Honours Differential Equations - Week 7 - Solving Partial Differential Equations

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Contents

1 Partial Differential Equations

1.1 Defining Partial Differential Equations

We are going to consider 3 partial differential equations:

- **Heat Equation**: models how heat is transferred across a medium (diffusive process)
- **Wave Equation**: models how waves propagate across a medium (oscillatory process)
- **Laplace's (Potential) Equation**: models processes which are time-independent/steady state

This is because **all linear, second order PDEs** can be classified into one of the categories above.

1.2 Solving Partial Differential Equations: Separation of Variables

- **When do we use separation of variables?**
  - when we have linear, separable PDEs
- **What do we assume as the form of the solution?**
  - if we have a PDE in terms of a function $u(x_1, x_2, \ldots, x_n)$, we assume a solution of the form:

$$X_1(x_1)X_2(x_2)\ldots X_n(x_n)$$

- **What does separation of variables lead to?**
  - given a set of boundary and initial conditions (this depend on the specifics of the problem), we obtain a set of eigenvalue problems, whose solutions, when multiplied, produce a solution to the PDE

DERIVATIONS FOR THE HEAT AND WAVE EQUATIONS CAN BE FOUND IN THE BOYCE TEXTBOOK, AS APPENDIX A OF CHAPTER 10.

2 The Heat Equation

Let $u(x, t)$ denote the temperature across a 1 dimensional rod of length L. Then, the distribution of heat across the rod is modelled by the PDE:

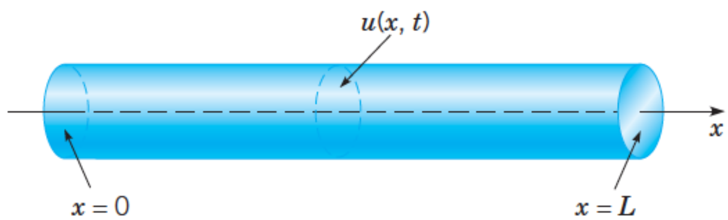
$$\boldsymbol{\alpha^2 u_{xx} = u_t}$$

where α is the **thermal diffusivity**, dependent on the material of the rod:

$$\alpha^2 = \frac{\kappa}{\rho s}$$

where:

- ρ: density of the rod
- κ: thermal conducitivity of the rod
- s: specific heat capacity of the rod



We can generalise this to higher dimensions, for example:

- in 2 dimensions, we have:
$$\alpha^2(u_{xx} + u_{yy}) = u_t$$
- in 3 dimensions, we have:
$$\alpha^2(u_{xx} + u_{yy} + u_{zz}) = u_t$$

2.1 The Heat Equation: Homogeneous Boundary Conditions

2.1.1 Defining the Heat Equation

If we have **homogeneous boundary conditions** the Heat Equation is defined by:

$$\alpha^2 u_{xx} = u_t$$

subject to the boundary conditions:

- $u(x, 0) = f(x), \quad 0 \leq x \leq L$
- $u(0, t) = u(L, t) = 0, \quad t > 0$
- **What do the initial/boundary conditions imply about the solution we expect?**
  - the initial condition states that the heat across the bar is initially distributed according to some function $f(x)$
  - the boundary condition states that the ends of the rod are kept at a fixed temperature of 0
  - overall, we have a IVP in the variable t, and a BVP in the variable x

2.1.2 Applying Separation of Variables to Heat Equation With Homogeneous Boundary Conditions

If we assume a solution of the form:
$$u(x, t) = X(x)T(t)$$
then the PDE $\alpha^2 u_{xx} = u_t$ implies:
$$\alpha^2 X''T = XT'$$
If we rearrange this to have X and T on each side:

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}$$

Notice that the RHS depends solely on x, whils the LHS depends solely on t. These 2 can only be equal iff they are consatnat (think about fixing x, and changing t; the equality will only hold if changing t has no effect). Thus, we claim that there exists some $\lambda \in \mathbb{R}$ such that:

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = \lambda$$

2.1.3 Solving the Eigenvalue Problem for $X(x)$

We first have an eigenvalue problem, given by:

$$\frac{X''}{X} = \lambda \implies X'' = \lambda X$$

subject to the boundary condition:

$$u(0,t) = u(L,t) = 0, \quad t > 0$$

We first notice that:

$$u(0,t) = 0 \implies X(0)T(t) = 0$$

since we want non-trivial solutions, in particular we require that $\forall t, T(t) = 0$ is not the case. Hence, we must have that $X(0) = 0$. Using similar logic, we have $X(L) = 0$.

To solve the BVP, we consider 3 cases (this is the same as the work last week with BVPs). Before this, we introduce the $\cosh(x), \sinh(x)$ functions:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

---

(1) $\boldsymbol{\lambda = \mu^2 > 0}$

The ODE becomes:

$$X'' - \mu^2 X = 0$$

which is solved by:

$$X(x) = A\cosh(\mu x) + B\sinh(\mu x)$$

If $X(0) = 0$ and $X(L) = 0$, we obtain:

$$\begin{aligned} &A = 0 \\ &A\cosh(\mu L) + B\sinh(\mu L) = 0 \end{aligned}$$

But $\sinh(\mu L) \neq 0$, so $B\sinh(\mu L) = 0$ if and only if $B = 0$. Hence, if $\lambda = \mu^2$ there are no non-trivial solutions.

---

(2) $\boldsymbol{\lambda = 0}$

The ODE becomes:

$$X'' = 0$$

which is solved by:

$$X(x) = Ax + B$$

If $X(0) = 0$ and $X(L) = 0$, we obtain:

$$\begin{aligned} &B = 0 \\ &AL + B = 0 \end{aligned}$$

But $AL \neq 0$, unless $A = 0$, since $L > 0$. Hence, if $\lambda = 0$ there are no non-trivial solutions.

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③ $\boldsymbol{\lambda = -\mu^2 < 0}$

The ODE becomes:

$$X'' + \mu^2 X = 0$$

which is solved by:

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

If $X(0) = 0$ and $X(L) = 0$, we obtain:

$$A = 0$$
$$A\cos(\mu L) + B\sin(\mu L) = 0$$

Notice, $\sin(\mu L) = 0$ infinitely many times, so long as:

$$\mu = \frac{n\pi}{L}, \quad n \geq 1$$

(we don't want $\mu = 0$, since that case was covered before).

Hence, we obtain:

$$\lambda_n = -\frac{n^2\pi^2}{L^2}$$
$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

2.1.4 Solving the Eigenvalue Problem for $T(t)$

We now turn to the second problem, but using λ_n and T_n (since any T must satisfy any value for X):

$$\frac{1}{\alpha^2}\frac{T_n'}{T_n} = \lambda_n \implies T_n' = \alpha^2\lambda_n T_n$$

Solving the ODE for a general solution is standard:

$$T_n = \exp\left(\alpha^2\lambda_n t\right) = \exp\left(-\frac{n^2\pi^2\alpha^2}{L^2}t\right)$$

2.1.5 Solving the Heat Equation: Homogeneous Boundary Conditions

Hence, it follows that for any particular $n \geq 1$, we have:

$$u(x,t) = X_n(x)T_n(x) = \exp\left(-\frac{n^2\pi^2\alpha^2}{L^2}t\right)\sin\left(\frac{n\pi x}{L}\right)$$

If we use the fact that the ODE is **linear**, the most general solution must consist of a **linear combination** of the above solution, so:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{n^2\pi^2\alpha^2}{L^2}t\right)\sin\left(\frac{n\pi x}{L}\right)$$

We still require that $u(x,t)$ satisfies the initial condition:

$$u(x,0) = f(x), \quad 0 \leq x \leq L$$

Substituting in $t = 0$:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

But notice, this is as if we were expressing f as a Fourier Series, which makes finding c_n trivial: its just an Euler-Fourier Coefficient, so:

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

(intuitively, we can think of f being periodic with period L, by considering its **odd** extension, which means that the above is applicable)

One can see that the presence of the negative exponential in:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{n^2\pi^2\alpha^2}{L^2}t\right)\sin\left(\frac{n\pi x}{L}\right)$$

basically **guarantees** that the series converges to $U(x,t)$.

2.1.6 Implications of the Solution

- even if f is discontinuous, the temperature distribution is continuous; this makes sense intuitively, as we expect heat to distribute evenly through the rod, independent of whether at the start there are "cold" spots
- as $t \to \infty$, it is easy to see that due to the presence of the negative exponential, $u(x,t) \to 0$: in other words, the rod eventually has an even heat distribution

2.2 The Heat Equation: Non-Homogeneous Boundary Conditions

2.2.1 Defining the Heat Equation

We now consider the same heat equation:

$$\alpha^2 u_{xx} = u_t$$

but we change the boundary conditions:

- $u(x,0) = f(x), \quad 0 \leq x \leq L$
- $u(0,t) = a,\ u(L,t) = b, \quad t > 0$
- **What do the initial/boundary conditions imply about the solution we expect?**
  - the IVP has not changed: we still expect an initial temperature distribution
  - the BVP changes, this time allowing for the ends of the rods to be at different, non-zero temperatures
  - the strategy to solve this will be to reduce the problem to a homogeneous BVP, which we can solve

2.2.2 The Steady-State Temperature Distribution $v(x)$

From the previous section, we saw that as $t \to \infty$, we expect any solution $u(x,t)$ to tend to some stable solution, independent of t or any initial condition (except the temperature at the endpoints, which is constant throughout).

Let $v(x)$ be this steady-state. If we use it in the PDE (that is, let $u(x,t) = v(x)$, we get:

$$\alpha^2 u_{xx} = u_t \implies \alpha^2 v'' = 0$$

In other words, we expect v to be **linear**:

$$v(x) = Ax + B$$

Furthermore, we require that v satisfies the boundary conditions (since these hold for any t), so:

$$v(0) = a \qquad v(L) = b$$

which lead to the system:

$$\begin{aligned} a &= B \\ b &= AL + B \end{aligned}$$

$$\begin{aligned} \implies \quad b &= AL + a \\ \implies \quad A &= \frac{b-a}{L} \end{aligned}$$

Hence, the steady-state solution must be:

$$v(x) = \frac{b-a}{L}x + a$$

(Notice, this is the equation of the line joining the points $u(0,t), u(L,t)$. Notice also that if $b = a = 0$, we get $v(x) = 0$, which corresponds to our homogeneous BVP)

2.2.3 The Transient-State Temperature Distribution $w(x,t)$

We have now defined the behaviour of any solution as $t \to \infty$. Now we need to find what happens in between. Without loss of generality, we can assume there exists some function $w(x,t)$, such that:

$$u(x,t) = w(x,t) + v(x)$$

This reduces finding $u(x,t)$ to determining $w(x,t)$.

2.2.4 Solving the Heat Equation: Non-Homogeneous Boundary Conditions

Assuming a solution of the form $u(x,t) = w(x,t) + v(x)$, lets see if it satisfies the PDE:

$$\alpha^2 u_{xx} = u_t \implies \alpha^2 w_{xx} = w_t$$

Thus, the original PDE is now in terms of w.

We can update the initial/boundary conditions, by using $w(x,t) = u(x,t) - v(x)$:

$$\begin{aligned} w(x,0) &= u(x,0) - v(x) = f(x) - \frac{b-a}{L}x + a \\ w(0,t) &= u(0,t) - v(0) = a - a = 0 \\ w(L,t) &= u(L,t) - v(L) = b - b = 0 \end{aligned}$$

Hence, by converting the PDE to w, we obtain a **homogeneous system**, which we know how to solve (2.1.5).

Hence, if we have non-homogeneous boundary conditions, it follows that:

$$u(x,t) = v(x) + w(x,t) = \frac{b-a}{L}x + a + \sum_{n=1}^{\infty} c_n \exp\left(-\frac{n^2\pi^2\alpha^2}{L^2}t\right)\sin\left(\frac{n\pi x}{L}\right)$$

where:

$$c_n = \frac{2}{L}\int_0^L (f(x) - v(x))\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L \left(f(x) - \frac{b-a}{L}x - a\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

2.3 The Heat Equation: Insulating Ends

2.3.1 Defining the Heat Equation

We consider the heat equation:

$$\alpha^2 u_{xx} = u_t$$

with different boundary conditions:

- $u(x,0) = f(x), \quad 0 \leq x \leq L$
- $u_x(0,t) = u_x(L,t) = 0, \quad t > 0$
- **What do the initial/boundary conditions imply about the solution we expect?**
  - the IVP has not changed. we still expect an initial temeprature distribution
  - the BVP changes, this time acting as if the endpoints of the rod are insulated, and temperature can't flow to them

2.3.2 Applying Separation of Variables to Heat Equation With Insulated Ends

If we assume a solution of the form:

$$u(x,t) = X(x)T(t)$$

then the PDE $\alpha^2 u_{xx} = u_t$ implies:

$$\alpha^2 X''T = XT'$$

If we rearrange this to have X and T on each side:

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}$$

Notice that the RHS depends solely on x, whils the LHS depends solely on t. These 2 can only be equal iff they are consatnat (think about fixing x, and changing t; the equality will only hold if changing t has no effect). Thus, we claim that there exists some $\lambda \in \mathbb{R}$ such that:

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = \lambda$$

2.3.3 Solving the Eigenvalue Problem for $X(x)$

We first have an eigenvalue problem, given by:

$$\frac{X''}{X} = \lambda \implies X'' = \lambda X$$

subject to the boundary condition:

$$u_x(0,t) = u_x(L,t) = 0$$

Notice, if we differentiate $u = XT$ with respect to x, we get:

$$u_x = X'T$$

So:

$$u_x(0,t) = X'(0)T(t) = 0$$

Again, since we want no non-trivial solutions, this is only possible if $X'(0) = 0$. Similarly, we must also have $X'(L) = 0$.

To solve the BVP, we consider 3 cases (this is the same as the work last week with BVPs).

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(1) $\boldsymbol{\lambda = \mu^2 > 0}$

The ODE becomes:

$$X'' - \mu^2 X = 0$$

which is solved by:

$$X(x) = A\cosh(\mu x) + B\sinh(\mu x)$$

and differentiating:

$$X'(x) = A\mu\sinh(\mu x) + B\mu\cosh(\mu x) = K_1\sinh(\mu x) + K_2\cosh(\mu x)$$

If $X'(0) = 0$ and $X'(L) = 0$, we obtain:

$$\begin{aligned} &K_1 = 0 \\ &K_1\sinh(\mu L) + K_2\cosh(\mu L) = 0 \end{aligned}$$

But $\cosh(\mu L) \neq 0$, so $K_2\cosh(\mu L) = 0$ if and only if $K_2 = 0$. Hence, if $\lambda = \mu^2$ there are no non-trivial solutions.

---

(2) $\boldsymbol{\lambda = 0}$

The ODE becomes:

$$X'' = 0$$

which is solved by:

$$X(x) = Ax + B$$

and differentiating:

$$X'(x) = A$$

If $X'(0) = 0$ and $X'(L) = 0$, we obtain:

$$\begin{aligned} A &= 0 \\ A &= 0 \end{aligned}$$

This means that we can set B to whatever, and the boundary condition is satisfied. $\lambda = 0$ is an eigenvalue corresponding to eigenfunction $X = 1$.

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(3) $\boldsymbol{\lambda = -\mu^2 < 0}$

The ODE becomes:

$$X'' + \mu^2 X = 0$$

which is solved by:

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

and differentiating:

$$X'(x) = -A\mu\sin(\mu x) + B\mu\cos(\mu x) = K_1\sin(\mu x) + K_2\cos(\mu x)$$

If $X'(0) = 0$ and $X'(L) = 0$, we obtain:

$$\begin{aligned} &K_2 = 0 \\ &K_1\sin(\mu L) + K_2\cos(\mu L) = 0 \end{aligned}$$

Notice, $\sin(\mu L) = 0$ infinitely many times, so long as:

$$\mu = \frac{n\pi}{L}, \quad n \geq 1$$

(we don't want $\mu = 0$, since that case was covered before).

Hence, we obtain:

$$\begin{aligned} \lambda_n &= -\frac{n^2\pi^2}{L^2} \\ X_n(x) &= \cos\left(\frac{n\pi x}{L}\right) \end{aligned}$$

2.3.4 Solving the Eigenvalue Problem for $T(t)$

We now turn to the second problem, but using λ_n and T_n (since any T must satisfy any value for X):

$$\frac{1}{\alpha^2}\frac{T_n'}{T_n} = \lambda_n \implies T_n' = \alpha^2\lambda_n T_n$$

Solving the ODE for a general solution is standard:

$$T_n = \exp\left(\alpha^2\lambda_n t\right) = \exp\left(-\frac{n^2\pi^2\alpha^2}{L^2}t\right)$$

Alternatively, if $\lambda = 0$, the problem gets reduced to:

$$T' = 0$$

so T will just be constant, corresponding to the eigenvalue $\lambda = 0$.

2.3.5 Solving the Heat Equation: Insulated Ends

Hence, it follows that for any particular $n \geq 1$, we have:

$$u(x,t) = X_n(x)T_n(x) = \exp\left(-\frac{n^2\pi^2\alpha^2}{L^2}t\right)\cos\left(\frac{n\pi x}{L}\right)$$

and if $n = 0$:

$$u(x,t) = XT = u_0(x,t) = 1$$

(we just pick an arbitrary constant, since it can be scaled up or down)

If we use the fact that the ODE is **linear** and **homogeneous**, the most general solution must consist of a **linear combination** of the above solution, so:

$$\begin{aligned} u(x,t) &= \frac{c_0}{2}u_0(x,t) + \sum_{n=1}^{\infty} c_n \exp\left(-\frac{n^2\pi^2\alpha^2}{L^2}t\right)\cos\left(\frac{n\pi x}{L}\right) \\ &= \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \exp\left(-\frac{n^2\pi^2\alpha^2}{L^2}t\right)\cos\left(\frac{n\pi x}{L}\right) \end{aligned}$$

We still require that $u(x,t)$ satisfies the initial condition:

$$u(x,0) = f(x), \quad 0 \leq x \leq L$$

Substituting in $t = 0$:

$$u(x,0) = f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right)$$

But notice, this is as if we were expressing f as a Fourier Series, which makes finding c_n trivial: its just an Euler-Fourier Coefficient, so:

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

for $n = 0, 1, 2, \ldots$. (intuitively, we can think of f being periodic with period L, by considering its **even** extension, which means that the above is applicable)

We can think of $\frac{c_0}{2}$ as representing a steady state solution (given as the average value of the original temperature distribution); the infinite series is thus a transient solution.

2.4 The Heat Equation: Further Cases

Different initial/boundary conditions lead to different, intersting problems, which can still be solved by separation of variables:

- we can make it so that one end of the bar is insulated, whilst the other one holds constant temperature:

  $$u_x(0,t) = 0, \quad u(L,t) = K$$

  We can solve this problem by reducing the boundary conditions to homogeneous ones by subtracting the steady-state solution. The resulting problem is solved by similar procedures as in the work above. However, the extension of the initial function f outside of the interval [0, L] is somewhat different from that in any case considered so far.

- alternatively, we can make it so that the rate of flow of heat at the endpoints is proportional to the temperature of the bar:

  $$u_x(0,t) = h_1 u(0,t) = 0, \quad u_x(L,t) = h_2 u(L,t) = 0$$

3 The Wave Equation

3.1 Defining the Wave Equation

Let $u(x,t)$ denote the (vertical) displacement of an elastic string of length L in 1 dimension. Then, the vibration resulting from plucking said string (neglecting damping and assuming low amplitude) can be modelled by the PDE:

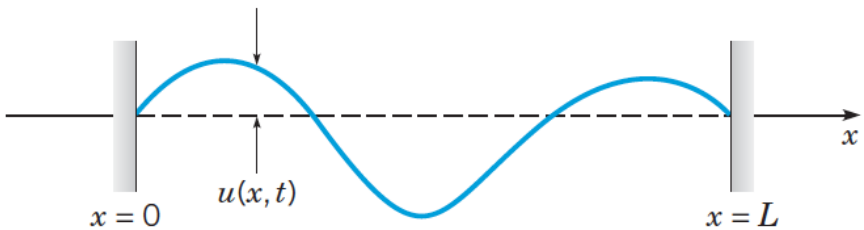
$$\boldsymbol{a^2 u_{xx} = u_{tt}}$$

where a is the velocity of propagation of the wave across the string:

$$a^2 = \frac{T}{\rho}$$

where:

- T is the tension of the string
- ρ is the mass per unit length of the string (material)



We can extend the PDE to 2 dimensions (i.e for a drum):

$$a^2(u_{xx} + u_{yy}) = u_{tt}$$

and 3 dimensions:

$$a^2(u_{xx} + u_{yy} + u_{zz}) = u_{tt}$$

For shallow water waves, we have:

$$a^2 = gH$$

3.2 General Boundary/Initial Conditions

We consider the simplest example, in which a string is bound at both ends. This is governed by the PDE:

$$a^2 u_{xx} = u_{tt}$$

given boundary/initial conditions:

- $u(x,0) = f(x), \quad 0 \leq x \leq L$
- $u_t(x,0) = g(x), \quad 0 \leq x \leq L$
- $u(0,t) = u(L,t) = 0, \quad t \geq 0$

- **What do the initial/boundary conditions imply about the solution we expect?**
  - we have 2 IVPs, since we are considering the second partial derivative of u with respect to time. These dictate the initial position and velocity of the string, before being plucked.
  - the BVP shows that we are fixing the string at its endpoints
  - the BVP enforces that:
$$f(0) = f(L) = g(0) = g(L) = 0$$

3.3 The Wave Equation: Elastic String with Nonzero Initial Displacement

3.3.1 Defining the Wave Equation

We consider a system governed by the PDE:

$$a^2 u_{xx} = u_{tt}$$

given boundary/initial conditions:

- $u(x, 0) = f(x), \quad 0 \le x \le L$
- $u_t(x, 0) = 0, \quad 0 \le x \le L$
- $u(0, t) = u(L, t) = 0, \quad t \ge 0$
- **What do the initial/boundary conditions imply about the solution we expect?**
  - the IVP changes: whilst the wave is initially displaced, its initial velocity is required to be 0 (i.e we disturb the string, keeping it taut with our finger, and release it at $t = 0$)
  - the BVP shows that we are fixing the string at its endpoints

3.3.2 Applying Separation of Variables to Wave Equation With Nonzero Initial Displacements

If we assume a solution of the form:
$$u(x, t) = X(x)T(t)$$
then the PDE $a^2 u_{xx} = u_{tt}$ implies:
$$a^2 X''T = XT''$$
So rearranging:
$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{t}$$
Again, this is only possible if both sides are equal to a constant, so:
$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{t} = \lambda$$
Hence, we obtain 2 eigenvalue problems:
$$X'' = \lambda X$$
$$T'' = a^2 \lambda T$$

3.3.3 Solving the Eigenvalue Problem for $X(x)$

We need to solve:

$$X'' = \lambda X$$

subject to the boundary condition:

$$u(0,t) = u(L,t) = 0$$

Notice, if we use $u(x,t) = X(x)T(t)$, the boundary problem leads to:

$$X(0)T(t) = 0$$

which as discussed above only has non-trivial solutions if $X(0) = 0$. Similarly, we require $X(L) = 0$

Notice, this is the exact same problem as we had defined for the Heat Equation under Homogeneous Boundary Conditions (2.1.3), which has solutions:

$$\lambda_n = -\frac{n^2\pi^2}{L^2}$$

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

3.3.4 Solving the Eigenvalue Problem for $T(t)$

We need to solve:

$$T_n'' = -\frac{n^2\pi^2 a^2}{L^2} T_n$$

subject to the initial condition:

$$u_t(x,0) = 0$$

Notice, if we use $u(x,t) = X(x)T(t)$, the initial value problem leads to:

$$X(x)T'(0) = 0$$

which with similar logic as before, require $T'(0) = 0$ for non-trivial solutions.

If we rearrange the ODE above, we have:

$$T_n'' + \frac{n^2\pi^2 a^2}{L^2} T_n = 0$$

which we know has sinusoidal solutions:

$$T_n(t) = A\cos\left(\frac{n\pi a}{L}t\right) + B\sin\left(\frac{n\pi a}{L}t\right)$$

and differentiating:

$$T_n'(t) = K_1 \sin\left(\frac{n\pi a}{L}t\right) + K_2\cos\left(\frac{n\pi a}{L}t\right)$$

Since we require $T'(0) = 0$ to satisfy the IVP, this implies that:

$$K_2 = 0$$

So we must have:

$$T(t) \propto \cos\left(\frac{n\pi a}{L}t\right)$$

(the initial condition requires $K_2 = 0$, so the *sin* term must disappear)

3.3.5 Solving the Wave Equation: Nonzero Initial Displacement

Using all of the above, we get that:

$$u_n(x,t) = \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right)$$

satisfies the BVP and a IVP. These u_n conform a fundamental set of solutions, so using **linearity** and **homogeneity**, we use superposition to obtain a general solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right)$$

We want to satisfy the initial condition $u(x,0) = f(x)$, so substituting $t = 0$:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)$$

Once again, this reduces satisfying the IVP to computing the Euler-Fourier Coefficients, which are given by:

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

for $n = 1, 2, \ldots$.

3.3.6 Implications of the Solution

- the quantities $\frac{n\pi a}{L}$ represent the **natural frequencies** of the string
- in the book they show that the series does indeed converge
- if f is discontinuous, the wave equation will produce solutions which periodically repeat this discontinuity (as opposed to the Heat Equation, in which discontinuities were smoothed out)

3.4 The Wave Equation: Elastic String with Nonzero Initial Velocity

3.4.1 Defining the Wave Equation

We consider a system governed by the PDE:

$$a^2 u_{xx} = u_{tt}$$

given boundary/initial conditions:

- $u(x,0) = 0, \quad 0 \leq x \leq L$
- $u_t(x,0) = g(x), \quad 0 \leq x \leq L$
- $u(0,t) = u(L,t) = 0, \quad t \geq 0$
- **What do the initial/boundary conditions imply about the solution we expect?**
  - the IVP changes: this time, the string is at rest, and set into motion with a given velocity (i.e the string is plucked)
  - the BVP shows that we are fixing the string at its endpoints

3.4.2 Applying Separation of Variables to Wave Equation With Nonzero Initial Displacements

If we assume a solution of the form:

$$u(x,t) = X(x)T(t)$$

then the PDE $a^2 u_{xx} = u_{tt}$ implies:

$$a^2 X''T = XT''$$

So rearranging:

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{t}$$

Again, this is only possible if both sides are equal to a constant, so:

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{t} = \lambda$$

Hence, we obtain 2 eigenvalue problems:

$$X'' = \lambda X$$

$$T'' = a^2 \lambda T$$

3.4.3 Solving the Eigenvalue Problem for $X(x)$

We need to solve:

$$X'' = \lambda X$$

subject to the boundary condition:

$$u(0,t) = u(L,t) = 0$$

Notice, if we use $u(x,t) = X(x)T(t)$, the boundary problem leads to:

$$X(0)T(t) = 0$$

which as discussed above only has non-trivial solutions if $X(0) = 0$. Similarly, we require $X(L) = 0$

Notice, this is the exact same problem as we had defined for the Heat Equation under Homogeneous Boundary Conditions (**??**), which has solutions:

$$\lambda_n = -\frac{n^2\pi^2}{L^2}$$

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

3.4.4 Solving the Eigenvalue Problem for $T(t)$

We need to solve:

$$T_n'' = -\frac{n^2\pi^2 a^2}{L^2} T_n$$

subject to the initial condition:

$$u(x,0) = 0$$

Notice, if we use $u(x,t) = X(x)T(t)$, the initial value problem leads to:

$$X(x)T(0) = 0$$

which with similar logic as before, require $T(0) = 0$ for non-trivial solutions.

If we rearrange the ODE above, we have:

$$T_n'' + \frac{n^2\pi^2a^2}{L^2}T_n = 0$$

which we know has sinusoidal solutions:

$$T_n(t) = A\cos\left(\frac{n\pi a}{L}t\right) + B\sin\left(\frac{n\pi a}{L}t\right)$$

Since we require $T(0) = 0$ to satisfy the IVP, this implies that:

$$A = 0$$

So we must have:

$$T(t) \propto \sin\left(\frac{n\pi a}{L}t\right)$$

3.4.5 Solving the Wave Equation: Nonzero Initial Velocity

Using all of the above, we get that:

$$u_n(x,t) = \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi a}{L}t\right)$$

satisfies the BVP and a IVP. These u_n conform a fundamental set of solutions, so using **linearity** and **homogeneity**, we use superposition to obtain a general solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi a}{L}t\right)$$

We want to satisfy the initial condition $u_t(x,0) = g(x)$, so we first differentiate termwise:

$$u_t(x,t) = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{L}\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right)$$

So substituting $t = 0$:

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{L}\sin\left(\frac{n\pi}{L}x\right)$$

Once again, this reduces satisfying the IVP to computing the Euler-Fourier Coefficients, which are given by:

$$c_n\frac{n\pi a}{L} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

for $n = 1, 2, \ldots$. This then implies that to find the c_n:

$$c_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

3.5 The Wave Equation: General Case

3.5.1 Defining the Wave Equation

We consider a system governed by the PDE:

$$a^2 u_{xx} = u_{tt}$$

given boundary/initial conditions:

- $u(x, 0) = f(x), \quad 0 \leq x \leq L$
- $u_t(x, 0) = g(x), \quad 0 \leq x \leq L$
- $u(0, t) = u(L, t) = 0, \quad t \geq 0$
- **What do the initial/boundary conditions imply about the solution we expect?**
  - the IVP changes: this time, both the displacement and velocity can be modelled by any function
  - the BVP shows that we are fixing the string at its endpoints

3.5.2 Solving the Wave Equation: General Case

Whilst separation of variables could be used, we could use the **principle of superposition** to obtain the general solution.

Let:

$$v(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi a}{L}t\right)$$

with:

$$c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

Similarly, let:

$$w(x, t) = \sum_{n=1}^{\infty} k_n \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi a}{L}t\right)$$

with:

$$k_n = \frac{2}{n\pi a} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

Define $u(x, t) = v(x, t) + w(x, t)$. The wave equation is:

$$a^2 u_{xx} = u_{tt}$$

If we plug in $u(x, t)$:

$$a^2(v_{xx} + w_{xx}) = v_{tt} + w_{tt} \implies (a^2 v_{xx} - v_{tt}) + (a^2 w_{xx} - w_{tt}) = 0 + 0 = 0$$

so u satisfies the wave equation.

We now check the initial/boundary conditions:

$$u(0, t) = v(0, t) + w(0, t) = 0$$
$$u(L, t) = v(L, t) + w(L, t) = 0$$
$$u(x, 0) = v(x, 0) + w(x, 0) = f(x) + 0 = f(x)$$
$$u_t(x, 0) = v_t(x, 0) + w_t(x, 0) = 0 + g(x) = g(x)$$

So u also satsifies the IVP/BVP. Hence, u is a general solution to the wave equation.

4 Laplace's Equation

4.1 Describing Laplace's Equations

Laplace's Equation is used to describe systems which don't change in time. The **Laplacian** is an operator. For 2 dimensions:

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Laplace's Equations is defined for some time independent function $u(x, y)$:

$$\nabla^2 u = 0 \implies u_{xx} + u_{yy} = 0$$

4.2 Applications of Laplace's Equation

- **When is Laplace's Equation used?**
  - steady solutions to heat equation (if $u(x, y, t)$ is time independent, then two dimensional heat conduction becomes $\alpha^2(u_{xx} + u_{yy}) = 0$)
  - minimal surfaces (i.e soap films)
  - fluid dynamics (i.e aerofoils)
  - it is sometimes called the potential equation, since its solutions are gravitational/electrostatic potential functions for particles in gravitational/electric fields in the absence of matter/electric charges

4.3 Boundary Conditions for Laplace's Equations

- **Does Laplace's Equation require initial conditions?**
  - since we assume time independence, we don't expect the solution to change as time progresses
  - hence, we don't expect any initial condition to be satisfied
- **How are boundary values given for Laplace's Equation?**
  - we need to think about the nature of boundary conditions
  - for the heat equation, in 1 dimension, we need to specify a condition for each boundary point of the interval
  - it makes sense that in 2 dimension, we need to specify values of u at each point of the boundary of the region - that is, the boundary condition requires that $u(x, y, y)$ satisfies a 2 dimensional ´´line" as a boundary condition
  - for example, we can consider a wire frame, which holds a bubble: the sections of the wire frame conform the boundary conditions (i.e at the boundary the soap bubble should be in the shape of the boundary)
  - for 3 dimensions, we would specify a boundary surface
- **What is the Dirichlet Problem?**
  - a BVP in which we specify the value of the function $u(, xy, t)$ at the boundary
- **What is the Neumann Problem?**
- a BVP in which we specify the value of the normal derivative of $u(x, y, t)$ at the boundary

4.4 Laplace's Equation: Dirichlet Problem in a Rectangle

4.4.1 Defining the Dirichlet Problem

We consider the PDE:

$$u_{xx} + u_{yy} = 0$$

subject to the boundary conditions, for $0 \leq x \leq a$ and $0 \leq x \leq b$:

- $u(x, 0) = u(x, b) = 0, \quad 0 < x < a$
- $u(0, y) = 0, \quad 0 < y < b$
- $u(a, y) = f(y), \quad 0 < y < b$
- **What do the boundary conditions imply about the solution we expect?**
  - the BVP is essentially describing a partial rectangle in the xy plane: there are 2 horizontal lines at $y = 0$ and $y = b$; there is a vertical line at $x = 0$; lastly, there is line, defined by $f(y)$, which can be thought of as "extending" to the z plane (i.e $z = f(y)$)

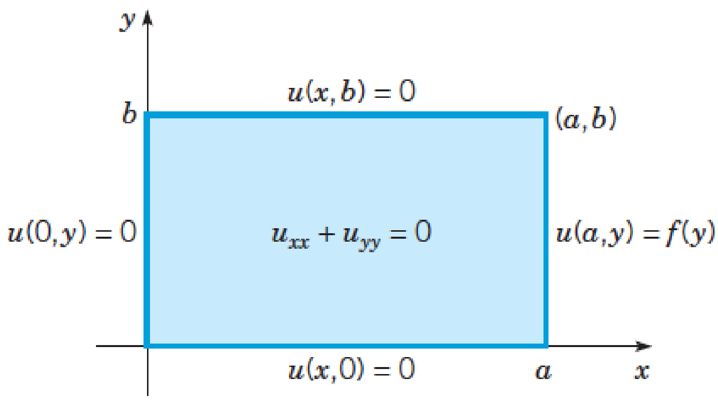


Figure 1: Visualising the Dirichlet Problem in a Rectangular Domain

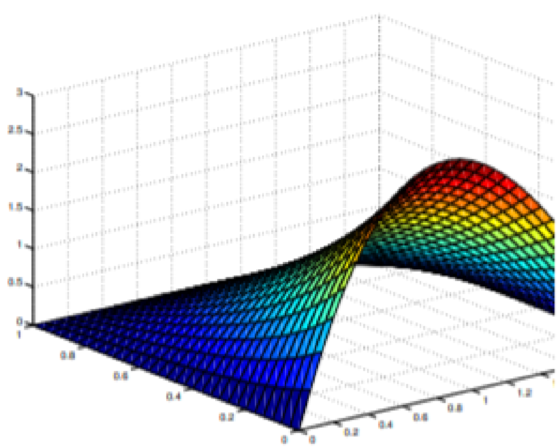


Figure 2: An example of a solution. Think of the boundary as a wire, and of the solution as a soap film.

4.4.2 Applying Separation of Variables to Dirichlet Problem in a Rectangle

Assuming a solution of the form $u(x,y) = X(x)Y(y)$, we plug this in to the PDE $u_{xx} + u_{yy} = 0$, and get:

$$X''Y + XY'' = 0 \implies \frac{X''}{X} = -\frac{Y''}{Y}$$

This is only true if both sides are constants, so:

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

This leads to 2 eigenvalue problems:

$$X'' = \lambda X$$

$$Y'' = -\lambda Y$$

If we consider the boundary conditions:

- $u(x,0) = u(x,b) = 0, \quad 0 < x < a$
- $u(0,y) = 0, \quad 0 < y < b$
- $u(a,y) = f(y), \quad 0 < y < b$

we get:

- $X(x)Y(0) = X(x)Y(b) = 0$
- $X(0)Y(y) = 0$
- $X(a)Y(y) = f(y)$

which, for non-trivial solutions, implies:

- $Y(0) = Y(b) = 0$
- $X(0) = 0$
- $X(a)Y(y) = f(y)$

4.4.3 Solving the Eigenvalue Problem for $Y(y)$

We solve:

$$Y'' = -\lambda Y$$

subject to:

$$Y(0) = Y(b) = 0$$

However, notice this is the same problem as in the Heat Equation with Homogeneous Boundary Conditions (2.1.3). This means that:

$$\lambda_n = \frac{n^2\pi^2}{b^2}$$

(this time since λ_n is taken as positive, we don't need to include the negative sign as in (2.1.3))

$$Y_n(y) = \sin\left(\frac{n\pi}{b}y\right)$$

4.4.4 Solving the Eigenvalue Problem for $X(x)$

We now consider:

$$X_n'' - \frac{n^2\pi^2}{b^2} X_n = 0$$

which we know has exponential solutions. We alternatively express them as cosh and sinh (this is the same ODE as in (2.1.3) with $\lambda = \mu^2 > 0$):

$$X_n(x) = A\cosh\left(\frac{n\pi}{b}x\right) + B\sinh\left(\frac{n\pi}{b}x\right)$$

There is a boundary condition requiring $X(0) = 0$, which implies:

$$A = 0$$

Since $\sinh(0) = 0$ it follows that the boundary condition will always be satisfied, so long as $A = 0$:

$$X_n(x) \propto \sinh\left(\frac{n\pi}{b}x\right)$$

4.4.5 Solving the Dirichlet Problem in a Rectangle

Putting all of the above together, it follows that solutions are of the form:

$$u_n(x,y) = \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

for $n = 1, 2, \ldots$

The general solution must be given by using **linearity** and **homogeneity**:

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

To compute the coefficients, we use the last of the boundary conditions:

$$u(a,y) = f(y)$$

which means that:

$$u(a,y) = f(y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b}a\right)\sin\left(\frac{n\pi}{b}y\right)$$

We can thus compute the c_n by using the Euler-Fourier Coefficients:

$$c_n \sinh\left(\frac{n\pi}{b}a\right) = \frac{2}{b}\int_0^b f(y)\sin\left(\frac{n\pi}{b}y\right)dy$$

which means that:

$$c_n = \frac{2}{b\sinh\left(\frac{n\pi}{b}a\right)}\int_0^b f(y)\sin\left(\frac{n\pi}{b}y\right)dy$$

It can be shown that $c_n \sinh\left(\frac{n\pi}{b}a\right)\sin\left(\frac{n\pi}{b}y\right)$ grows similarly to a negative exponential, which indicates that the series will converge.

4.5 Laplace's Equation: Dirichlet Problem in a Circle

4.5.1 Defining the Dirichlet Problem

We now consider Laplace´s Equation, but using a circle as boundary.

It is easier to work with polar coordinates:

$$r^2 = x^2 + y^2$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

If we have $u(x, y)$ satisfying:

$$u_{xx} + u_{yy} = 0$$

If we use polar coordiantes:

$$u(x, y) = u(r\cos(\theta), r\sin\theta)$$

So differentiating:

$$u_r = u_x x_r + u_y y_r = u_x \cos(\theta) + u_y \sin(\theta)$$

$$u_\theta = u_x x_\theta + u_y y_\theta = -u_x r\sin(\theta) + u_y r\cos(\theta)$$

We are interested in second derivatives, so we differentiate again:

$$\begin{aligned}
u_{rr} &= (u_{xx}x_r + u_{xy}y_r)\cos(\theta) + (u_{yy}y_r + u_{yx}x_r)\sin(\theta) \\
&= (u_{xx}\cos(\theta) + u_{xy}\sin(\theta))\cos(\theta) + (u_{yy}\sin(\theta) + u_{yx}\cos(\theta))\sin(\theta) \\
&= u_{xx}\cos^2(\theta) + u_{yy}\sin^2(\theta) + 2u_{xy}\cos(\theta)\sin(\theta)
\end{aligned}$$

$$\begin{aligned}
u_{\theta\theta} &= -(u_{xx}x_\theta + u_{xy}y_\theta)r\sin(\theta) + (u_{yy}y_\theta + u_{yx}x_\theta)r\cos(\theta) \\
&= -(-u_{xx}r\sin(\theta) + u_{xy}r\cos(\theta))r\sin(\theta) - u_x r\cos(\theta) + (u_{yy}r\cos(\theta) - u_{yx}r\sin(\theta))r\cos(\theta) - u_y r\sin(\theta) \\
&= u_{xx}r^2\sin^2(\theta) + u_{yy}r^2\cos^2(\theta) - 2u_{xy}r^2\sin(\theta)\cos(\theta) - r(u_x\cos(\theta) + u_y\sin(\theta)) \\
\implies \frac{1}{r^2}u_{\theta\theta} &= u_{xx}\sin^2(\theta) + u_{yy}\cos^2(\theta) - 2u_{xy}\sin(\theta)\cos(\theta) - \frac{1}{r}(u_x\cos(\theta) + u_y\sin(\theta)) \\
&= u_{xx}\sin^2(\theta) + u_{yy}\cos^2(\theta) - 2u_{xy}\sin(\theta)\cos(\theta) - \frac{1}{r}u_r
\end{aligned}$$

So if we add u_{rr} and $\frac{1}{r^2}u_{\theta\theta}$ we get:

$$\begin{aligned}
&u_{rr} + \frac{1}{r^2}u_{\theta\theta} \\
&\qquad = u_{xx}\cos^2(\theta) + u_{yy}\sin^2(\theta) + 2u_{xy}\cos(\theta)\sin(\theta) + u_{xx}\sin^2(\theta) + u_{yy}\cos^2(\theta) - 2u_{xy}\sin(\theta)\cos(\theta) - \frac{1}{r}u_r \\
&\qquad = u_{xx} + u_{yy} - \frac{1}{r}u_r
\end{aligned}$$

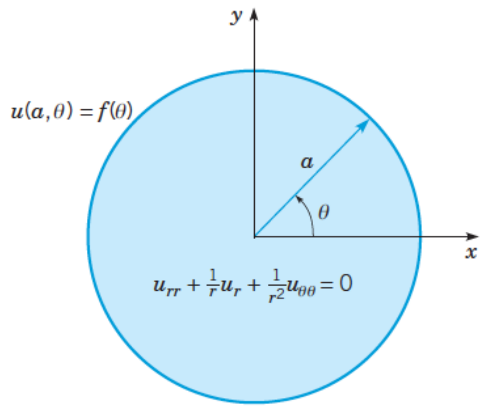
So it follows that in polar coordinates, Laplace's Equation becomes:

$$u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r = 0$$

for some $r < a$ and $0 < \theta < 2\pi$. The boundary conditions are updated:

$$u(a, \theta) = f(\theta), \quad 0 < \theta < 2\pi$$

and f must satisfy $f(0) = f(2\pi)$ (since we want the boundary to be closed, and this ensures that the periodic extension for the Fourier Series is continuous and bounded)



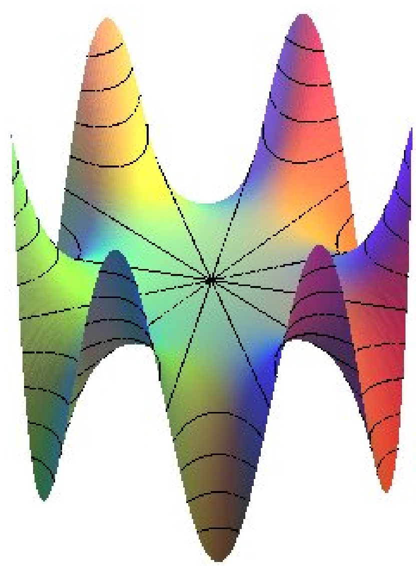


Figure 3: The boundary is like a wire, with the curve as a soap film.

4.5.2 Applying Separation of Variables to Dirichlet Problem in a Circle

Assume a solution to the PDE:

$$u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r = 0$$

of the form:

$$u(r, \theta) = R(r)\Theta(\theta)$$

where $u(r, \theta)$ is a **periodic** (in θ, with period 2π), **bounded** function.

Then, we must have:

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \implies r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta}$$

which is only possible if:

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

This leads to the following eigenvalue problems:

$$r^2 R'' + rR' = \lambda R$$

$$\Theta'' = -\lambda\Theta$$

4.5.3 Solving the Eigenvalue Problem for $\Theta(\theta)$

Whilst there are no explicity boundary conditions, we require that Θ be 2π periodic wrt θ (that is, $\Theta(\theta) = \Theta(\theta + 2\pi)$. Imposing this will mean that $\lambda \in \mathbb{R}$. Hence, finding Θ requires considering 3 cases.

---

① $\boldsymbol{\lambda = -\mu^2 < 0}$

The ODE becomes:

$$\Theta'' - \mu^2\Theta = 0$$

which has solution:

$$\Theta(\theta) = Ae^{\mu\theta} + Be^{-\mu\theta}$$

this is never periodic, unless $A = B = 0$ (we can think of the above as being expressed in terms of cosh and sinh which aren't periodic).

② $\boldsymbol{\lambda = 0}$

The ODE becomes:

$$\Theta'' = 0$$

which we know has a linear solution:

$$\Theta(\theta) = A\theta + B$$

This is only periodic if $A = 0$, in which case $\Theta(\theta) = B$.

③ $\boldsymbol{\lambda = \mu^2 > 0}$

The ODE becomes:

$$\Theta'' + \mu^2\Theta = 0$$

which has solutions:

$$\Theta(\theta) = A\cos(\mu\theta) + B\sin(\mu\theta)$$

If we consider $\Theta(\theta + 2\pi)$, we get:

$$A\cos(\mu(\theta + 2\pi)) + B\sin(\mu(\theta + 2\pi)) = A\cos(\mu\theta + 2\mu\pi) + B\sin(\mu\theta + 2\mu\pi)$$

We can use the sum of angle formulae:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\beta\sin\alpha$$

So:

$$\begin{aligned}&A\cos(\mu\theta+2\mu\pi)+B\sin(\mu\theta+2\mu\pi)\\=&A\cos(\mu\theta)\cos(2\mu\pi)-A\sin(\mu\theta)\sin(2\mu\pi)+B\sin(\mu\theta)\cos(2\mu\pi)+B\cos(\mu\theta)\sin(2\mu\pi)\end{aligned}$$

For this to be equal to $A\cos(\mu\theta)+B\sin(\mu\theta)$, this means that we must have:

$$\sin(2\mu\pi)=0$$

$$\cos(2\mu\pi)=1$$

This is true if and only if $\mu\in\mathbb{Z}$, so we let $\mu=n=1,2,\ldots$. Hence, for $\lambda=n^2$, we have:

$$\Theta_n(\theta)=A\cos(n\theta)+B\sin(n\theta)$$

4.5.4 Solving the Eigenvalue Problem for $R(r)$

If we have $\lambda_0=0$, then we have the ODE:

$$r^2R''+rR'=0$$

Using $R'(r)=Z(r)$:

$$r^2Z'+rZ=0 \implies rZ'+Z=0$$

This is known as a Cauchy-Euler Equation. If we assume that $Z=kr^m$, then:

$$k(mr^m+r^m)=0 \implies m=-1$$

So it follows that:

$$Z(r)=\frac{k}{r} \implies R(r)=\int\frac{k}{r}=k\ln(r)+C$$

But notice, this means that $R(r)$ is unbounded (as $r\to 0$, $ln(r)\to-\infty$), unless $k=0$, in which case $R(r)=C$ must be constant. Hence, if $\lambda=0$ we set:

$$u(r,\theta)=u_0(r,\theta)\propto 1$$

If $\lambda\neq 0$, then we have $\lambda_n=n^2$, so the ODE becomes:

$$r^2R''+rR'=n^2R \implies r^2R''+rR'-n^2R=0$$

Again, we try $R(r)=r^m$, so:

$$R'(r)=mr^{m-1}$$

$$R''(r)=m(m-1)r^{m-2}$$

So the ODE becomes:

$$m(m-1)r^m+mr^m-n^2r^m=r^m(m^2-n^2)$$

Hence, $R(r)=r^m$ satisfies the ODE if and only if $m^2=n^2$, which indicates that:

$$R_n(r)=Ar^n+Br^{-n}$$

Since we want a bounded solution, and $0<r<a$, notice that as $r\to 0$, $r^{-n}\to\infty$, so we require $B=0$. Hence:

$$R_n(r)\propto r^n$$

4.5.5 Solving the Dirichlet Problem in a Circle

If we put all of the above together, we get that:

$$u_0(r,\theta) = 1$$

If $n \geq 1$, there are 2 possibilities, since we found that $\Theta_n(\theta)$ could be expressed as any linear combination of $\cos(n\theta)$ and $\sin(n\theta)$ so:

$$u_n(r,\theta) = r^n \cos(n\theta)$$

or

$$v_n(r,\theta) = r^n \sin(n\theta)$$

Again, due to **linearity** and **homogeneity**, the general solution must be of the form:

$$u(r,\theta) = \frac{c_0}{2}u_0(r,\theta) + \sum_{n=1}^{\infty} c_n u_n(r,\theta) + k_n v_n(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n(c_n\cos(n\theta) + k_n\sin(n\theta))$$

We can then apply the fact that we require $u(a,\theta) = f(\theta)$ to find c_n, k_n:

$$u(a,\theta) = f(\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n(c_n\cos(n\theta) + k_n\sin(n\theta))$$

Since we have enforced that f be periodic with period 2π, we can compute c_n, k_n by using the Euler-Fourier Coefficients, such that:

$$a^n c_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \qquad n \geq 0$$

$$a^n k_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta, \qquad n \geq 1$$

so the coefficients are:

$$c_n = \frac{1}{a^n\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \qquad n \geq 0$$

$$k_n = \frac{1}{a^n\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta, \qquad n \geq 1$$

We require both the cosine and sine terms, since we are considering boundary data ($f(\theta)$) which lies on a circle, and is 2π periodic