Honours Differential Equations - Week 9 - The Laplace Transform

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Contents

1 Wave Equation in 2 Dimensions

1.1 Bessel's Equation

- **What is Bessel's Equation?**
  - a second order, non-linear, homogeneous ODE:
  $$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$
  where ν is a constant
- **What is Bessel's Equation of Order 0?**
  - Bessel's Equation with $\nu = 0$:
  $$x^2y'' + xy' + x^2y = 0$$
- **What is the general solution to Bessel's Equation of ORder 0?**
  - in order to derive a solution, we employ power series
  - the general solution is given by:
  $$y = c_1J_0(x) + c_2Y_0(x)$$
  where $J_0(x)$ is known as **the Bessel function of the first kind of order zero**, whilst $Y_0(x)$ is known as **the Bessel function of the second kind of order zero**
- **What are J_0 and Y_0**
  - both are constructed as power series:
  $$J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$
  $$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right)J_0(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}H_n}{2^{2n}(n!)^2}\right]$$
  where:
    * H_n denotes the nth partial sum of the harmonic series:
    $$H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$$
    * γ is the **Euler-Mascheroni Constant**:
    $$\gamma = \lim_{n\to\infty}(H_n - \ln n)$$

We will now consider the wave equation in 2 dimensions, given by:
$$u_{tt} = \alpha^2\nabla^2u = \alpha^2(u_{xx} + u_{yy})$$

1.2 Rectangular Membrane

For the rectangular membrane, we consider the following initial/boundary conditions:

- $u(0, y, t) = u(a, y, t) = 0$
- $u(x, 0, t) = u(x, b, t) = 0$
- $u(x, y, 0) = f(x, y)$
- $u_t(x, y, 0) = 0$

In other words, we consider a rectangular membrane, taut at the ends of a rectangle, with initial displacement $f(x, y)$ and 0 initial velocity.

1.2.1 Separation of Variables

We assume:

$$u(x, y, t) = X(x)Y(y)T(t)$$

Plugging into the ODE:

$$XYT'' = \alpha^2(X''YT + XY''T)$$

Dividing through by XYT:

$$\frac{1}{\alpha^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}$$

The RHS is a function of x, y, whilst the LHS is a function of t. This can only be possible if each of the ratios is constant. In other words:

$$X'' = \lambda X$$

$$Y'' = \mu Y$$

$$T'' = \alpha^2(\mu + \lambda)T$$

1.2.2 Solving for $X(x)$

Consider:

$$X'' = \lambda X$$

Notice, we are subject to the boundary conditions:

$$u(0, y, t) = u(a, y, t) = 0$$

So:

$$X(0)Y(y)T(t) = X(a)Y(y)T(t) = 0 \iff X(0) = X(a) = 0$$

As we saw in Week 7, $X'' = \lambda X$ has non-trivial solutions satisfying the boundary conditions if and only if $\lambda < 0$, in which case:

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

After applying the boundary conditions, we can solve the eigenvalue problem, and obtain:

$$\lambda_n = \frac{n^2\pi^2}{a^2}$$

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right)$$

for $n \geq 1$.

1.2.3 Solving for $Y(y)$

Consider:

$$Y'' = \mu Y$$

In the same way as above, the boundary conditions require $Y(0) = Y(b) = 0$, and the eigenvalue problem is solved via:

$$\lambda_m = \frac{m^2\pi^2}{b^2}$$

$$Y_m(y) = \sin\left(\frac{m\pi y}{b}\right)$$

for $m \geq 1$.

1.2.4 Solving for $T(t)$

Consider:

$$T'' = \alpha^2(\lambda + \mu)T = \alpha^2 \left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} \right) T$$

Lets denote:

$$\omega_{m,n}^2 = \alpha^2 \left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} \right)$$

We end up with the same ODEs as above, with general solutions:

$$T_{m,n}(t) = A_{m,n}\cos(\omega_{m,n}t) + B_{m,n}\sin(\omega_{m,n}t)$$

1.2.5 Solving the PDE

We employ the principle of superposition to see that:

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (A_{m,n}\cos(\omega_{m,n}t) + B_{m,n}\sin(\omega_{m,n}t)) \times \sin\left(\frac{n\pi x}{a}\right) \times \sin\left(\frac{m\pi y}{b}\right)$$

Firstly, recall one of the initial conditions:

$$u_t(x,y,0) = 0$$

In particular this implies that:

$$\frac{d}{dt}A_{m,n}\cos(\omega_{m,n}t) + B_{m,n}\sin(\omega_{m,n}t) = -C_{m,n}\sin(\omega_{m,n}t) + D_{m,n}\cos(\omega_{m,n}t) = 0$$

Which implies:

$$D_{m,n}\cos(\omega_{m,n}0) = 0 \iff D_{m,n} \iff B_{m,n} = 0$$

Hence, the general solution becomes:

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{m,n}\cos(\omega_{m,n}t) \times \sin\left(\frac{n\pi x}{a}\right) \times \sin\left(\frac{m\pi y}{b}\right)$$

If we now consider the final initial condition:

$$u(x,y,0) = f(x,y)$$

we get:

$$u(x,y,0) = f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{m,n} \times \sin\left(\frac{n\pi x}{a}\right) \times \sin\left(\frac{m\pi y}{b}\right)$$

This is like a Fourier Problem in 2D, and we can find $A_{m,n}$ analogously:

$$A_{m,n} = \frac{4}{ab}\int_0^a \int_0^b f(x,y)\sin\left(\frac{n\pi x}{a}\right) \times \sin\left(\frac{m\pi y}{b}\right) dydx$$

(see here for more)

This explains why a drum doesn't sound harmonic: the fundamental frequencies in the solution are not multiples of each other.

1.3 Circular Membrane

Perhaps reading this is more clear; it also contains the solution to Bessel's Equation

We now consider a circular membrane. We change to polar coordinates:

$$u_{tt} = \alpha^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right)$$

given the following initial/boundary conditions:

- $u(1, \theta, t) = 0$
- $u_t(r, \theta, 0) = 0$
- $u(r, \theta, 0) = f(r, \theta)$

1.3.1 Separation of Variables

Taking $u(r, \theta, t) = R(r)\Theta(\theta)T(\theta)$:

$$R\Theta T'' = \alpha^2 (R''\Theta T + \frac{1}{r} R'\Theta T + \frac{1}{r^2} R\Theta'' T$$

Dividing through by $R\Theta T$:

$$\frac{1}{\alpha^2}\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{(rR')'}{rR} + \frac{1}{r^2}\frac{\Theta''}{\Theta}$$

Notice, the LHS is a function of t, and we can make the RHS a function of r enforcing that $\frac{\Theta''}{\Theta} = -m^2$. Then, all the ratios must be constants, so:

$$\Theta'' = -m^2\Theta$$

$$(rR)' = (-\mu^2 r + \frac{m^2}{r})R$$

(notice a factor of r cancels)

$$T'' = -\mu^2\alpha^2 T$$

1.3.2 Solving for $\Theta(\theta)$

We enforce that Θ be 2π periodic in θ. The ODE $\Theta'' = -m^2\Theta$ is the same as above, which we know has solution:

$$\Theta = A\cos(m\theta) + B\sin(m\theta)$$

for $m \geq 0$.

1.3.3 Solving for $R(r)$

Notice, we will require that u be bounded, so in particular R must be bounded as $r \to 0^+$, and $R(1) = 0$ (the latter condition is derived from the initial condition $u(1, \theta, t)$). The ODE can be written as:

$$R'' + \frac{1}{r}R' + (\mu^2 - \frac{m^2}{r^2})R = 0$$

If we make a change of variables $t = \mu r$, we have $R\left(\frac{t}{\mu}\right)$ so:

$$\frac{dR}{dr} = \frac{dR}{dt} \times \frac{dt}{dr} = \frac{dR}{dt} \times \mu$$

$$\frac{d^2R}{dr^2} = \mu^2\frac{d^2R}{dt^2}$$

Hence, the ODE becomes:

$$\mu^2\frac{d^2R}{dt^2} + \frac{\mu}{t}\frac{dR}{dt} \times \mu + \left(\mu^2 - \frac{\mu^2 m^2}{t^2}\right)R\left(\frac{t}{\mu}\right) = 0$$

Dividing through by μ^2:

$$\frac{d^2R}{dt^2} + \frac{1}{t}\frac{dR}{dt} \times \mu + \left(1 - \frac{m^2}{t^2}\right)R\left(\frac{t}{\mu}\right) = 0$$

Multiplying by t^2 shows us that this is a Bessel Equation of order m (if $m = 0$, then we'd get the equation shown at the very top), and so, it follows that, going from t to r coordinates:

$$R(r) = AJ_m(\mu r) + BY_m(\mu r)$$

Y_m is an unbounded term, but R must be bounded, so we enforce $B = 0$:

$$R(r) = AJ_m(\mu r)$$

Recall the initial condition boundary condition $R(1) = 0$:

$$R(1) = AJ_m(\mu) = 0$$

In other words, the eigenvalues are all the zeros of J_m, $\mu_{m,n}$, and the eigenfunctions will be $R_{m,n} = J_m(\mu_{m,n}r)$.

The J_m are oscillatory, so there are infinitely manye igenvalues and eigenfunctions. Moreover, notice that $(rR)' = (-\mu^2 r + \frac{m^2}{r})R$ is a (singular) Sturm-Liouville Problem, which means that the $R_{m,n}$ will be **orthogonal**.

1.3.4 Solving for $T(t)$

For T we have the same ODE as in teh rectangular case, so we know:

$$T_{m,n}(t) = C_{m,n}\cos(\mu_{m,n}\alpha t) + D_{m,n}\sin(\mu_{m,n}\alpha t)$$

Similarly, $u_t(r, \theta, 0)$ will imply that $D_{m,n} = 0$, so:

$$T_{m,n}(t) = C_{m,n}\cos(\mu_{m,n}\alpha t)$$

1.3.5 Solving the PDE

Applying superposition:

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(A_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)) \times J_m(\mu_{m,n}r) \times \cos(\mu_{m,n}\alpha t)$$

(we have rewritten $A = A_{m,n}$ and $B = B_{m,n}$ in anticipation to when we compute them, and for consistency with the summation)

As an initial condition we had that $u(r, \theta, 0) = f(r, \theta)$, so:

$$u(r,\theta,0) = f(r,\theta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(A_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)) \times J_m(\mu_{m,n}r)$$

We can then show that:

$$A_{m,n} \propto \int_0^{2\pi} \int_0^1 f(r,\theta)\cos(m\theta) J_m(\mu_{m,n} r) r dr d\theta$$
$$B_{m,n} \propto \int_0^{2\pi} \int_0^1 f(r,\theta)\sin(m\theta) J_m(\mu_{m,n} r) r dr d\theta$$

Actual values of $A_{m,n}$ and $B_{m,n}$ can be found here.

Again, the frequencies of each of the components in the solutions are not multiples of each other, so the sound won't be harmonic.

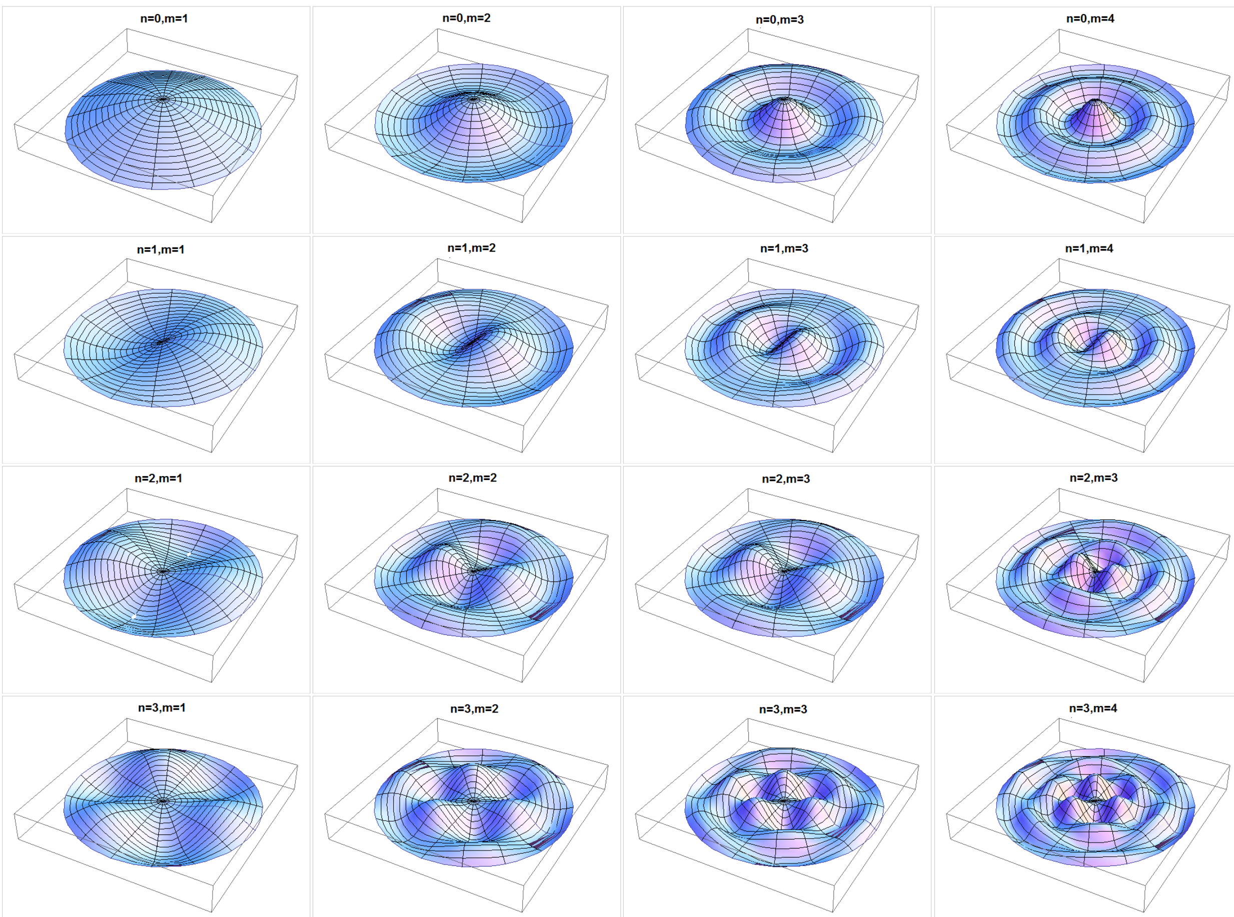

Figure 1: The derivation was excruciating, pedantic, repetitive and full of gaps, but at least the solutions look good. If I look back on these notes, I wrote this severely sleep deprived at 03:20 on Friday the 19th of November, 2021, whilst listening to Clocks by Coldplay.

Paul's Online Math Notes for Laplace Transform
Khan Academy's Series on the Laplace Transform - where I first learn from

2 The Laplace Transform

2.1 Defining the Laplace Transform

Definition (Laplace Transform)
Suppose that:

1. *f is a piecewise continuous function on the interval $0 \leq t \leq A$ for any positive A*
2. *there exist real constants K, a, M with $K, M > 0$ such that:*

$$|f(t)| \leq Ke^{at}, \quad t \geq M$$

*Then, we define the **Laplace Transform** of f via:*

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

given $s > a$.

- **Are the conditions above necessary for the Laplace Transform to be defined?**
  - yes, otherwise the improper integral might not be defined

    Proof: Laplace Transform. By the second assumption, we can bound:

    $$f(t) \leq |f(t)| < Ke^{at}$$

    for $t \geq M$. But then:

    $$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty e^{-st} f(t)dt < K \int_0^\infty e^{-st} e^{at} dt = K \int_0^\infty e^{t(a-s)} dt$$

    If we integrate using the fact that f is piecewise continuous on $0 \leq A$:

    $$\begin{aligned} K \int_0^\infty e^{t(a-s)} dt &= \lim_{A \to \infty} K \int_0^A e^{t(a-s)} dt \\ &= \lim_{A \to \infty} \frac{K}{a-s} \left[e^{t(a-s)} \right]_0^A \\ &= \lim_{A \to \infty} \frac{K}{a-s} \left[e^{A(a-s)} \right] \end{aligned}$$

    Intuitively, if $a - s < 0$, it is easy to see that the limit converges; otherwise, it will diverge (either because the exponential goes to infinity, or if $a - s = 0$, the fraction is undefined). In other words, for convergence we require $s > a$ □

- **What are functions of exponential order?**
  - a function is of exponential order if they satisfy the conditions defined above for the existence of Lapalce Transform

2.2 Basics of the Laplace Transform

2.2.1 A Linear Operator

One of the most useful properties of the Lapalce Transform is that it is linear, meaning that we can decompose complicated functions into its terms.

Proof: Linearity of Laplace Transform.

$$\begin{aligned}\mathcal{L}\left\{c_1 f_1(t) + c_2 f_2(t)\right\} &= \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t))dt \\ &= c_1 \int_0^\infty e^{-st} f_1(t)dt + c_2 \int_0^\infty f_2(t)dt \\ &= c_1 \mathcal{L}\left\{f_1(t)\right\} + c_2 \mathcal{L}\left\{f_2(t)\right\}\end{aligned}$$

□

2.2.2 Laplace Transform of Derivative

The second most useful property of the Laplace Transform is that it is easy to apply it to derivatives. We consider a first order derivative, which can then be generalised.

Laplace Transform of Derivative.

$$\mathcal{L}\left\{f'(t)\right\} = \int_0^\infty e^{-st} f'(t)dt$$

If we use integration by parts:

$$u = e^{-st} \qquad du = -se^{-st}$$
$$dv = f'(t) \qquad v = f(t)$$

Hence:

$$\begin{aligned}\mathcal{L}\left\{f'(t)\right\} &= \int_0^\infty e^{-st} f'(t)dt \\ &= \lim_{A\to\infty} \left[f(t)e^{-st}\right]_0^A + s\int_0^\infty f(t)e^{-st}dt \\ &= \lim_{A\to\infty} \left[f(A)e^{-sA} - f(0)\right] + s\mathcal{L}\left\{f(t)\right\} \\ &= -f(0) + s\mathcal{L}\left\{f(t)\right\}\end{aligned}$$

Using similar arguments we can show that, assuming each derivative can have the Laplace Transform applied:

$$\boldsymbol{\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n \mathcal{L}\left\{f(t)\right\} - s^{n-1}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)}$$

(we obtain this formula by recursively applying the theorem above. For example:)

$$\begin{aligned}\mathcal{L}\left\{f''(t)\right\} &= s\mathcal{L}\left\{f'(t)\right\} - f'(0) \\ &= s(s\mathcal{L}\left\{f(t)\right\} - f(0)) - f'(0) \\ &= s^2\mathcal{L}\left\{f(t)\right\} - sf(0) - f'(0)\end{aligned}$$

□

2.2.3 Laplace Transform of t^n

By knowing this Laplace Tranform, we can find the Laplace Transform of any polynomial by linearity:

$$\mathcal{L}\left\{t^n\right\} = \int_0^\infty e^{-st}t^n dt$$

If we apply integration by parts:

$$u = t^n \qquad du = nt^{n-1}$$
$$dv = e^{-st} \qquad v = -\frac{1}{s}e^{-st}$$

Hence:

$$\begin{aligned}\mathcal{L}\left\{t^n\right\} &= \int_0^\infty e^{-st}t^n dt \\ &= \lim_{A\to\infty}\left[-\frac{t^n}{s}e^{-st}\right]_0^A + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st} \\ &= \lim_{A\to\infty}\left[-\frac{A^n}{s}e^{-sA}\right] + \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}\end{aligned}$$

If we use the argument that exponentials grow faster than polynomials, it follows that:

$$\lim_{A\to\infty}\left[-\frac{A^n}{s}e^{-sA}\right]$$

So:

$$\mathcal{L}\left\{t^n\right\} = \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}$$

It follows that[1], if we repeatedly apply this recursion:

$$\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^n}\mathcal{L}\left\{1\right\}$$

The Laplace Transform of 1 is easy:

$$\mathcal{L}\left\{1\right\} = \int_0^\infty e^{-st}dt = -\frac{1}{s}\lim_{A\to\infty}\left[e^{-st}\right]_0^A = -\frac{1}{s}\lim_{A\to\infty}\left[e^{-sA}-1\right] = \frac{1}{s}$$

Hence, it follows that:

$$\boldsymbol{\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}}$$

2.2.4 Laplace Transform of Exponential

$$\begin{aligned}\mathcal{L}\left\{e^{at}\right\} &= \int_0^\infty e^{-(s-a)t}dt \\ &= -\frac{1}{s-a}\lim_{A\to\infty}\left[e^{-(s-a)t}\right]_0^A \\ &= -\frac{1}{s-a}\lim_{A\to\infty}\left[e^{-(s-a)A}-1\right] \\ &= \frac{1}{s-a}\end{aligned}$$

[1] This all can be proven by induction, but we would require knowledge of what we are trying to prove

where we have assume that $s > a$, as otherwise the limit would diverge.

Hence, it follows that:

$$\boldsymbol{\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}}$$

2.2.5 Laplace Transform of Sine and Cosine

To save some work, we first acknowledge that:

$$\cos(at) = Re(e^{iat})$$

$$\sin(at) = Im(e^{iat})$$

Hence, we consider:

$$\mathcal{L}\left\{e^{iat}\right\} = \frac{1}{s-ia}$$

Then:

$$\begin{aligned}\mathcal{L}\left\{\cos(at)\right\} &= Re\left(\frac{1}{s-ia}\right)\\ &= Re\left(\frac{s+ia}{s^2+a^2}\right)\\ &= \frac{s}{s^2+a^2}\end{aligned}$$

We can find $\mathcal{L}\left\{\sin(at)\right\}$ by considering the imaginary part. It is then easy to see that:

$$\boldsymbol{\mathcal{L}\left\{\cos(at)\right\} = \frac{s}{s^2+a^2}}$$

$$\boldsymbol{\mathcal{L}\left\{\sin(at)\right\} = \frac{a}{s^2+a^2}}$$

More on the following can be found here

2.2.6 Laplace Transform of $f(ct)$

$$\mathcal{L}\left\{f(ct)\right\} = \int_0^\infty e^{-st}f(ct)dt$$

Consider the change of variables $u = ct$. Then:

$$\frac{du}{dt} = c \implies \frac{1}{c}du = dt$$

$$u(0) = c \times 0 = 0$$

$$u(\infty)^2 = \infty$$

[2] Ouch!

Hence, from u-substitution:

$$\begin{aligned}
\mathcal{L}\{f(ct)\} &= \int_0^\infty e^{-st} f(ct)dt \\
&= \frac{1}{c}\int_0^\infty e^{-u\left(\frac{s}{c}\right)} f(u)du \\
&= \frac{1}{c}\mathcal{L}\{f(t)\}\left(\frac{s}{c}\right) \\
&= \frac{1}{c}F\left(\frac{s}{c}\right)
\end{aligned}$$

2.2.7 s - shift

$$\begin{aligned}
\mathcal{L}\left\{e^{-ct}f(t)\right\} &= \int_0^\infty e^{-st}e^{-ct} f(t)dt \\
&= \int_0^\infty e^{-(s+c)t} f(t)dt \\
&= \mathcal{L}\{f(t)\}(s+c) \\
&= F(s+c)
\end{aligned}$$

2.2.8 Derivative in s

$$\begin{aligned}
\mathcal{L}\{tf(t)\} &= \int_0^\infty te^{-st} f(t)dt \\
&= \int_0^\infty -\frac{d}{ds}\left(e^{-st}\right) f(t)dt \\
&= -\frac{d}{ds}\int_0^\infty e^{-st} f(t)dt \\
&= -\frac{d}{ds}\mathcal{L}\{f(t)\}(s) \\
&= -F'(s)
\end{aligned}$$

2.2.9 Laplace Transform of $t^n f(t)$

We saw above that:

$$\mathcal{L}\{tf(t)\} = -F'(s)$$

Consider:

$$\begin{aligned}
\mathcal{L}\left\{t^2 f(t)\right\} &= \int_0^\infty t^2 e^{-st} f(t)dt \\
&= \int_0^\infty \frac{d^2}{ds^2}\left(e^{-st}\right) f(t)dt \\
&= \frac{d^2}{ds^2}\int_0^\infty e^{-st} f(t)dt \\
&= \frac{d^2}{ds^2}\mathcal{L}\{f(t)\}(s) \\
&= F''(s)
\end{aligned}$$

where we have use the fact that:

$$\frac{d}{ds}e^{-st} = -te^{-st} \qquad \frac{d}{ds}(-te^{-st}) = -t^2e^{-st}$$

Overall, it follows that:

$$\boldsymbol{\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\left\{f(t)\right\}(s) = (-1)^n F^{(n)}(s)}$$

2.2.10 Laplace Transform of Products of Exponentials and Cosines/Sines

As we will see, when applying the Laplace Transform to an ODE, we obtain solutions as algebraic expressions, typically rational. It is thus useful to consider these products, as means of reversing said rational fractions.

$$\begin{aligned}
\mathcal{L}\left\{e^{bt}\cos(at)\right\} &= \mathcal{L}\left\{\cos(at)\right\}(s-b) = \frac{s-b}{(s-b)^2+a^2} \\
\mathcal{L}\left\{e^{bt}\sin(at)\right\} &= \mathcal{L}\left\{\sin(at)\right\}(s-b) = \frac{a}{(s-b)^2+a^2} \\
\mathcal{L}\left\{t^n e^{bt}\cos(at)\right\} &= (-1)^n \frac{d^n}{ds^n}\mathcal{L}\left\{e^{bt}\cos(at)\right\} = (-1)^n \frac{d^n}{ds^n}\frac{s-b}{(s-b)^2+a^2}
\end{aligned}$$

2.2.11 Laplace Transform Table

A very nice, thorough Laplace Transform Table

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt = \lim_{b\to\infty} \int_0^b e^{-st} f(t)\,dt.$$

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\ n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\ n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\ n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\ n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

2.2.12 The Inverse Laplace Transform

- **Is the Laplace Transform unique?**

- it can b ehsown that if f and g are continuous functions, and:

$$\mathcal{L}\{f\} = \mathcal{L}\{g\}$$

then:

$$f = g$$

- **What if f and g are only pointwise continuous?**
  - then it is possible that f and g are distinct, but:

$$\mathcal{L}\{f\} = \mathcal{L}\{g\}$$

  - this is a consequence of the fact that at the points of discontinuity, f and g can be defined in different ways

- **What is the Inverse Laplace Transform?**
  - an operator, which given $F(s)$ return $f(t)$. In other words:

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

- **Is the Inverse Laplace Transform a linear operator?**
  - the Inverse Laplace Transform is also linear
  - if:

$$f_1(t) = \mathcal{L}^{-1}\{F_1(s)\} \quad \dots \quad f_n(t) = \mathcal{L}^{-1}\{F_n(s)\}$$

Then, the function:

$$f(t) = f_1(t) + f_2(t) + \dots + f_n(t)$$

has a Laplace Transform $F(s)$ given by:

$$F(s) = F_1(s) + F_2(s) + \dots + F_n(s)$$

  - by the Uniqueness of the Lapalce Transform, we can write:

$$f(t) = \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}(F_1(s)) + \mathcal{L}^{-1}(F_2(s)) + \dots + \mathcal{L}^{-1}(F_n(s))$$

- **How can we find the Inverse Laplace Transform?**
  - there is a formula available, but it requires knowledge of complex analysis
  - it is just easier to look at the table of Laplace Transforms. For example,

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2 + a^2}\right\} = \sin(at)$$

2.3 Partial Fraction Decomposition

S.no	Rational Fraction	Partial Fraction
1.	$\frac{p(x)+q}{(x-a)(x-b)}$	$\frac{A}{(x-a)}+\frac{B}{(x-b)}$
2.	$\frac{p(x)+q}{(x-a)^2}$	$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)}+\frac{B}{(x-b)}+\frac{C}{(x-c)}$
4.	$\frac{px^2+q(x)+r}{(x-a)^2(x-b)}$	$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\frac{B}{(x-b)}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{\frac{A}{(x-a)}+Bx+C}{(x^2+bx+c)}$

2.4 Laplace Transform for Solving ODEs

- **What is the result of applying the Laplace Transform to an ODE?**
  - lets consider a second order, linear, non-homogeneous ODE:

$$ay'' + by' + cy = f(t)$$

  - if we apply the Laplace Transform to the LHS:

$$\begin{aligned}\mathcal{L}\{ay'' + by' + cy\} &= a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y\} \\ &= as^2\mathcal{L}\{y\} - asy(0) - ay'(0) + bs\mathcal{L}\{y\} - by(0) + c\mathcal{L}\{y\} \\ &= \mathcal{L}\{y\}(as^2 + bs + c) - y(0)(as + b) - ay'(0)\end{aligned}$$

  - let $F(s) = \mathcal{L}\{f(t)\}$
  - then, the ODE becomes:

$$\mathcal{L}\{y\}(as^2 + bs + c) - y(0)(as + b) - ay'(0) = F(s)$$

  - solving for $\mathcal{L}\{y\}$:

$$\mathcal{L}\{y\} = \frac{F(s) + y(0)(as + b) + ay'(0)}{as^2 + bs + c}$$

  - notice the denominator is the **characteristic polynomial** of the ODE
  - we can then solve this by finding the Inverse Laplace Transform, which is usually done by employing partial fraction decomposition

- **Why use Laplace Transform instead of standard methods?**
  1. It turns solving the ODE into an algebraic equation in s, which is easy to solve
  2. Solutions involving initial conditions are embedded in solutions using Laplace Transform by the presence of $y(0), y'(0), \ldots$, so less steps
  3. In solving non-homogeneous ODE, we don't need to compute the homogeneous ODE and then find a particular solution
  4. The method is generalisable for higher-order ODEs

3 Examples

1. **Solve the ODE:**

$$y'' - y' - 2y = 0$$

satisfying $y(0) = 1, y'(0) = 0$**.**

Applying the Laplace Transform, we get:

$$\mathcal{L}\{y\} = \frac{s-1}{s^2 - s - 2}$$

where we have used:

- $a = 1$
- $b = c = -1$
- $F(s) = 0$
- $y(0) = 1$
- $y'(0) = 0$

Notice:

$$s^2 - s - 2 = (s+1)(s-2)$$

Thus:

$$\mathcal{L}\{y\} = \frac{s-1}{(s+1)(s-2)}$$

Lets apply partial fraction decomposition:

$$\begin{aligned}
&\frac{s-1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} \\
\implies &\frac{s-1}{(s+1)(s-2)} = \frac{As - 2A + Bs + B}{(s+1)(s-2)} \\
\implies &1 = A + B \\
&-1 = B - 2A \\
\implies &2 = 3A \ \therefore \ A = \frac{2}{3} \\
\implies &1 = \frac{2}{3} + B \ \therefore \ B = \frac{1}{3}
\end{aligned}$$

Hence:

$$\mathcal{L}\{y\} = \frac{s-1}{s^2 - s - 2} = \frac{2}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}$$

and so, recalling that:

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$$

it follows that:

$$y = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$$

2. **Solve the ODE:**

$$y'' + y = \sin(2t)$$

subject to $y(0) = 2, y'(0) = 1$

Applying the Laplace transform:

$$\mathcal{L}\{y\} = \frac{\frac{2}{s^2+4} + 2s + 1}{s^2+1} = \frac{2 + 2s^3 + 8s + s^2 + 4}{(s^2+1)(s^2+4)} = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

We then consider partial fraction decomposition:

$$\begin{aligned}
&\frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \\
\Longrightarrow\ &\frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} = \frac{(As+B)(s^2+4)) + (Cs+D)(s^2+1)}{(s^2+1)(s^2+4)} \\
\Longrightarrow\ &\frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} = \frac{As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D}{(s^2+1)(s^2+4)} \\
\Longrightarrow\ &\frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} = \frac{s^3(A+C) + s^2(B+D) + s(4A+C) + (4B+D)}{(s^2+1)(s^2+4)} \\
\Longrightarrow\ &A + C = 2 \\
&B + D = 1 \\
&4A + C = 8 \\
&4B + D = 6
\end{aligned}$$

Substracting the 3rd equation from the first equation:

$$3A = 6 \implies A = 2$$

So $A + C = 2$ implies:

$$C = 0$$

Substracting the 4th equation from the second equation:

$$3B = 5 \implies B = \frac{5}{3}$$

So $B + D = 1$ implies:

$$D = -\frac{2}{3}$$

Thus:

$$\begin{aligned}
\mathcal{L}\{y\} &= \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} \\
&= \frac{2s + \frac{5}{3}}{s^2+1} + \frac{-\frac{2}{3}}{s^2+4} \\
&= 2\frac{s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1} + -\frac{1}{3}\frac{2}{s^2+4}
\end{aligned}$$

But recall:

$$\begin{aligned}
\mathcal{L}\{\cos(at)\} &= \frac{s}{s^2+a^2} \\
\mathcal{L}\{\sin(at)\} &= \frac{a}{s^2+a^2}
\end{aligned}$$

Hence, it follows that:

$$y = 2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t)$$