}{k}}^{u} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}$$

where k is a constant that you should compute in terms of a, α, β.

We shall use the following identities:

$$\cos^2(\theta) + \sin^2(\theta) = 1 \implies \cos^2(\theta) = 1 - \sin^2(\theta)$$

$$\cos^2(2\theta) = 1 - 2\sin^2(\theta) \implies \sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

We have that:

$$
\begin{aligned}
2a\cos(t)+\beta^2-2a\cos(\alpha) &= 4a\left(\frac{2a\cos(t)+\beta^2-2a\cos(\alpha)}{4a}\right)\\
&= 4a\left(\frac{\beta^2}{4a}+\frac{1-\cos(\alpha)}{2}-\frac{1-\cos(t)}{2}\right)\\
&= 4a\left(\frac{\beta^2}{4a}+\sin^2(\alpha/2)-\sin^2(t/2)\right)
\end{aligned}
$$

Now, let:

$$k^2=\frac{\beta^2}{4a}+\sin^2(\alpha/2)$$

and

$$s(t)=\frac{\sin(t/2)}{k}$$

Then:

$$\frac{ds}{dt}=\frac{\cos(t/2)}{2k}$$

Hence, we can apply integration by substitution:

$$
\begin{aligned}
\int_\alpha^y \frac{dt}{\sqrt{2a\cos(t)+\beta^2-2a\cos(\alpha)}} &= \int_{s(\alpha)}^{s(y)} \frac{2k}{\cos(t/2)}\frac{ds}{\sqrt{4a\left(k^2-\sin^2(t/2)\right)}}\\
&= \int_{s(\alpha)}^{s(y)} \frac{k}{\sqrt{a}}\frac{ds}{\sqrt{\cos^2(t/2)\left(k^2-\sin^2(t/2)\right)}}\\
&= \int_{s(\alpha)}^{s(y)} \frac{k}{\sqrt{a}}\frac{ds}{\sqrt{k^2\cos^2(t/2)\left(1-\frac{\sin^2(t/2)}{k^2}\right)}}\\
&= \int_{s(\alpha)}^{s(y)} \frac{1}{\sqrt{a}}\frac{ds}{\sqrt{\cos^2(t/2)(1-s^2)}}\\
&= \int_{s(\alpha)}^{s(y)} \frac{1}{\sqrt{a}}\frac{ds}{\sqrt{\left(1-\sin^2(t/2)\right)(1-s^2)}}\\
&= \int_{s(\alpha)}^{s(y)} \frac{1}{\sqrt{a}}\frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}
\end{aligned}
$$

Letting $s(y)=u$, it follows that:

$$x-x_0=\int_\alpha^y \frac{dt}{\sqrt{2a\cos(t)+\beta^2-2a\cos(\alpha)}} \implies \sqrt{a}(x-x_0)=\int_{\frac{\sin(\alpha/2)}{k}}^{u}\frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}$$

as required.

6. **In this problem we describe the method of characteristics for the inhomogeneous first order equations:**

$$au_x+bu_y=c$$

where a, b, c are given coefficients.
This equation says that the vector $(u_x, u_y, -1)$ is perpendicular to:

$$(a(x, y, u), b(x, y, u), c(x, y, u))$$

Since the normal to the graph of u is $\frac{(u_x,u_y,-1)}{\sqrt{1+u_x^2+u_y^2}}$ it follows that the vector (a, b, c) lies on the tangent plane to the graph at the point $(x, y, u(x, y))$.
Within a specific surface $u(x, y)$ solving the PDE we can consider the field of directions defined by the tangnetial vectors (a, b, c). This field of directions is composed of the tangents of a one-parameter family of curves in that surface, called characteristics, which are determined by the system of ordinary differential equations:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$

Using this observation, show that the solution to the nonlinear equation:

$$u_x + u_y = u^2$$

passing through the initial curve:

$$x = t, y = -t, u = t$$

(that is, $u(t, -t) = t$) becomes infinite along the hyperbola:

$$x^2 - y^2 = 4$$

We assume that the characteristics are defined by some parameter s, such that $x = x(s), y = y(s)$. Then:

$$\frac{d}{ds}(u(x(s), y(s)) = u_x \frac{dx}{ds} + u_y \frac{dy}{ds}$$

This means that the solution will satisfy the system:

$$\frac{dx}{ds} = 1$$

$$\frac{dy}{ds} = 1$$

$$\frac{du}{ds} = u^2$$

or equivalently:

$$\frac{dy}{dx} = 1$$

$$\frac{dy}{du} = \frac{1}{u^2}$$

We can solve these ODEs:

$$\frac{dy}{dx} = 1 \implies y = x + C, \qquad C \in \mathbb{R}$$

$$\frac{dy}{du} = \frac{1}{u^2}$$

$$\implies \int dy = \int \frac{1}{u^2} du$$

$$\implies y + D = -\frac{1}{u}$$

$$\implies u = -\frac{1}{y + D}, \qquad D \in \mathbb{R}$$

Notice $y(x)$ and $u(x, y(x))$ are entirely determined by the values of C, D. In particular, we can think of D as a function of C, such that $D = f(C)$. Then:

$$u(x,y) = -\frac{1}{y+D} = -\frac{1}{y+f(C)} = -\frac{1}{y+f(y-x)}$$

We are given the initial condition:

$$u(t,-t) = t$$

We can then find f for this initial condition:

$$\begin{aligned} & t = u(t,-t) \\ \implies & t = -\frac{1}{-t+f(-2t)} \\ \implies & f(-2t) = \frac{t^2-1}{t} \end{aligned}$$

Hence, define:

$$f(\alpha) = \frac{\left(-\frac{\alpha}{2}\right)^2 - 1}{\left(-\frac{\alpha}{2}\right)} = \frac{4-\alpha^2}{2\alpha}, \qquad \alpha \in \mathbb{R}$$

It thus follows that:

$$\begin{aligned} u(x,y) &= -\frac{1}{y+f(y-x)} \\ &= -\frac{1}{y+\frac{4-(y-x)^2}{2(y-x)}} \\ &= -\frac{1}{\frac{2y(y-x)+4-(y-x)^2}{2(y-x)}} \\ &= -\frac{2(y-x)}{2y^2-2yx+4-(y^2-2xy+x^2)} \\ &= \frac{2(x-y)}{4-(x^2-y^2)} \end{aligned}$$

In particular, u along the parabola $x^2 - y^2 = 4$ goes to infinity, as expected.