Honours Differential Equations - Week 8 - Sturm-Liouville Theory

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Contents

1 Motivation

1.1 Symmetric Matrices

- **What is a symmetric matrix?**
  - a matrix $\boldsymbol{A}$ is **symmetric** if and only if:
  $$\boldsymbol{A} = \boldsymbol{A}^T$$
- **What is an alternative definition for a symmetric matrix?**
  - a perhaps more versatile definition for symmetric matrices is that:
  $$\underline{u} \cdot (\boldsymbol{A}\underline{v}) = \underline{v} \cdot (\boldsymbol{A}\underline{u})$$
  - (the following is derived without any formal check from books or notes, so it might be incorrect)

  If $\boldsymbol{A} = \boldsymbol{A}^T$:
  $$\begin{aligned} & \underline{u} \cdot (\boldsymbol{A}\underline{v}) \\ =& \underline{u}^T \boldsymbol{A}\underline{v} \\ =& \underline{u}^T \boldsymbol{A}^T \underline{v} \\ =& (\boldsymbol{A}\underline{u})^T \underline{v} \\ =& (\boldsymbol{A}\underline{u}) \cdot \underline{v} \\ =& \underline{v} \cdot (\boldsymbol{A}\underline{u}) \end{aligned}$$

  If $\underline{u} \cdot (\boldsymbol{A}\underline{v}) = \underline{v} \cdot (\boldsymbol{A}\underline{u})$:
  $$\begin{aligned} & \underline{u} \cdot (\boldsymbol{A}\underline{v}) = \underline{v} \cdot (\boldsymbol{A}\underline{u}) \\ \implies & \underline{u}^T (\boldsymbol{A}\underline{v}) = \underline{v}^T (\boldsymbol{A}\underline{u}) \\ \implies & (\boldsymbol{A}\underline{v})^T \underline{u} = (\boldsymbol{A}\underline{u})^T \underline{v} \\ \implies & \underline{v}^T \boldsymbol{A}^T \underline{u} = \underline{v}^T \boldsymbol{A}^T \underline{v} \\ \implies & \underline{v} \cdot (\boldsymbol{A}^T \underline{u}) = \underline{v} \cdot (\boldsymbol{A}^T \underline{v}) \end{aligned}$$

  This can only be true if and only if $A = A^T$, so A is symmetric.
- **What is the Spectral Theorem?**
  - the Spectral Theorem states that if we have a **symmetric** matrix with real entries, then its eigenvectors are **orthogonal**
  - in proving this, it can be shown that eigenvalues must be **real**. Read more on this here
  - we can establish orthogonality by using the second definition of symmetric matrices. If $\underline{u}, \underline{v}$ are distinct eigenvectors:
  $$\begin{aligned} & \underline{u} \cdot (\boldsymbol{A}\underline{v}) = \underline{v} \cdot (\boldsymbol{A}\underline{u}) \\ \implies & \underline{u} \cdot (\lambda_v \underline{v}) = \underline{v} \cdot (\lambda_u \underline{u}) \\ \implies & (\lambda_v - \lambda_u)(\underline{u} \cdot \underline{v}) = 0 \end{aligned}$$

  and since eigenvalues will be distinct, this is true if and only if $\underline{u}$ and $\underline{v}$ are orthogonal.

1.2 Sine as a Basis Eigenfunction

Recall, if we have an ODE:

$$X_n^{''} = \lambda X_n$$

with boundary conditions:

$$X_n(0) = X_n(L) = 0$$

the fundamental set of eigenfunctions is:

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

We know that the set of X_n is mutually orthogonal:

$$(X_m, X_n) = \int X_m X_n dx = 0, \quad m \neq 0$$

Notice, when se computed the eigenfuncition, we only considered $\lambda \in \mathbb{R}$, even if the ODE is perfectly valid if we have a complex λ. We can justify this.

If we let $\mathcal{L}[y]$ denote a 2nd order differential operator, and consider $X_m \neq X_n$, then:

$$\mathcal{L}\left[X_m\right] = \lambda_m X_m \qquad \mathcal{L}\left[X_n\right] = \lambda_n X_n$$

Using the dot product and its linearity:

$$\begin{aligned}
&(X_n, \mathcal{L}\left[X_m\right]) = \lambda_m (X_n, X_m) \qquad (X_m, \mathcal{L}\left[X_n\right]) = \lambda_n (X_n, X_m) \\
\implies &(X_n, \mathcal{L}\left[X_m\right]) - (X_m, \mathcal{L}\left[X_n\right]) = (\lambda_m - \lambda_n)(X_n, X_m) \\
\implies &(X_n, \mathcal{L}\left[X_m\right]) - (X_m, \mathcal{L}\left[X_n\right]) = 0 \\
\implies &(X_n, \mathcal{L}\left[X_m\right]) = (X_m, \mathcal{L}\left[X_n\right])
\end{aligned}$$

Notice, this heavily ressembles the formula above for **symmetric** matrices. In this case, we can say that the operator $\mathcal{L}[y]$ is **self-adjoint**. Not only this, but a self-adjoint operator has the same properties as a symmetric matrices:

- the eigenvalues will be real
- the eigenvectors will be orthogonal

For brevity, we won't show it for this example, but will show it when we develop Sturm-Liouville Theory.

1.3 Generalised Heat Equation

- **What is the heat equation?**
  - given a positive constant α^2:

    $$\alpha^2 u_{xx} = u_t$$

    for $0 < x < L, t > 0$ and given boundary conditions:

    $$u(0,t) = u(L,t) = 0$$

    and initial condition:

    $$u(x,0) = f(x)$$

- **How can we make the heat equation more general?**

- consider the PDE:

$$r(x)u_t = (p(x)u_x)_x - q(x)u + F(x,t)$$

given boundary conditions:

$$u_x(0,t) - h_1 u(0,t) = 0$$
$$u_x(L,t) - h_2 u(L,t) = 0$$

- the PDE itself now allows for information regarding the presence of heat sources, and changing material properties of the rod
- the boundary conditions enforce that the temperature at the ends of the rod are **proportional** to the heat flow at the ends

In attempting to solve the PDE, we will obtain an ODE with solutions with very interesting properties. In fact it is this ODE which is the focus of the study of Sturm-Liouville Theory. This ODE appears in many other applications beyond heat flow.

Lets consider the PDE with $F(x,t) = 0$ (we can solve this inhomogeneous case later, once we have developed Sturm-Liouville Theory). We can apply separation of variables:

$$u(x,t) = X(x)T(t)$$

which leads to:

$$r(x)X(x)T'(t) = (p(x)X'(x))'T(t) - q(x)X(x)T(t)$$
$$\implies \frac{(p(x)X'(x))' - q(x)X(x)}{r(x)X(x)} = \frac{T'(t)}{T(t)} = -\lambda$$

(We anticipate that only $\lambda < 0$ will yield non-trivial solutions, so we use $-\lambda$ as a constant). We then obtain 2 ODEs:

$$(p(x)X'(x))' - q(x)X(x) + \lambda r(x)X(x) = 0$$
$$T'(t) + \lambda T(t) = 0$$

The boundary conditions also change, since $u(x,t) = X(x)T(t)$:

$$X'(0)T(t) - h1X'(0)T(t) = 0 \implies X'(0) - h1X'(0) = 0$$
$$X'(L)T(t) - h1X'(L)T(t) = 0 \implies X'(L) - h1X'(L) = 0$$

Solving this BVP is the subject of **Sturm-Liouville Theory**, which we shall develop below.

2 Sturm-Liouville Theory

2.1 The Sturm-Liouville Boundary Value Problem

A **Sturm-Liouville Boundary Value Problem** is a BVP given by the second order ODE:

$$(p(x)y')' - q(x)y + \lambda r(x)y = 0$$

where:

- we consider the **interval**:

$$0 < x < 1$$

- the **boundary conditions** are:

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0$$
$$\beta_1 y(1) + \beta_2 y'(1) = 0$$

- the functions p, p', q, r are **continuous** on $[0, 1]$
- $\forall x \in [0, 1], p(x) > 0, r(x) > 0$

Further points to note:

- the conditions on p, p', q, r make this a **regular** BVP; this is typically called the **Regular Sturm-Liouville Boundary Problem**
- the assumptions simplify the problem, but allow it to remain generalised
- the boundary conditions are **separated**: each depends on only 1 of the endpoints
- at least one of α_1/β_1 and α_2/β_2 must be **non-zero**
- we can write the ODE using the operator:

$$\mathcal{L}[y] = -(p(x)y')' + q(x)y$$

2.2 Lagrange's Identity

We will use **Lagrange's Identity** very extensively. In particular, because we can use it to show that $\mathcal{L}[y]$ will be **self-adjoint**, which from the motivation section, we know has nice properties (real eigenvalues, orthogonal eigenvectors).

2.2.1 Recapping: Dot Product and New Definition

It will be useful to recall the definition of the dot product of 2 (possibly complex) functions over some interval $[0, L]$:

$$(u, v) = \int_0^L u(x)v(x)dx$$

We can introduce an alternative "definition", just for convenience when discussing Sturm-Liouville Theory:

$$\langle u, v \rangle = \int_0^L u(x)v(x)r(x)dx$$

where $r(x)$ is the weight factor in the Sturm-Liouville BVP.

2.2.2 Deriving Lagrange's Identity With Sturm-Liouville Operator

Let u, v be functions with continuous second derivatives for $x \in [0, 1]$. For completeness we shall allow complex u, v. Then:

$$\begin{aligned}
(\mathcal{L}[v], u) &= \int_0^1 \mathcal{L}[v] \times \bar{u}(x)dx \\
&= \int_0^1 (-(p(x)v'(x))' + q(x)v(x)) \times \bar{u}(x)dx \\
&= \int_0^1 -(p(x)v'(x))'\bar{u}(x) + q(x)v(x)\bar{u}(x)dx
\end{aligned}$$

We can apply integration by parts twice. With the first integral:

$$\begin{aligned} a &= \bar{u}(x) & da &= \bar{u}'(x) \\ db &= -(p(x)v'(x))' & b &= -p(x)v'(x) \end{aligned}$$

So:

$$\int_0^1 -(p(x)v'(x))'\bar{u}(x)dx = [-p(x)v'(x)\bar{u}(x)]_0^1 + \int_0^1 p(x)v'(x)\bar{u}'(x)dx$$

We can apply it a second time with the integral we get:

$$\begin{aligned} a &= p(x)\bar{u}'(x) & da &= (p(x)\bar{u}'(x))' \\ db &= v'(x) & b &= v(x) \end{aligned}$$

So:

$$\int_0^1 p(x)v'(x)\bar{u}'(x)dx = [p(x)\bar{u}'(x)v(x)]_0^1 - \int_0^1 v(x)(p(x)\bar{u}'(x))'dx$$

Thus, overall we have:

$$\begin{aligned} (\mathcal{L}[v], u) &= [-p(x)v'(x)\bar{u}(x)]_0^1 + [p(x)\bar{u}'(x)v(x)]_0^1 + \int_0^1 -v(x)(p(x)\bar{u}'(x))' + q(x)v(x)\bar{u}(x)dx + \\ &= [-p(x)v'(x)\bar{u}(x) + p(x)\bar{u}'(x)v(x)]_0^1 + \int_0^1 v(x)(-(p(x)\bar{u}'(x))' + q(x)\bar{u}(x))dx \\ &= [p(x)(\bar{u}'(x)v(x) - v'(x)\bar{u}(x)]_0^1 + \int_0^1 v(x) \times \mathcal{L}[\bar{u}]\, dx \\ &= [p(x)(\bar{u}'(x)v(x) - v'(x)\bar{u}(x))]_0^1 + (v(x), \mathcal{L}[u]) \end{aligned}$$

The expression:

$$(\mathcal{L}[v], u) - (v, \mathcal{L}[u]) = [p(x)(\bar{u}'(x)v(x) - v'(x)\bar{u}(x))]_0^1$$

is **Lagrange's Identity**.

In the notes and the book, we have:

$$(v, \mathcal{L}[u]) - (\mathcal{L}[v], u) = [-p(x)(\bar{u}'(x)v(x) - v'(x)\bar{u}(x))]_0^1$$

Or in integral form:

$$\int_0^1 v\mathcal{L}[\bar{u}] - \mathcal{L}[v]\,\bar{u}dx = [-p(x)(\bar{u}'(x)v(x) - v'(x)\bar{u}(x))]_0^1$$

2.2.3 Applying Boundary Conditions to Lagrange's Equation

Consider the boundary conditions for the Sturm-Liouville BVP:

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0$$

$$\beta_1 y(1) + \beta_2 y'(1) = 0$$

If we expand the non-integral term in Lagrange's Identity (ignore the p term, since its strictly positive, and doesn't appear in the boundary conditions):

$$\begin{aligned} &[\bar{u}'(x)v(x) - v'(x)\bar{u}(x)]_0^1 \\ &= \bar{u}'(1)v(1) - v'(1)\bar{u}(1) - \bar{u}'(0)v(0) + v'(0)\bar{u}(0) \end{aligned}$$

Both u and v must satisfy the boundary conditions, so it follows that (assuming that none of $\alpha_1, \alpha_2, \beta_1, \beta_2$ are 0):

$$\bar{u}'(0) = -\frac{\alpha_1 \bar{u}(0)}{\alpha_2}$$
$$\bar{u}'(1) = -\frac{\beta_1 \bar{u}(1)}{\beta_2}$$
$$v'(0) = -\frac{\alpha_1 v(0)}{\alpha_2}$$
$$v'(1) = -\frac{\beta_1 v(1)}{\beta_2}$$

From which it follows that:

$$\begin{aligned}
&[\bar{u}'(x)v(x) - v'(x)\bar{u}(x)]_0^1 \\
&= \bar{u}'(1)v(1) - v'(1)\bar{u}(1) - \bar{u}'(0)v(0) + v'(0)\bar{u}(0) \\
&= -\frac{\beta_1 \bar{u}(1)}{\beta_2}v(1) + \frac{\beta_1 v(1)}{\beta_2}\bar{u}(1) + \frac{\alpha_1 \bar{u}(0)}{\alpha_2}v(0) - \frac{\alpha_1 v(0)}{\alpha_2}\bar{u}(0) \\
&= 0
\end{aligned}$$

A much simpler proof is possible with 2 of the constants being 0. For example, taking $\alpha_2 = \beta_2 = 0$, the boundary conditions imply that $y(0) = y(1) = 0$ so:

$$\begin{aligned}
&[\bar{u}'(x)v(x) - v'(x)\bar{u}(x)]_0^1 \\
&= \bar{u}'(1)v(1) - v'(1)\bar{u}(1) - \bar{u}'(0)v(0) + v'(0)\bar{u}(0) \\
&= 0
\end{aligned}$$

(notice, this is equivalent to setting the boundary conditions to enforce periodicity)

Thus, if u, v satisfy the Sturm-Liouville ODE and its boundary conditions, it follows that:

$$(\mathcal{L}[v], u) = (v, \mathcal{L}[u])$$

by construction.

2.3 Eigenvalues for Sturm-Liouville BVP

2.3.1 Eigenvalues are Real

Theorem (Eigenvalues of Sturm-Liouville Boundary Value Problem)
Consider the Sturm-Liouville BVP:

$$(p(x)y')' - q(x)y + \lambda r(x)y = 0$$

where:

- *we consider the* ***interval****:*
$$0 < x < 1$$
- *the* ***boundary conditions*** *are:*
$$\alpha_1 y(0) + \alpha_2 y'(0) = 0$$
$$\beta_1 y(1) + \beta_2 y'(1) = 0$$

- *the functions p, p', q, r are **continuous** on $[0,1]$*
- $\forall x \in [0,1], p(x) > 0, r(x) > 0$

*Then, the eigenvalues of the BVP are **real**.*

Proof: Eigenvalues of Sturm-Liouville Boundary Value Problem. Let:

$$u = v = \phi = M(x) + iN(x)$$

be solutions (eigenfunctions) to the Sturm-Liouville BVP. Let $\lambda \in \mathbb{C}$ be their eigenvalue.

The Sturm-Liouville BVP can be formulated via:

$$\mathcal{L}[y] = \lambda r(x) y$$

So from Lagrange's Identity:

$$(\mathcal{L}[\phi], \phi) = (\phi, \mathcal{L}[\phi]) \implies (\lambda r(x)\phi, \phi) = (\phi, \lambda r(x)\phi)$$

(alternatively we could've written this as $\langle \lambda\phi, \phi \rangle = \langle \phi, \lambda\phi \rangle$, since $r(x) \in \mathbb{R}$, so $r(x) = \bar{r}(x)$)

Using the definition of the function dot product:

$$\lambda \int_0^1 r(x)\phi(x)\bar{\phi}(x)dx = \bar{\lambda} \int_0^1 \bar{r}(x)\bar{\phi}(x)\phi(x)dx$$
$$\implies (\lambda - \bar{\lambda}) \int_0^1 r(x)(M^2(x) + N^2(x))dx = 0$$

The term in the integral is clearly psoitive, so this can only be possible if and only if:

$$\lambda = \bar{\lambda}$$

so it must be the case that $\lambda \in \mathbb{R}$, and this will be true for any eigenfunction satisfying the BVP. □

- **Why is having real eigenvalues important?**
  - this justifies how we have only been considering only the real eigenvalues of BVP up until now (we always used $\lambda = \mu^2$)
  - as in general ODEs, independent of there being complex eigenvalues, we can always use a real alternative
  - it can also be shown that the corresponding eigenfunctions will also be real

2.3.2 Eigenvalues are Increasing

Theorem (Increasing Eigenvalues)
The eigenvalues of the Sturm-Liouville BVP:

$$(p(x)y')' - q(x)y + \lambda r(x)y = 0$$

where:

- *we consider the **interval**:*

$$0 < x < 1$$

- *the* ***boundary conditions*** *are:*

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0$$
$$\beta_1 y(1) + \beta_2 y'(1) = 0$$

- *the functions* p, p', q, r *are* ***continuous*** *on* $[0, 1]$
- $\forall x \in [0, 1], p(x) > 0, r(x) > 0$

are ***simple*** *(one eigenvalue for each eigenvector).*

Moreover, the eigenvalues form an ***infinite, monotonically increasing*** *sequence:*

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$$

such that:

$$\lambda_n \to \infty$$

2.4 Eigenfunctions for Sturm-Liouville BVP

2.4.1 Orthogonality of Eigenfunctions

Theorem (Orthogonality of Eigenvectors)
Consider the Sturm-Liouville BVP:

$$(p(x)y')' - q(x)y + \lambda r(x)y = 0$$

where:

- *we consider the* ***interval****:*

$$0 < x < 1$$

- *the* ***boundary conditions*** *are:*

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0$$
$$\beta_1 y(1) + \beta_2 y'(1) = 0$$

- *the functions* p, p', q, r *are* ***continuous*** *on* $[0, 1]$
- $\forall x \in [0, 1], p(x) > 0, r(x) > 0$

Then, the eigenfunctions corresponding to distinct eigenvalues are ***orthogonal with respect to the weight function*** $r(x)$*. In other words:*

$$\langle \phi_m, \phi_n \rangle = 0$$

where $\lambda_m \neq \lambda_n$ *are the corresponding eigenvalues.*

Proof: Orthogonality of Eigenvectors. Let ϕ_m, ϕ_n be eigenfunctions which solve the Sturm-Liouville BVP:

$$\mathcal{L}[y] = \lambda r(x) y$$

So from Lagrange's Identity:

$$(\mathcal{L}[\phi_m], \phi_n) = (\phi_m, \mathcal{L}[\phi_n]) \implies (\lambda_m r(x)\phi_m, \phi_n) = (\phi_m, \lambda_n r(x)\phi_n)$$

Using the definition of the function dot product:

$$\lambda_m \int_0^1 r(x)\phi_m(x)\bar{\phi}_n(x)dx = \bar{\lambda}_n \int_0^1 \bar{r}(x)\bar{\phi}_n(x)\phi_m(x)dx$$
$$\implies (\lambda_m - \lambda_n) \int_0^1 r(x)\phi_m(x)\bar{\phi}_n(x)dx = 0$$
$$\implies (\lambda_m - \lambda_n) \langle \phi_m, \phi_n \rangle = 0$$

Since we are assuming $\lambda_m \neq \lambda_n$, it follows that:

$$\langle \phi_m, \phi_n \rangle = 0$$

so distinct eigenfunctions are orthogonal.

□

2.4.2 Orthornormal Eigenfunctions

- **What are orthonormal functions?**
  - a set of functions is said to be orthonormal, if they are orthogonal, and their magnitude (in terms of inner product) is 1
- **Why are orthonormal functions useful?**
  - mainly, they help simplify calculations, by getting rid of unnecessary constant
- **What is a normalisation condition?**
  - consider an eigenfunction ϕ_n. A **normalisation condition** is given by:

    $$\langle \Phi_n, \Phi_n \rangle = \int_0^1 r(x)\Phi_n^2(x)dx = 1$$

    where $\Phi_n(x) = k_n\phi_n(x)$ has been scaled by a constant, as to ensure that the orthonormality requirmenet is met
  - the eigenfunction $\Phi_n(x)$ is **normalised**
  - we can form an **orthonormal set with respect to** r by normalising each eigenfunction
- **How can we succinetly describe the orthonormal set for the Sturm-Liouville BVP?**
  - if we have a set of functions $\{\Phi_n\}$ which are eigenfunctions of the Sturm-Liouville BVP, they are orthonormal **if and only if**:

    $$\langle \Phi_n(x), \Phi_m(x) \rangle = \int_0^1 r(x)\Phi_n(x)\Phi_m(x)dx = \delta_{mn}$$

    where δ_{mn} is the Kronecker Delta (1 if $m = n$, 0 otherwise)

2.5 Worked Example

Determine the normalised eigenfunctions of:

$$y'' + \lambda y = 0$$

subject to:

$$y(0) = 0$$

$$y'(1) + y(1) = 0$$

3 Generalising Fourier Series: Sturm-Liouville Basis Functions

3.1 Recapping Fourier Series

We saw that if we had an ODE:

$$y'' = \lambda y$$

subject to:

$$y(0) = y(1) = 0$$

we obtained eigenfunctions of the form:

$$y_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

where $L = 1$, and eigenvalues $\lambda_n = n^2\pi^2$. Using these eigenfunctions as an orthogonal basis, we could express a function $f(x)$ defined with the same boundary conditions via a **Fourier Series**:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

where:

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)dx$$

We could do a similar thing with $\cos(n\pi x)$.

The key here is that given a function defined with some boundaries/conditions, we could expand it in terms of a series of orthogonal eigenfunctions. The Sturm-Liouville BVP is an example of this, so we can maybe use it to expand functions, via a richer set of orthogonal functions, beyond just sines and cosines.

3.2 Expanding Functions Using Sturm-Liouville Eigenfunctions

- **How can we expand a function via eigenfunctions of the Sturm-Liouville BVP?**
  - consider an orthonormal set of eigenfunctions satisfying the Sturm-Liouville BVP:

$$\{\phi_n\}_{n=1}^{\infty}$$

  - we now assume that we can find some c_n, such that:

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

- **If we can expand a function using Sturm-Liouville eigenfunctions, how can we compute the coefficients?**
  - lets assume that $f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$ holds. We can find c_n using similar methods to the ones

used in Fourier Series:

$$
\begin{aligned}
&\langle f(x), \phi_m(x)\rangle \\
=&\int_0^1 r(x)f(x)\phi_m(x)dx \\
=&\int_0^1 r(x)\left(\sum_{n=1}^{\infty} c_n\phi_n(x)\right)\phi_m(x)dx \\
=&\int_0^1 \left(\sum_{n=1}^{\infty} r(x)c_n\phi_n(x)\phi_m(x)\right)dx \\
=&\sum_{n=1}^{\infty}\left(\int_0^1 r(x)c_n\phi_n(x)\phi_m(x)dx\right) \\
=&\sum_{n=1}^{\infty} c_n\langle\phi_n(x), \phi_m(x)\rangle \\
=&\sum_{n=1}^{\infty} c_n\delta_{mn} \\
=&c_m
\end{aligned}
$$

where we have assumed that we can integrate term by term

– hence, it follows that if we can express f as in the series above, the coefficients must be:

$$c_m = \langle f(x), \phi_m(x)\rangle = \int_0^1 r(x)f(x)\phi_m(x)dx$$

3.3 Sturm-Liouville and Completeness

We now give a theorem which extends the Fourier Convergence Theorem for the eigenfunctions of the Sturm-Liouville BVP

Theorem (Convergence Theorem for Sturm-Liouville Eigenfunctions)
Let $\phi_1, \phi_2, \ldots$ be the normalised eigenfunctions of the Sturm-Liouville BVP:

$$(p(x)y')' - q(x)y + \lambda r(x)y = 0$$

where:

- *we consider the* ***interval****:*

$$0 < x < 1$$

- *the* ***boundary conditions*** *are:*

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0$$
$$\beta_1 y(1) + \beta_2 y'(1) = 0$$

- *the functions p, p', q, r are* ***continuous*** *on $[0, 1]$*
- $\forall x \in [0, 1], p(x) > 0, r(x) > 0$

Now, let f and f' be ***piecewise continuous*** *on $0 \leq x \leq 1$. Then, the series:*

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$$

where:

$$c_n = \langle f(x), \phi_n(x) \rangle = \int_0^1 r(x) f(x) \phi_n(x) dx$$

converges to:

$$\frac{f(x^+) + f(x^-)}{2}$$

at each point on $(0, 1)$*.*

Moreover,

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

converges at each point in $[0, 1]$ *if* f *is* ***continuous*** *and* f' *is* ***piecewise continuous*** *on* $[0, 1]$*, and* f *satisfies:*

- *if* $\phi_n(0) = 0$ *for every* n *(that is, if* $\alpha_2 = 0$*), then* $f(0) = 0$
- *if* $\phi_n(1) = 0$ *for every* n *(that is, if* $\beta_2 = 0$*), then* $f(1) = 0$

4 The Self-Adjoint Problem

4.1 Relation Between Sturm-Liouville and Symmetric Matrices

- we have seen that a symmetric matrix and $\mathcal{L}[y]$ have similar properties
- they are both symmetric, in the sense that they can be defined by a similar identity:

$$\underline{u} \cdot (\boldsymbol{A}\underline{v}) = \underline{v} \cdot (\boldsymbol{A}\underline{u})$$

$$(\mathcal{L}[v], u) = (v, \mathcal{L}[u])$$

- the eigenvalues of the eigenvalue problem are **real**
- the eigenvectors of the eigenvalue problem are **orthogonal**
- the principal difference is that in the matrix space, we have finitely many eigenvalues (at most the number of rows/columns in the matrix), whilst in function space there are infinitely many eigenvalues
- **linear operator theory** is the area of maths related to studying the connection between the matrix problem and Sturm-Liouville BVP

4.2 Generalising Stum-Liouville: Self-Adjoint Problem

- **What is a self-adjoint problem?**
  - a BVP, defined by:

    $$\mathcal{L}[y] = \lambda r(x) y, \quad 0 < x < 1$$

    where:

    $$\mathcal{L}[y] = P_n(x) y^{(n)} + \ldots + P_1(x) y' + P_0(x) y$$

    and we have **n linear homogeneous** boundary conditions at the endpoints
  - the above problem is said to be **self-adjoint** if $\mathcal{L}$ satisfies certain conditions
- **What are the restriction on $\mathcal{L}$?**
  - it must satisfy Lagrange's Identity:

    $$(\mathcal{L}[v], u) = (v, \mathcal{L}[u])$$

- for this, $\mathcal{L}$ must have **even** order
- we have already seen the 2nd order operator; for 4 order, we must be able to express it as:

$$\mathcal{L}[y] = (p(x)y'')'' - (q(x)y')' + s(x)y$$

- we must also ensure that the boundary conditions allow us to remove the term resulting in integration by parts when deriving Lagrange's Identity

- **Do all self-adjoint problems have the properties of the Strum-Liouville BVP?**
  - for this to be the case, we require:
    * $P_n, \ldots, P_1, P_0$ must be **continuous** on $0 \leq x \leq 1$
    * the derivatives appearing in terms of the form

$$\mathcal{L}[y] = (p(x)y)'' - (q(x)y)' + s(x)y$$

      (in this case p'', q') must be **continuous**
    * $p(x), r(x) > 0$ for $0 \leq x \leq 1$
  - under these conditions:
    * the eigenvalues will be **real**, and tend to infinity
    * the eigenvectors will be **orthogonal** with respect to the weight function $r(x)$
    * arbitrary functions can be represented via a series involving the eigenfunctions
    * however, there is no guarantee that the eigenvalues will be **simple** (one for each eigenvector)

5 Non-Homogeneous Sturm-Liouville Problem

5.1 Defining the Non-Homogeneous BVP

We have considered homogeneous Sturm-Liouville BVP, with homogeneous boundary conditions. Now, we consider the **non-homogeneous** version (still with homogeneous boundary conditions).

Consider $\mathcal{L}[y] = -(p(x)y')' + q(x)y$, such that:

$$\mathcal{L}[y] = \mu r(x)y + f(x)$$

where:

- μ is a constant (not necessarily an eigenvalue)
- $f(x)$ is a function defined on $0 \leq x \leq 1$
- we consider the **interval**:

$$0 < x < 1$$

- the **boundary conditions** are:

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0$$
$$\beta_1 y(1) + \beta_2 y'(1) = 0$$

- the functions p, p', q, r are **continuous** on $[0, 1]$
- $\forall x \in [0, 1], p(x) > 0, r(x) > 0$

The strategy to solve the BVP is:

1. Assume a inhomogeneous solution as a infinite series with unknown coefficient
2. Plug the solution back into the ODE
3. Obtain a series expansion for $f(x)$ with known coefficients
4. Use the coefficients in the last step to obtain the series expansion to the inhomogeneous solution, using this to solve the inhomogeneous BVP

5.2 Solving the Homogeneous Problem

- **What are the solutions to the homogeneous BVP?**
  - consider the eigenvalues $\lambda_1, \lambda_2, \ldots$ and **normalised** eigenfunctions $\phi_1(x), \phi_2(x), \ldots$ which solve the homogeneous problem:
$$\mathcal{L}[y] = \lambda r(x) y$$
- **How can we express a series solution for the inhomogeneous problem?**
  - assume that $\phi(x)$ is a solution to the inhomogeneous BVP:
$$\mathcal{L}[y] = \mu r(x) y + f(x)$$
  - we can try expanding $\phi(x)$ via the orthonormal eigenfunctions:
$$\phi(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$
- **Can we directly compute b_n?**
  - if we use the coefficeint formula, we see that:
$$b_n = \langle \phi(x), \phi_n(x) \rangle$$
  - we can't currently compute this: we are trying to find $\phi(x)$, so we can't use it to compute b_n

5.3 Plugging Back to ODE

- **Since we can't find b_n directly, how can we compute it indirectly?**
  - we notice that $\phi(x)$ must satisfy the boundary conditions (it is compoesed of ϕ_n, each of which satisfy it, since the conditions are identical to the homogeneous BVP)
  - this hints at plugging in $\phi(x)$ into the inhomogeneous BVP

If we plug in $\phi(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$ into $\mathcal{L}[y] = \mu r(x) y + f(x)$, we get:

$$\begin{aligned}
&\mathcal{L}[y] = \mu r(x) y + f(x) \\
\implies &\mathcal{L}\left[\phi(x)\right] = \mu r(x) \phi(x) + f(x) \\
\implies &\mathcal{L}\left[\sum_{n=1}^{\infty} b_n \phi_n(x)\right] = \mu r(x) \sum_{n=1}^{\infty} b_n \phi_n(x) + f(x) \\
\implies &\sum_{n=1}^{\infty} b_n \mathcal{L}\left[\phi_n(x)\right] = \mu r(x) \sum_{n=1}^{\infty} b_n \phi_n(x) + f(x)
\end{aligned}$$

Noticing that $\phi_n(x)$ satisfies the homogeneous problem, we must have:

$$\mathcal{L}\left[\phi_n(x)\right] = \lambda_n r(x)\phi_n(x)$$

so:

$$\begin{aligned}
&\mathcal{L}[y] = \mu r(x)y + f(x) \\
\implies &\sum_{n=1}^{\infty} b_n \mathcal{L}\left[\phi_n(x)\right] = \mu r(x) \sum_{n=1}^{\infty} b_n \phi_n(x) + f(x) \\
\implies &\sum_{n=1}^{\infty} b_n (\lambda_n r(x)\phi_n(x)) = \mu r(x) \sum_{n=1}^{\infty} b_n \phi_n(x) + f(x) \\
\implies &\sum_{n=1}^{\infty} b_n (\lambda_n r(x)\phi_n(x)) = \mu r(x) \sum_{n=1}^{\infty} b_n \phi_n(x) + f(x) \\
\implies &\lambda_n r(x) \sum_{n=1}^{\infty} b_n \phi_n(x) = \mu r(x) \sum_{n=1}^{\infty} b_n \phi_n(x) + f(x) \\
\implies &r(x)(\lambda_n - \mu) \sum_{n=1}^{\infty} b_n \phi_n(x) = f(x) \\
\implies &\frac{f(x)}{r(x)} = (\lambda_n - \mu) \sum_{n=1}^{\infty} b_n \phi_n(x)
\end{aligned}$$

where we are using the fact that $r(x) > 0$.

5.4 Expanding f/r

- **How can we compute b_n from the formula derived above?**
  - once we have reached the step above, we have reduced all the uncertainty to b_n
  - however, it is hard to "extract" it from the summation
  - in order to do this, and assuming that the conditions in (3.3) are satisfied by $\frac{f(x)}{r(x)}$, we can write it as:

$$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

- **How can we find the c_n?**
  - once again, we use the coefficient formula:

$$c_n = \left\langle \frac{f(x)}{r(x)}, \phi_n \right\rangle$$

  - this has terms which we know, so c_n is computable:

$$c_n = \int_0^1 r(x)\frac{f(x)}{r(x)}\phi_n(x)dx = \int_0^1 f(x)\phi_n(x)dx$$

We showed that:

$$\frac{f(x)}{r(x)} = (\lambda_n - \mu) \sum_{n=1}^{\infty} b_n \phi_n(x)$$

If we then employ the series expansion of $\frac{f(x)}{r(x)}$ we get:

$$\sum_{n=1}^{\infty} c_n \phi_n(x) = (\lambda_n - \mu) \sum_{n=1}^{\infty} b_n \phi_n(x)$$

which reduces to:

$$\sum_{n=1}^{\infty} [(\lambda_n - \mu) b_n - c_n] \phi_n(x) = 0$$

But then notice, each $\phi_n(x)$ is orthonormal, or in other words, **linearly independent**. Straight from the definition, any linear combination of linearly independent functions is 0 **if and only if** each of the coefficients is 0. Hence, for each $n \in \mathbb{N}$, if $\phi(x)$ satsifies the inhomogeneous BVP, it must be the case that:

$$(\lambda_n - \mu) b_n - c_n = 0$$

5.5 Solving the Non-Homogeneous Problem

Depending on the realtion between λ_n and μ we can obtain different solutions, which affect b_n. We consider each case separately.

5.5.1 Eigenvalues Not Present in BVP Definition

If $\forall n \in \mathbb{N}, \mu \neq \lambda_n$ this is the easiest case, since:

$$b_n = \frac{c_n}{\lambda_n - \mu}$$

so:

$$y = \phi(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} \phi_n(x)$$

(We don't necessarily know if $\phi(x)$ converges; provided that f is continuous, the theorem on convergence will indeed prove the convergence)

5.5.2 Eigenvalues Present in BVP Definition

Now we need to consider the case in which for $n = m$, $\lambda_m = \mu$. If this is the case, we have that:

$$0 \times b_m - c_m = 0$$

There are 2 cases to consider:

(1) $\boldsymbol{c_m \neq 0}$

If this is the case, $0 \times b_m - c_m = 0$ becomes $c_m = 0$. This is clearly impossible, so the inhomogeneous BVP has **no solutions**.

(2) $\boldsymbol{c_m = 0}$

If this is the case, $0 \times b_m - c_m = 0$ is always satisfied, independent of b_m. This means that solutions will not be unique, and we can consider **any multiple** of the eigenfunction $\phi_m(x)$.

However, further notice that $c_m = 0$ implies that:

$$\left\langle \frac{f(x)}{r(x)}, \phi_m(x) \right\rangle = \int_0^1 f(x) \phi_m(x) dx = 0$$

Thus, $c_m = 0$ if and only if $f(x)$ is orthogonal to $\phi_m(x)$; in other words, if $\mu = \lambda_m$, solutions only exist if $f(x)$ is orthogonal to $\phi_m(x)$

5.6 Theorem Summary for the Non-Homogeneous Sturm-Liouville Problem

We can summarise the results above in the following theorems:

Theorem
The nonhomogeneous boundary value problem:

$$\mathcal{L}[y] = \mu r(x)y + f(x)$$

where:

- μ *is a constant (not necessarily an eigenvalue)*
- $f(x)$ *is a function defined on* $0 \leq x \leq 1$
- *we consider the* ***interval****:*
$$0 < x < 1$$
- *the* ***boundary conditions*** *are:*
$$\alpha_1 y(0) + \alpha_2 y'(0) = 0$$
$$\beta_1 y(1) + \beta_2 y'(1) = 0$$
- *the functions* p, p', q, r *are* ***continuous*** *on* $[0, 1]$
- $\forall x \in [0, 1], p(x) > 0, r(x) > 0$

has a ***unique*** *solution for each* ***continuous*** f *whenever* μ *is different from all the eigenvalues of the corresponding homogeneous problem; the solution is given by :*

$$y = \phi(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu}\phi_n(x)$$

and the series converges for each $x \in [0, 1]$*.*

If μ *is equal to an eigenvalue* λ_m *of the corresponding homogeneous problem, then the nonhomogeneous boundary value problem has* ***no solution*** *unless:*

$$\int_0^1 f(x)\phi_m(x)dx = 0$$

That is, unless f *is* ***orthogonal*** *to* ϕ_m*. In that case, the solution is* ***not unique*** *and contains an arbitrary multiple of* $\phi_m(x)$*.*

A more succint version of the theorem is given by:

Theorem 1 (Fredholm Alternative Theorem)
For a given value of μ*, either the nonhomogeneous problem:*

$$\mathcal{L}[y] = \mu r(x)y + f(x)$$

where:

- μ *is a constant (not necessarily an eigenvalue)*
- $f(x)$ *is a function defined on* $0 \leq x \leq 1$
- *we consider the* ***interval****:*
$$0 < x < 1$$

- *the **boundary conditions** are:*

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0$$
$$\beta_1 y(1) + \beta_2 y'(1) = 0$$

- *the functions p, p', q, r are **continuous** on $[0,1]$*
- $\forall x \in [0,1], p(x) > 0, r(x) > 0$

*has a **unique** solution for each **continuous** f (if μ is not equal to any eigenvalue λ_m of the corresponding homogeneous problem), or else the homogeneous problem:*

$$\mathcal{L}[y] = \mu r(x) y$$

with the boundary conditions as above, has a nontrivial solution (the eigenfunction corresponding to λ_m).

6 Solving Non-Homogeneous Heat Conduction PDEs

In solving the Non-Homogeneous Sturm-Liouville BVP, we can now solve Non-Homogeneous PDEs. We exemplify this via a (extended) example involving the **Non-Homogeneous Heat Conduction PDE**.

6.1 Defining the non-Homogeneous Heat Conduction PDE

We consider the PDE:

$$r(x)u_t = (p(x)u_x)_x - q(x)u + F(x,t)$$

given boundary conditions:

$$u_x(0,t) - h_1 u(0,t) = 0$$
$$u_x(1,t) - h_2 u(1,t) = 0$$

and initial conditions:

$$u(x,0) = f(x)$$

This becomes a Sturm-Liouville type problem if we require:

- the functions p, p', q, r are **continuous** on $[0,1]$
- $\forall x \in [0,1], p(x) > 0, r(x) > 0$

We solve in a similar way as with the inhomogeneous ODEs.

6.2 Finding Eigenvectors

From the Motivation Section (1.3) we know that the **homogeneous** problem can be solved via separation of variables:

$$u(x,t) = X(x)T(t)$$

which lead to the ODEs:

$$(p(x)X'(x))' - q(x)X(x) + \lambda r(x) X(x) = 0$$
$$T'(t) + \lambda T(t) = 0$$

The boundary conditions also change, since $u(x,t) = X(x)T(t)$:

$$X'(0)T(t) - h1X'(0)T(t) = 0 \implies X'(0) - h1X'(0) = 0$$
$$X'(L)T(t) - h1X'(L)T(t) = 0 \implies X'(L) - h1X'(L) = 0$$

We let $\phi_1, \phi_2, \ldots$ be the set of eigenvectors, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots$. Notice this is just solving the standard, homogeneous Sturm-Liouville BVP.

6.3 Deriving Solution to Inhomogeneous Problem

Using the above, we assume a solution of the form:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

where $b_n = b_n(t)$, since u is a function of x, t. If u is as above, it satisfies the boundary conditions of the inhomogeneous problem.

6.4 Plugging Derived Solution to Inhomogeneous Problem

Assuming a solution $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)$ it must satisfy the BVP, so:

$$\begin{aligned}
&\implies r(x)u_t = (p(x)u_x)_x - q(x)u + F(x,t) \\
&\implies r(x)\frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} b_n(t)\phi_n(x)\right) = \frac{\partial}{\partial x}\left(p(x)\frac{\partial}{\partial x}\sum_{n=1}^{\infty} b_n(t)\phi_n(x))\right) - q(x)\sum_{n=1}^{\infty} b_n(t)\phi_n(x) + F(x,t) \\
&\implies r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = \frac{\partial}{\partial x}\left(p(x)\sum_{n=1}^{\infty} b_n(t)\phi_n'(x))\right) - q(x)\sum_{n=1}^{\infty} b_n(t)\phi_n(x) + F(x,t) \\
&\implies r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\phi_n'(x) + p(x)\sum_{n=1}^{\infty} b_n(t)\phi_n''(x) - q(x)\sum_{n=1}^{\infty} b_n(t)\phi_n(x) + F(x,t) \\
&\implies r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)(p'(x)\phi_n'(x) + p(x)\phi_n''(x) - q(x)\phi_n(x)) + F(x,t) \\
&\implies r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)((p(x)\phi_n'(x))' - q(x)\phi_n(x)) + F(x,t)
\end{aligned}$$

But notice,$\phi_n(x)$ satisfies the homogeneous ODE:

$$(p(x)X'(x))' - q(x)X(x) + \lambda r(x)X(x) = 0$$

So we must have:

$$\begin{aligned}
&r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)((p(x)\phi_n'(x))' - q(x)\phi_n(x)) + F(x,t) \\
\implies &r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = -\sum_{n=1}^{\infty} b_n(t)\lambda_n r(x)\phi_n(x) + F(x,t) \\
\implies &\frac{F(x,t)}{r(x)} = \sum_{n=1}^{\infty} b_n'(t)\phi_n(x) + \sum_{n=1}^{\infty} b_n(t)\lambda_n\phi_n(x) \\
\implies &\frac{F(x,t)}{r(x)} = \sum_{n=1}^{\infty}(b_n'(t) + \lambda_n b_n(t))\phi_n(x)
\end{aligned}$$

6.5 Expanding F/r

We can write:

$$\frac{F(x,t)}{r(x)} = \sum_{n=1}^{\infty} \gamma_n(t)\phi_n(x)$$

where we can compute $\gamma_n(t)$ via:

$$\gamma_n(t) = \left\langle \frac{F(x,t)}{r(x)}, \phi_n \right\rangle = \int_0^1 F(x,t)\phi_n(x)dx$$

Then the expression above can be further reduced:

$$\begin{aligned}
\frac{F(x,t)}{r(x)} &= \sum_{n=1}^{\infty}(b_n'(t) + \lambda_n b_n(t))\phi_n(x) \\
\sum_{n=1}^{\infty} \gamma_n(t)\phi_n(x) &= \sum_{n=1}^{\infty}(b_n'(t) + \lambda_n b_n(t))\phi_n(x) \\
\sum_{n=1}^{\infty}(b_n'(t) + \lambda_n b_n(t) - \gamma_n(t))\phi_n(x) &= 0
\end{aligned}$$

6.6 Solving the Inhomogeneous PDE

We have then shown that for $u(x,t)$ to solve the PDE, we shall require:

$$b_n'(t) + \lambda_n b_n(t) - \gamma_n(t) = 0$$

since the ϕ_n are **linearly independent**. Since we know γ_n, this is just an inhomogeneous ODE in $b_n(t)$:

$$b_n'(t) + \lambda_n b_n(t) = \gamma_n(t)$$

This ODE is solvable by using **integrating factors**. In particular, $\mu = e^{\lambda_n t}$:

$$\begin{aligned}
& b_n'(t) + \lambda_n b_n(t) = \gamma_n(t) \\
\implies & e^{\lambda_n t} b_n'(t) + e^{\lambda_n t}\lambda_n b_n(t) = e^{\lambda_n t}\gamma_n(t) \\
\implies & \frac{d}{dt}\left(b_n(t)e^{\lambda_n t}\right) = e^{\lambda_n t}\gamma_n(t) \\
\implies & b_n(t)e^{\lambda_n t} = \int e^{\lambda_n t}\gamma_n(t)dt \\
\implies & b_n(t) = Ce^{-\lambda_n t} + e^{-\lambda_n t}\int_0^t e^{\lambda_n s}\gamma_n(s)ds \\
\implies & b_n(t) = Ce^{-\lambda_n t} + \int_0^t e^{\lambda_n (s-t)}\gamma_n(s)ds
\end{aligned}$$

To solve this we need an initial condition. Consider $t = 0$:

$$b_n(0) = C + \int_0^0 e^{\lambda_n (s)}\gamma_n(s)ds \implies b_n(0) = C$$

In other words, if we can find $b_n(0)$, we will have a solution which gives us $b_n(t)$.

6.7 Finding Initial Condition for b_n

Define $B_n = b_n(0)$. We can find B_n by noticing that the initial conditions of the PDE give us:

$$u(x,0) = f(x)$$

We have:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x) \implies f(x) = \sum_{n=1}^{\infty} B_n\phi_n(x)$$

Thus, B_n is just given by the coefficient formula:

$$B_n = \langle f(x), \phi_n \rangle = \int_0^1 r(x)f(x)\phi_n(x)dx$$

6.8 Solving the Inhomogeneous PDE - Part 2

If we have:

$$B_n = \langle f(x), \phi_n \rangle = \int_0^1 r(x)f(x)\phi_n(x)dx$$

then:

$$b_n(t) = B_n e^{-\lambda_n t} + \int_0^t e^{\lambda_n(s-t)}\gamma_n(s)ds$$

with the first term dependent on f, and the second term dependent on F.

We can thus write the solution to:

$$r(x)u_t = (p(x)u_x)_x - q(x)u + F(x,t)$$

via:

$$u(x,t) = \sum_{n=1} b_n(t)\phi_n(x)$$

where:

- $\phi_n(x)$ are the eigenvectors of the homogeneous ODE:

$$(p(x)y'(x))' - q(x)y(x) + \lambda r(x)y(x) = 0$$

- $$\gamma_n(t) = \left\langle \frac{F(x,t)}{r(x)}, \phi_n \right\rangle = \int_0^1 F(x,t)\phi_n(x)dx$$

Overall, to solve the PDE:

1. Find the eigenvalues and eigenfunctions of the homogeneous BVP
2. Compute γ_n and B_n
3. Compute b_n by evaluating the integral
4. Sum the infinite series

7 Parseval's Theorem for Sturm-Liouville Problems

Say we write $f(x)$ as a sum of Sturm-Liouville Eigenfunctions:

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$$

where:

$$c_n = \langle f(x), \phi_n(x) \rangle = \int_0^1 r(x)f(x)\phi_n(x)$$

then Parseval's Theorem States that:

$$\langle f, f\rangle = \int_0^1 r(x)f^2(x)dx = \sum_{n=1}^{\infty} c_n^2$$

Proof: Parseval's Theorem.

$$\begin{aligned}\int_0^1 r(x)f^2(x)dx &= \int_0^1 r(x)\sum_{n=1}^{\infty} c_n\phi_n(x)\sum_{m=1}^{\infty} c_m\phi_m(x)dx \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^1 r(x)c_nc_m\phi_n(x)\phi_m(x)dx \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_nc_m\langle\phi_n(x),\phi_m(x)\rangle \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_nc_m\delta_{mn} \\ &= \sum_{n=1}^{\infty} c_n^2\end{aligned}$$

□

8 Singular Sturm-Liouville Problem

- **What is the regular Sturm-Liouville Problem?**
  - a BVP characterised by:
  $$(p(x)y')' - q(x)y + \lambda r(x)y = 0$$
  where:
    * we consider the **interval**:
    $$0 < x < 1$$
    * the **boundary conditions** are:
    $$\alpha_1 y(0) + \alpha_2 y'(0) = 0$$
    $$\beta_1 y(1) + \beta_2 y'(1) = 0$$
    * the functions p, p', q, r are **continuous** on $[0, 1]$
    * $\forall x \in [0, 1], p(x) > 0, r(x) > 0$
- **What is the singular Sturm-Liouville Problem?**
  - a BVP characterised by:
  $$(p(x)y')' - q(x)y + \lambda r(x)y = 0$$
  where:
    * we consider the **interval**:
    $$0 < x < 1$$

    * the **boundary conditions** are:

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0$$

$$\beta_1 y(1) + \beta_2 y'(1) = 0$$

    * the functions p, p', q, r are **continuous** on $(0, 1)$
    * $\forall x \in (0, 1), p(x) > 0, r(x) > 0$
    * at least one of p, q, r fail to satisfy positivity/continuity at 0 and/or 1

- **When is a singular BVP self-adjoint?**
  - consider 2 functions u, v which:
    * are twice continuously differentiable on $(0, 1)$
    * satisfy:

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0$$
$$\beta_1 y(1) + \beta_2 y'(1) = 0$$

      at each **regular** boundary point
    * if $x = 0$ is **singular**, they satisfy boundary conditions such that:

$$\lim_{\varepsilon \to 0^+} p(\varepsilon)(u'(\varepsilon)v(\varepsilon) - u(\varepsilon)v`(\varepsilon)) = 0$$

      (the derivation of these conditions is in the book)
    * if $x = 1$ is **singular**, they satisfy boundary conditions such that:

$$\lim_{\varepsilon \to 1^-} p(\varepsilon)(u'(\varepsilon)v(\varepsilon) - u(\varepsilon)v`(\varepsilon)) = 0$$

  - if we have any 2 such u, v, then we ensure that Lagrange's Identity holds:

$$(\mathcal{L}[u], v) - (u, \mathcal{L}[V]) \int_0^1 \mathcal{L}[u]\, v - u\mathcal{L}[v]\, dx = 0$$

    (this can be an improper integral), and then the singular BVP described by:

$$\mathcal{L}[y] = \lambda r(x) y$$

    will be self-adjoint

- **How do eigenvalues differ between regular and singular Sturm-Liouville BVP**
  - in **regular** Sturm-Liouville BVPs, the eigenvalues are **discrete**
  - in **singular** Sturm-Liouville BVPs, the eigenvalues can be **continuous**: there can be nontrivial solutions for every λ on the whole domain, or some subinterval
  - if we have continuous eigenvalues, the problem is said to have a **continuous spectrum**, and the set of eigenfunctions can't be enumerated

- **If we have a continuous spectrum, how can we use eigenfunctions to describe functions?**
  - instead of using a sum, we would use an integral