Honours Differential Equations - Week 6 - Fourier Series

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Contents

Paul's Online Math Notes pretty much cover everything about BVPs and Fourier Series

1 Linear Algebra Recap

1.1 Solutions to a Linear System

We consider the general system:

$$\boldsymbol{A}\underline{x} = \underline{b}$$

- **When does such a system have a unique solution?**
  - if $\boldsymbol{A}$ is nonsingular ($\boldsymbol{A}^{-1}$ exists), then the system has a unique solution:

    $$\underline{x} = \boldsymbol{A}^{-1}\underline{b}$$

  - otherwise, if the matrix is not square, or its determinant is 0, there can be either infinitely many solutions, or no solutions
- **What is the *only* solution to the corresponding homogeneous system $\boldsymbol{A}\underline{x} = \underline{0}$ in which $\boldsymbol{A}$ is invertible?**
  - $\underline{x} = \underline{0}$ is always a solution, so if $\boldsymbol{A}$ is nonsingular, this will be the only solution
- **How are the solutions to the homogeneous system related to the solution of the non-homogeneous system?**
  - if the only solution to $\boldsymbol{A}\underline{x} = \underline{0}$ is $\underline{0}$, then it must be the case that $\boldsymbol{A}$ is invertible, and so, the non-homogeneous system has a unique solution
  - otherwise, the homogeneous system has a no or infinitely many solutions if and only if the homogeneous system has a non-zero solution

1.2 The Dot Product and Vectors as Basis

We shall use $(\underline{a}, \underline{b})$ to represent the dot product between 2 vectors, equivalent to using $\underline{a} \cdot \underline{b}$

- **How is the dot product of 2 vectors computed?**
  - for ease of computation, we shall consider vectors in $\mathbb{R}^2$. Define:

    $$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

  - the dot product is computed by performing elementwise multiplication, and adding each resulting element:

    $$(\underline{a}, \underline{b}) = a_1 b_1 + a_2 b_2$$

  - alternatively,

    $$(\underline{a}, \underline{b}) = \|\underline{a}\|\|\underline{b}\| \cos\theta$$

    where θ is the angle between the 2 vectors
- **What is the dot product of a vector with itself?**
  - we can easily compute this:

    $$(\underline{a}, \underline{a}) = a_1 a_1 + a_2 a_2 = a_1^2 + a_2^2 = \|\underline{a}\|^2$$

- **What is a vector basis?**
  - consider a **vector space** V (the set of all vectors which contain $\underline{0}$, and are closed under addition and scalar multiplication)
  - a **basis** for V, is the smallest set of linearly independent vectors which **span** V (that is, any vector in V can be expressed as a linear combination of the basis vectors)
- **How can we express a vector in terms of basis vectors?**
  - we can use a linear combination. Let n be the dimension of a vector space V, with basis:

$$\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$$

  - if $\underline{x} \in V$, then there exist unique $c_i \in \mathbb{R}, i \in [1, n]$ such that:

$$\underline{x} = \sum_{i=1}^{n} c_i \underline{v}_i$$

- **How can we find the coefficients for the basis expression of a vector in a given basis?**
  - let $\underline{c}$ denote the set of coefficients $c_i \in \mathbb{R}, i \in [1, n]$, and assume we have:

$$\underline{x} = \sum_{i=1}^{n} c_i \underline{v}_i$$

  - consider the matrix $\boldsymbol{A}$ constructed by taking $\underline{v}_i$ as column vectors:

$$\boldsymbol{A} = \begin{pmatrix} \underline{v}_1 & \underline{v}_2 & \dots & \underline{v}_n \end{pmatrix}$$

  - then, we can express $\underline{x}$ via:

$$\underline{x} = \boldsymbol{A}\underline{c}$$

  - since each $\underline{v}_i$ is linearly independent, and $\boldsymbol{A}$ is a square matrix, $\underline{A}$ is invertible, so:

$$\underline{c} = \boldsymbol{A}^{-1}\underline{x}$$

- **When are 2 vectors orthogonal?**
  - 2 vectors are orthogonal if and only if their dot product is 0
- **Why are orthogonal basis vectors desirable?**
  - if we can construct a basis for a vector space consisting of solely orthogonal vectors, then we have that, $\forall i, j \in [1, n]$:

$$(\underline{v}_i, \underline{v}_j) = \begin{cases} 0, & i \neq j \\ \|\underline{v}_i\|^2, & i = j \end{cases}$$

  - this makes it extremely easy to find $\underline{c}$, since if:

$$\underline{x} = \sum_{i=1}^{n} c_i \underline{v}_i$$

we can find c_j by computing the dot product of the above expression with $\underline{v}_j$:

$$\begin{aligned} \underline{x} &= \sum_{i=1}^{n} c_i \underline{v}_i \\ \implies \quad \underline{v}_j \cdot \underline{x} &= \underline{v}_j \cdot \sum_{i=1}^{n} c_i \underline{v}_i \\ \implies \quad (\underline{v}_j, \underline{x}) &= \sum_{i=1}^{n} c_i (\underline{v}_j, \underline{v}_i) \\ \implies \quad (\underline{v}_j, \underline{x}) &= c_j (\underline{v}_j, \underline{v}_j) \\ \implies \quad c_j &= \frac{(\underline{v}_j, \underline{x})}{\|\underline{v}_j\|^2} \end{aligned}$$

- **What is the eigenvalue problem?**
  - the eigenvalue problem is the problem of finding non-trivial eigenvectors $\underline{v}_i$ which satisfy:

$$\boldsymbol{A}\underline{v}_i = \lambda \underline{v}_i$$

- **Are eigenvectors basis vectors?**
  - if each eigenvalue is distinct, we can guarantee we can obtain distinct, linearly independent eigenvectors [1], so in particularly, they will span a subspace
- **When are eigenvectors orthogonal?**
  - if we have a **symmetric matrix** ($\boldsymbol{A} = \boldsymbol{A}^T$, then its eigenvectors will be orthogonal

2 Boundary Value Problems

2.1 Defining Boundary Value Problems

- **How do initial value problems differ from boundary value problems, in terms of the initial conditions provided**
  - in an **initial value problem**, we seek to solve a ODE/system given:
    * the value of the solution at a point
    * the value of the derivative of the solution at the same point

    (this is an example for second order ODEs)
  - in a **boundary value problem**, we seek to solve a ODE/system given the values of the solution at 2 points
  - for example, an IVP could be:

$$y'' + y = 0, \qquad y(0) = 0, y'(0) = 2$$

, whilst a BVP could be:

$$y'' + y = 0, \qquad y(0) = 0, y(2) = -\frac{3}{\pi}$$

[1] https://math.stackexchange.com/questions/29371/how-to-prove-that-eigenvectors-from-different-eigenvalues-are-linearly-independe

- **How do solutions differ between IVPs and BVPs?**
  - in an IVP, we are guaranteed that a **unique** solution exists
  - in a BVP, we can have no solutions, a unique solution, or infinitely many solutions
  - this is akin to how the system $\boldsymbol{A}\underline{x} = \underline{b}$ can have no solutions, a unique solution, or infinitely many solutions, depending on the properties of $\boldsymbol{A}$
- **What is a homogeneous BVP?**
  - a BVP is homogeneous if:
    - it seeks a solution to a homogeneous ODE (for example, $\ddot{y} + p(t)\dot{y} + q(t)y = 0$)
    - the **boundary conditions** are both 0 (for example, $y(3) = 0, y(-1.6) = 0$)

2.2 Eigenvalues and Eigenfunctions

- **What is the boundary eigenvalue problem?**
  - the eigenvalue problem is defined by a linear map (a matrix), acting on a vector, such that it only stretches it
  - this idea can be generalised to other linear maps; in particular, we can consider linear maps acting on functions, which leads to the **boundary eigenvalue problem**
  - in particular, if L is a linear operator, the boundary eigenvalue problem concerns itself with finding functions y_i such that:
  $$Ly_i(x) = \lambda y_i(x)$$
  given some set of **boundary conditions**
- **What are eigenvalues and eigenfunctions?**
  - consider a boundary eigenvalue problem:
  $$Ly_i(x) = \lambda y_i(x)$$
  then:
    - λ is an **eigenvalue**
    - $y_i(x)$ is its corresponding **eigenfunction**

2.3 Worked Example

We use these examples to outline the fact that BVPs can have no solutions/infinitely many solutions, and also how to find eigenfunctions. The second example in particular is used to illustrate how, if we impose periodicity as a boundary condition, we get that sin and cos appear as eigenfunctions, hinting at the fact that they could be used to construct a basis for any periodic functions.

2.3.1 Example 1

Determine the eigenfunctions for the BVP defined by:

$$y'' = \lambda y, \quad y(0) = y(L) = 0$$

This is an easily solvable ODE:

$$y'' - \lambda y = 0$$

which has characteristic polynomial:

$$r^2 - \lambda = 0$$

So solutions have the form:

$$y = Ae^{\sqrt{\lambda}t} + Be^{-\sqrt{\lambda}t}$$

(so long as $\lambda \neq 0$)

We can now consider 3 distinct cases.

$\underline{\lambda = \mu^2 > 0}$

If this is the case, our solution becomes[2]:

$$y = Ae^{\mu t} + Be^{-\mu t}$$

If we apply the boundary conditions:

$$y(0) = 0 \implies A + B = 0$$
$$y(L) = 0 \implies Ae^{\mu L} + Be^{-\mu L} = 0$$

If we set $A = -B$, the second equation becomes:

$$B(e^{-\mu L} - e^{\mu L}) = 0$$

which only has solution $B = 0$ ($e^{-\mu L} - e^{\mu L} = 0 \implies -\mu L = \mu L$ which is only true if $\mu = 0$ or $L = 0$, and none of these are possible). But if $B = 0$, then $A = 0$, so we have only the trivial solution.

Thus, if $\lambda > 0$, the BVP has **no non-trivial solutions**

$\underline{\lambda = 0}$

In this case our ODE is $y'' = 0$ which has solution:

$$y = At + B$$

Applying the boundary conditions:

$$y(0) = 0 \implies B = 0$$
$$y(L) = 0 \implies AL = 0$$

so if $\lambda = 0$, there are **no non-trivial solutions**.

$\underline{\lambda = -\mu^2 < 0}$

In this case (taking the real solution), our solution becomes:

$$y = A\cos(\mu t) + B\sin(\mu t)$$

If we apply the boundary conditions:

$$y(0) = 0 \implies A = 0$$
$$y(L) = 0 \implies B\sin(\mu L) = 0$$

This is a system with non-trivial solutions: independent of B, if we have:

$$\mu = \frac{n\pi}{L}$$

we are guaranteed that $B\sin(\mu L) = 0$. We have a free parameter n (and the solution is independent of B), so if $\lambda < 0$, there are **infinitely many solutions**. Moreover, it follows that the eigenvalues and eigenfunctions are:

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$
$$y_n(t) = \sin\left(\frac{n\pi t}{L}\right)$$

[2]this could also be expressed in terms of $\cosh(x), \sinh(x)$

2.3.2 Example 2

Determine the eigenfunctions for the BVP defined by:

$$y'' = \lambda y, \quad y(t) = y(t + 2L)$$

What we are seeking is a $2L$ - Periodic Function. However, notice that periodicity means that after an interval of length $2L$ the function looks the "same" (i.e seems to have been shifted). This is equivalent to requiring:

$$y(-L) = y(L)$$

and

$$y'(-L) = y'(L)$$

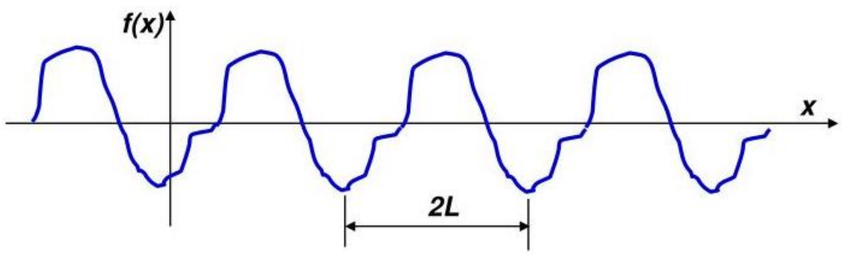


Figure 1: A periodic function will have the same value and same derivative at a point a period away.

<u>$\lambda = \mu^2 > 0$</u>

As before, the solution will be of the form:

$$y = Ae^{\mu x} + Be^{-\mu t}$$

and the derivative:

$$y' = A\mu e^{\mu t} - B\mu e^{-\mu t}$$

If we apply the (modified) boundary conditions:

$$y(-L) = y(L) \implies Ae^{-\mu L} + Be^{\mu L} = Ae^{\mu L} + Be^{-\mu L}$$

$$y'(-L) = y'(L) \implies A\mu e^{-\mu L} - B\mu e^{\mu L} = A\mu e^{\mu L} - B\mu e^{-\mu L}$$

If we multiply terms in the first equation by $e^{\mu L}$:

$$A + Be^{2\mu L} = Ae^{2\mu L} + B \implies e^{2\mu L}(B - A) = (B - A)$$

but since $e^{2\mu L} > 0$, this can only be satisfied if $B = A$

We can use this in the second equation, which, after removing the μ factor, becomes:

$$Ae^{-\mu L} - Ae^{\mu L} = Ae^{\mu L} - Ae^{-\mu L} \implies 2A(e^{-\mu L} - e^{\mu L}) = 0$$

which as discussed above is only satisfied if $A = 0$, which implies $B = 0$. Thus, if $\lambda > 0$, the system has **no non-trivial solutions**.

<u>$\lambda = 0$</u>

As discussed above, this has solution:

$$y = At + B$$

and derivative:

$$y' = A$$

If we apply the (modified) boundary conditions:

$$y(-L) = y(L) \implies -AL + B = AL + B$$

$$y'(-L) = y'(L) \implies A = A$$

The first equation implies that $2AL = 0$, so we must have $A = 0$. This then means that for any $B \in \mathbb{R}$, if $\lambda = 0$we can satisfy the BVP, with **infinitely many solutions** (namely a constant)

$\underline{\lambda = -\mu^2 < 0}$

In this case, the solution is of the form:

$$y = A\cos(\mu t) + B\sin(\mu t)$$

with derivative

$$y' = -A\mu\sin(\mu t) + B\mu\cos(\mu t)$$

If we apply the (modified) boundary conditions, and use the fact that cos is even and sin is odd:

$$y(-L) = y(L) \implies A\cos(\mu L) - B\sin(\mu L) = A\cos(\mu L) + B\sin(\mu L)$$

$$y'(-L) = y'(L) \implies A\mu\sin(\mu L) + B\mu\cos(\mu L) = -A\mu\sin(\mu L) + B\mu\cos(\mu L)$$

The first equation implies (assuming $B \neq 0$):

$$\sin(\mu L) = 0$$

which we know requires $\mu L = n\pi$.

The second equation implies (assuming $A \neq 0$):

$$\sin(\mu L) = 0$$

which we know requires $\mu L = n\pi$.

In particular, if $\lambda < 0$, there are **infinitely many solutions**. In particular, these must be given by any linear combination of $\sin\left(\frac{n\pi t}{L}\right)$ and $\cos\left(\frac{n\pi t}{L}\right)$ (since we can use any value of A and B).

Overall, it follows that:

$$\lambda = 0 \implies y_n(t) = 1$$

and

$$\lambda_n = \frac{n^2\pi^2}{L^2} \implies y_n(t) = \sin\left(\frac{n\pi t}{L}\right), \cos\left(\frac{n\pi t}{L}\right)$$

which we can interpret as $1, \sin\left(\frac{n\pi t}{L}\right), \cos\left(\frac{n\pi t}{L}\right)$ acting as basis eigenfunctions for periodic solutions of the ODE.

3 Even and Odd Functions

3.1 Defining Even Functions

Definition (Even Function)**.** *Let f be a function whose domain contains $-x$ if and only if it contains x. Then, f is an **even function** if:*

$$f(x) = f(-x)$$

Intuitively, an even function is a function which is symmetric on the y-axis, such as 5, x^2, $\cos(x)$ and $|x|$.

3.2 Defining Odd Functions

Definition (Odd Function)**.** *Let f be a function whose domain contains $-x$ if and only if it contains x. Then, f is an **odd function** if:*

$$f(x) = -f(-x)$$

Intuitively, an odd function is a function which is symmetric given a 180º rotation about the origin, such as x, $\sin(x)$ and $\frac{1}{x^3}$.

3.3 Properties of Even and Odd Functions

3.3.1 Sum and Product of Even/Odd Functions

- the **sum/difference** and **product/quotient** of 2 **even** functions are **even**
- the **sum/difference** of 2 **odd** functions is **odd**
- the **product/quotient** of 2 **odd** functions is **even**

3.3.2 Sum and Product of Even and Odd Functions

- the **sum/difference** of an **even** and **odd** function is neither even nor odd
- the **product/quotient** of an **even** and **odd** function is **odd**

3.3.3 Integral of Even Function Over Symmetric Interval

If f is even, then:

$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$$

3.3.4 Integral of Odd Function Over Symmetric Interval

If f is odd, then:

$$\int_{-L}^{L} f(x)dx = 0$$

3.4 The Odd and Even Extensions

(See https://people.math.carleton.ca/~mneufang/vorl/Fseries_1/node15.html)

Sometimes (*cough* when computing Fourier Series *cough*) we require functions to be odd/even over a given (symmetric) interval.

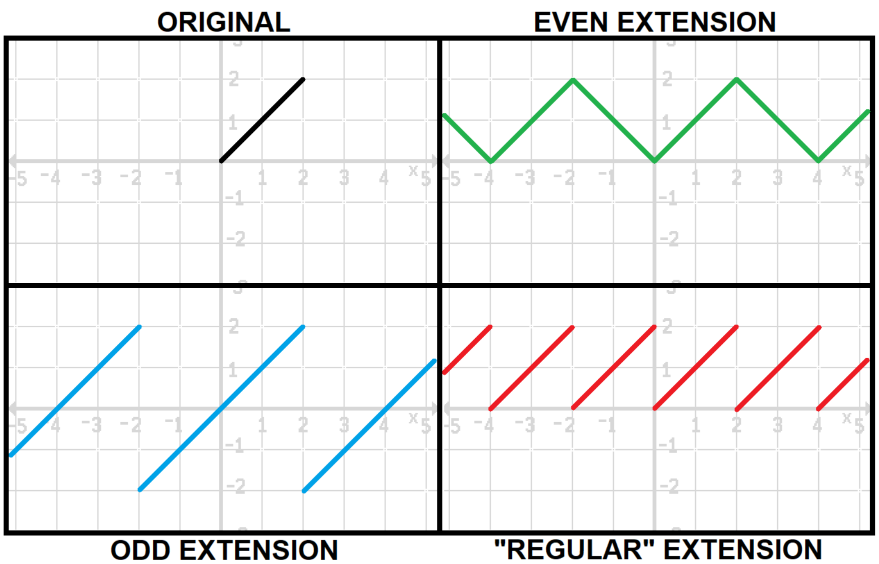
Consider a function f defined over some interval $(0, L)$. We can compute the **even** extension of f over $(-L, L)$ via:

$$f_{even}(x) = \begin{cases} f(x), & 0 < x < L \\ f(-x), & -L < x < 0 \end{cases}$$

Consider a function f defined over some interval $(0, L)$. We can compute the **odd** extension of f over $(-L, L)$ via:

$$f_{odd}(x) = \begin{cases} f(x), & 0 < x < L \\ -f(-x), & -L < x < 0 \end{cases}$$

Once we have the extension over $(-L, L)$, we can extend the function **periodically** by defining $f(x) = f(x + 2L)$.



4 Fourier Series

4.1 Periodic Functions

- **When is a function called periodic?**
  - a function is said to be **T-Periodic** if:

    $$f(x) = f(x + T)$$

    for some $T \in \mathbb{R}$
- **What is the fundamental period of a periodic function?**
  - the **fundamental period** is the smallest number T for which $f(x) = f(x + T)$
  - for $\sin(x)$ the fundamental period is 2π, whilst for constant functions, they have an arbitrary period, but no fundamental period

4.2 Inner Product of Function

4.2.1 Defining the Inner Product for Functions

Definition (Inner Product of Functions)**.** *Consider 2 non-zero functions $u(x), v(x)$, and let $a \leq x \leq b$. The **inner product** of u, v is:*

$$(u(x), v(x)) = \int_a^b u(x)v(x)dx$$

4.2.2 Orthogonality of Functions

Definition (Orthogonality of Functions)**.** *Consider 2 non-zero functions $u(x), v(x)$, and let $a \leq x \leq b$. u, v are **orthogonal** if:*

$$(u(x), v(x)) = 0$$

*A set of functions $\{f_i\}$ is said to be **mutually orthogonal** if each **distinct** pair of functions is orthogonal.*

4.2.3 Inner Product of Sine and Cosine

The identities used to integrate products of $sin(x)$ and $cos(x)$.

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$$
$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$
$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

We are particularly interested in the orthogonality of the set of functions:

$$\left\{1, \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right\}$$

over one period, $(-L, L)$. (technically this set contains infinitely many terms, as the 1 consists of all constant numbers, and we have infinitely many different sine and cosines, dependent on n)

Orthogonality of Constant Function

Clearly, if 1 is orthogonal with some other function, any constant will also be orthogonal with said function.

Constant + Constant

It is easy to see that:

$$(1, 1) = \int_{-L}^{L} 1dx = 2L > 0$$

so (technically) constant is mutually orthogonal with constant.

Constant + *cos*

If $n = 2k\pi, k \in \mathbb{N}$, we have $\cos\left(\frac{n\pi x}{L}\right) = 1$, so 1 and $\cos\left(\frac{n\pi x}{L}\right)$ are equal:

$$(1, 1) = \int_{-L}^{L} 1dx = 2L$$

For $n > 0$, the 2 functions are not equal, so:

$$\begin{aligned}\left(1, \cos\left(\frac{n\pi x}{L}\right)\right) &= \int_{-L}^{L} 1 \times \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{L}{n\pi}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} \\ &= \frac{L}{n\pi}\left[\sin(n\pi) - \sin(-n\pi)\right] \\ &= 0\end{aligned}$$

Thus, 1 is mutually orthogonal with $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$.

<u>Constant + *sin*</u>

If $n = \frac{\pi}{2}(4k+1), k \in \mathbb{N}$, we have $\sin\left(\frac{n\pi x}{L}\right) = 1$, so 1 and $\sin\left(\frac{n\pi x}{L}\right)$ are equal:

$$(1, 1) = \int_{-L}^{L} 1 dx = 2L$$

For $n > 0$, the 2 functions are not equal, so:

$$\begin{aligned}\left(1, \sin\left(\frac{n\pi x}{L}\right)\right) &= \int_{-L}^{L} 1 \times \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) dx \\ &= -\frac{L}{n\pi}\left[\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} \\ &= \frac{L}{n\pi}\left[\cos(n\pi) - \cos(-n\pi)\right] \\ &= \frac{L}{n\pi}\left[\cos(n\pi) - \cos(n\pi)\right] \\ &= 0\end{aligned}$$

Thus, 1 is mutually orthogonal with $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$.

Overall, 1 is mutually orthogonal with **everything** in the set.

Orthogonality of $\cos\left(\frac{n\pi x}{L}\right)$

We need to consider 3 cases:

1. $\left(\cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right), \quad m \neq n$
2. $\left(\cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right), \quad m = n$
3. $\left(\cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right)\right)$

cos + cos ($m \neq n$)

$$
\begin{aligned}
&\left(\cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right) \\
&= \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \times \cos\left(\frac{m\pi x}{L}\right) dx \\
&= \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(n-m)\pi x}{L}\right) + \cos\left(\frac{(n+m)\pi x}{L}\right) dx \\
&= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right) + \frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{L}\right)\right]_{-L}^{L} \\
&= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\left((n-m)\pi\right) + \frac{L}{(n+m)\pi}\sin\left((n+m)\pi\right) - \frac{L}{(n-m)\pi}\sin\left(-(n-m)\pi\right) - \frac{L}{(n+m)\pi}\sin\left(-(n+m)\pi\right)\right] \\
&= 0
\end{aligned}
$$

where we have used the fact that $n, m \in \mathbb{N}$ so $n-m, n+m \in \mathbb{N}$. Alternatively, we could've used the fact that cos is even, and integrated over $(0, L)$.

cos + cos ($m = n$)

$$
\begin{aligned}
&\left(\cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right) \\
&= \int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right) dx \\
&= \frac{1}{2}\int_{-L}^{L} 1 + \cos\left(\frac{2n\pi x}{L}\right) dx \\
&= \frac{1}{2}\left[x + \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]_{-L}^{L} \\
&= \frac{1}{2}\left[L + \frac{L}{2n\pi}\sin\left(2n\pi\right) - (-L) - \frac{L}{2n\pi}\sin\left(-2n\pi\right)\right] \\
&= L
\end{aligned}
$$

where we have used the identity:

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

Thus, from this and the previous result, it follows that the set:

$$\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$$

is mutually orthogonal.

cos + sin

$\left(\sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right)$
$= \int_{-L}^{L} sin\left(\frac{n\pi x}{L}\right) \times \cos\left(\frac{m\pi x}{L}\right) dx$
$= \frac{1}{2}\int_{-L}^{L} \sin\left(\frac{(n-m)\pi x}{L}\right) + \sin\left(\frac{(n+m)\pi x}{L}\right) dx$
$= -\frac{1}{2}\left[\frac{L}{(n-m)\pi}\cos\left(\frac{(n-m)\pi x}{L}\right) + \frac{L}{(n+m)\pi}\cos\left(\frac{(n+m)\pi x}{L}\right)\right]_{-L}^{L}$
$= -\frac{1}{2}\left[\frac{L}{(n-m)\pi}\cos((n-m)\pi) + \frac{L}{(n+m)\pi}\cos((n+m)\pi) - \frac{L}{(n-m)\pi}\cos(-(n-m)\pi) - \frac{L}{(n+m)\pi}\cos(-(n+m)\pi)\right]$
$= -\frac{1}{2}\left[\frac{L}{(n-m)\pi}\cos((n-m)\pi) + \frac{L}{(n+m)\pi}\cos((n+m)\pi) - \frac{L}{(n-m)\pi}\cos((n-m)\pi) - \frac{L}{(n+m)\pi}\cos((n+m)\pi)\right]$
$= 0$

Thus, from this it follows that the sets:

$$\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$$

and

$$\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$$

is mutually orthogonal. For sin we start with $n = 1$, as we are not interested in having 0 (its a trivial solution).

Orthogonality of $\sin\left(\frac{n\pi x}{L}\right)$

We need to consider 2 cases:

1. $\left(\sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right), \quad m \neq n$
2. $\left(\sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right), \quad m = n$

sin+ sin $(m \neq n)$

$\left(\sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right)\right)$
$= \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \times \sin\left(\frac{m\pi x}{L}\right) dx$
$= \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right) dx$
$= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right) - \frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{L}\right)\right]_{-L}^{L}$
$= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin((n-m)\pi) - \frac{L}{(n+m)\pi}\sin((n+m)\pi) - \frac{L}{(n-m)\pi}\sin(-(n-m)\pi) + \frac{L}{(n+m)\pi}\sin(-(n+m)\pi)\right]$
$= 0$

where we have used the fact that $n, m \in \mathbb{N}$ so $n - m, n + m \in \mathbb{N}$.

<u>$\sin + \sin$ $(m = n)$</u>

$$\begin{aligned}
&\left(\sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right) \\
&= \int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx \\
&= \frac{1}{2}\int_{-L}^{L} 1 - \cos\left(\frac{2n\pi x}{L}\right) dx \\
&= \frac{1}{2}\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]_{-L}^{L} \\
&= \frac{1}{2}\left[L - \frac{L}{2n\pi}\sin(2n\pi) - (-L) + \frac{L}{2n\pi}\sin(-2n\pi)\right] \\
&= L
\end{aligned}$$

where we have used the identity:

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

Thus, from this and the previous result, it follows that the set:

$$\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$$

is mutually orthogonal.

4.2.4 Sanity Check: $m \neq n$

We have been pretty lenient, as to the fact that the periodic functions described above are equal if and only if $m = n$. This might seem bizarre, since the functions are periodic, so there might exist some $m, n \in \mathbb{N}, m \neq n$ such that, for instance:

$$\cos\left(\frac{n\pi x}{L}\right) = \cos\left(\frac{m\pi x}{L}\right)$$

A quick computation quickly dismisses this; in particular, we consider periodicity, as that is the only instance in which the arguments of cos can be different and yet produce the same output:

$$\begin{aligned}
\cos\left(\frac{n\pi x}{L} + 2L\right) &= \cos\left(\frac{m\pi x}{L}\right) \\
\implies \frac{n\pi x}{L} + 2L &= \frac{m\pi x}{L} \\
\implies n\pi x + 2L^2 &= m\pi x \\
\implies n + \frac{2L^2}{x\pi} &= m
\end{aligned}$$

Which means that m would have to be irrational.

4.2.5 Orthogonality Summary

1. $$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

2. $$\int_{0}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L & \text{if } n = m = 0 \\ \frac{L}{2} & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

3. $$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

4. $$\int_{0}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} \frac{L}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

5. $$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = 0$$

Figure 2: We can see that cos is mutually orthogonal with itself and with sin. sin is mutually orthogonal with itself and cos.

4.3 Defining Fourier Series

Definition 1. *Given a periodic function $f(x)$ with period $2L$, it can be expressed as a* ***Fourier Series****:*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

given that certain conditions are satisfied.

a_n, b_n are ***constants****, known as* ***Fourier Coefficients***

4.4 The Euler-Fourier Coefficients

Assuming that a Fourier Series converges to $f(x)$ on $-L \leq x \leq L$, how do we do to find the Fourier Coefficients. Naturally, we exploit the orthogonality!

This is inspired by what we did in (1.2), where a orthogonal basis helped find coefficients in a simple, systematic way.

4.4.1 Cosine Fourier Coefficients

If we have:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right)$$

consider:

$$\left(f(x), \cos\left(\frac{n\pi x}{L}\right)\right)$$

We get:

$$\begin{aligned}
&\left(f(x), \cos\left(\frac{n\pi x}{L}\right)\right) \\
&= \left(\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right), \cos\left(\frac{n\pi x}{L}\right)\right) \\
&= \left(\frac{a_0}{2}, \cos\left(\frac{n\pi x}{L}\right)\right) + \left(\sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right), \cos\left(\frac{n\pi x}{L}\right)\right) \\
&= \left(\frac{a_0}{2}, \cos\left(\frac{n\pi x}{L}\right)\right) + \sum_{m=1}^{\infty} \left(\left(a_m \cos\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right) + \left(b_m \sin\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right)\right) \\
&= \left(\frac{a_0}{2}, \cos\left(\frac{n\pi x}{L}\right)\right) + \left(a_n \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right) \\
&= \frac{a_0}{2}\left(1, \cos\left(\frac{n\pi x}{L}\right)\right) + a_n\left(\cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right) \\
&= a_n L
\end{aligned}$$

Which thus means:

$$a_n = \frac{1}{L}\left(f(x), \cos\left(\frac{n\pi x}{L}\right)\right) = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

for $n \geq 1$.

For $n = 0$, we have:

$$\begin{aligned}
&(f(x), 1) \\
&= \left(\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right), 1\right) \\
&= \left(\frac{a_0}{2}, 1\right) + \left(\sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right), 1\right) \\
&= \left(\frac{a_0}{2}, 1\right) + \sum_{m=1}^{\infty} \left(\left(a_m \cos\left(\frac{m\pi x}{L}\right), 1\right) + \left(b_m \sin\left(\frac{m\pi x}{L}\right), 1\right)\right) \\
&= \frac{a_0}{2}(1, 1) \\
&= a_0 L
\end{aligned}$$

so:

$$a_0 = \frac{1}{L}\left(f(x), \cos\left(\frac{0\pi x}{L}\right)\right) = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{0\pi x}{L}\right) dx$$

Thus, by setting the first element to $\frac{a_0}{2}$, we can in fact compute $a_n, n \geq 0$ via:

$$a_n = \frac{1}{L}\left(f(x), \cos\left(\frac{n\pi x}{L}\right)\right) = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

4.4.2 Sine Fourier Coefficients

If we have:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right)$$

consider:

$$\left(f(x), \sin\left(\frac{n\pi x}{L}\right)\right)$$

We get:

$$\begin{aligned}
&\left(f(x), \sin\left(\frac{n\pi x}{L}\right)\right) \\
&= \left(\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right), \sin\left(\frac{n\pi x}{L}\right)\right) \\
&= \left(\frac{a_0}{2}, \sin\left(\frac{n\pi x}{L}\right)\right) + \left(\sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right), \sin\left(\frac{n\pi x}{L}\right)\right) \\
&= \left(\frac{a_0}{2}, \sin\left(\frac{n\pi x}{L}\right)\right) + \sum_{m=1}^{\infty} \left(\left(a_m \cos\left(\frac{m\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right) + \left(b_m \sin\left(\frac{m\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right)\right) \\
&= \left(\frac{a_0}{2}, \sin\left(\frac{n\pi x}{L}\right)\right) + \left(b_n \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right) \\
&= \frac{a_0}{2}\left(1, \sin\left(\frac{n\pi x}{L}\right)\right) + b_n \left(\sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right) \\
&= b_n L
\end{aligned}$$

Which thus means:

$$b_n = \frac{1}{L}\left(f(x), \sin\left(\frac{n\pi x}{L}\right)\right) = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

for $n \geq 1$.

4.4.3 Extra Pointers

- if the series converges, then a_n, b_n **must** be given by the formulae above
- the value of a_n, b_n depends solely on the value of f on $(-L, L)$; it is possible to show that we will get the same result by considering **any other** interval of length $2L$

4.4.4 Interpreting Fourier Series

- we have already seen that a Fourier Series is the result of using an orthogonal basis of functions to construct periodic functions
- one particular use of this is that we can decompose a signal into a sum of cosines and sines with different frequency. If we have many small a_n, b_n, we are able to compress a complex signal as a sum of a few cosines and sines. This is used for image and sound compression.

4.5 Fourier Cosine Series

It is possible to construct a Fourier Series using **only** cosine terms. This is the case if f is an **even** function.

We exploit the facts from (3.3), in particular the fact that a product of even and odd functions is odd, and the integral of an odd function over a symmetric interval is 0, whilst the integral of an even function over a symmetric interval is twice the value of the integral over half of the interval.

If we compute the Fourier Coefficients:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = 0$$

Thus, the Fourier Series of an **even** function is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

This is known as a **Fourier Cosine Series**.

We can use even extensions if a function is solely defined on $(0, L)$ in order to make use of the Cosine Series.

4.6 Fourier Sine Series

It is possible to construct a Fourier Series using **only** sine terms. This is the case if f is an **odd** function.

We exploit the facts from (3.3), in particular the fact that a product of even and odd functions is odd, the product of 2 odd functions is even, and the integral of an odd function over a symmetric interval is 0, whilst the integral of an even function over a symmetric interval is twice the value of the integral over half of the interval.

If we compute the Fourier Coefficients:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Thus, the Fourier Series of an **odd** function is:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

This is known as a **Fourier Sine Series**.

We can use odd extensions if a function is solely defined on $(0, L)$ in order to make use of the Cosine Series.

4.7 The Fourier Convergence Theorem

- **Does a Fourier Series always converge to its function?**
  - it is possible to find functions f whose Fourier Series do **not** converge to f, and might even diverge
  - even more common is to find functions which don't converge to the function at certain points
- **What is a piecewise continuous function?**
  - a function which is continuous, except possibly at finitely many points

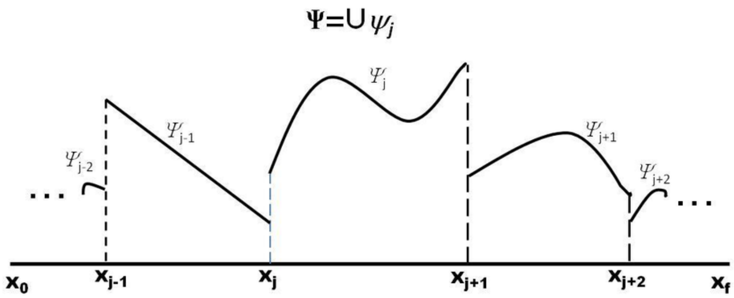
- in essence, we can split the function into finitely many subintervals; the function is continuous **on** the subintervals, but might not be continuous at the endpoints
- we still approach that as we approach the endpoints (left and right limits), we get a finite number
- we use:

$$f(c^+) = \lim_{x \to c^+} f(x)$$

to denote the right limit, and

$$f(c^-) = \lim_{x \to c^-} f(x)$$

to denote the left limit



Theorem (The Fourier Convergence Theorem)**.** *Suppose that f and f' are **piecewise continuous** on the interval $-L \leq x \leq L$. Further, assume that f is 2L periodic, such that it is defined outside the aforementioned interval. Then we can write f as a Fourier Series:*

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right)$$

whose coefficients are given by:

$$a_n = \frac{1}{L} \left(f(x), \cos\left(\frac{n\pi x}{L}\right) \right) = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$
$$b_n = \frac{1}{L} \left(f(x), \sin\left(\frac{n\pi x}{L}\right) \right) = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

In particular the Fourier Series:

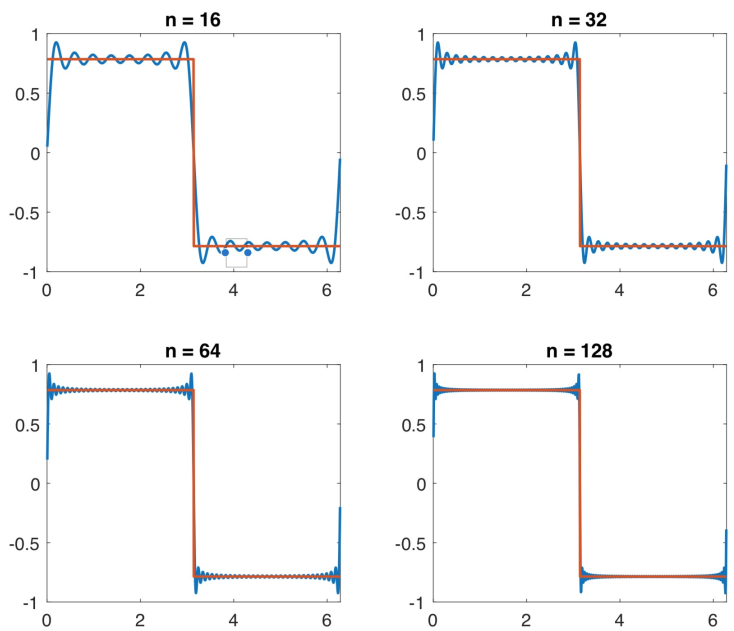
- *converges to f at all points in which f is continuous*
- *converges to:*

$$\frac{1}{2}(f(x^+) + f(x^-))$$

  at all points where f is discontinuous

- **Does the Fourier Convergence Theorem give necessary conditions for convergence?**
  - the Theorem only provides sufficient conditions to prove convergence
  - other more restrictive conditions exist

- **Which functions don't satisfy the conditions of the theorem?**
  - in general, functions which have infinitely many discontinuities (i.e $\tan(x), x \in \mathbb{R}$), or functions which have infinite discontinuities on the interval $[-L, L]$ (i.e $\frac{1}{x^2}$ as $x \to 0$ or $ln|x - L|$ as $x \to L$)
- **Can a Fourier Series converge to a non-differentiable or non-continuous function?**
  - consider a function which has jump discontinuities. If we define the value of the function at said discontinuities as
$$\frac{1}{2}(f(x^+) + f(x^-))$$
    then the Fourier Series will converge to $f(x)$; otherwise it won't
- **What is Gibbs Phenonmenon?**
  - the phenomenon arising from using Fourier Series with functions which has jump discontinuities
  - it can be observed that, close to these points, the Fourier Series will under/overshoot the point, since they can't accommodate the sudden change in the function
  - this over/undershooting is independent of n, so it can't be eliminated



- **If a Fourier Series is convergent, is the derivative of the Fourier Series convergent?**
  - it is possible to find a Fourier Series which converges to f, but if we differentiate the series, to attempt to get f', it might not be a convergent series

4.8 Parseval's Theorem

Derivation by blackpenredpen

Theorem (Parseval's Theorem). *Let $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$, and define:*

$$\|f\|^2 = (f, f) = \int_{-L}^{L} |f(x)|^2 dx$$

Then:

$$\|f\|^2 = 2L \sum_{n=-\infty}^{\infty} |c_n|^2 = L\left(\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2\right)$$

This can be interpreted as an infinite dimensional Pythagorean Theorem, and is particularly useful when finding infinite series.

Proof.

By definition and linearity,

$$\begin{aligned}(f, f) &= \sum_{n,m=-\infty}^{\infty} \int_{-L}^{L} c_n\, c_m^\star\, e^{i(n-m)\pi x/L}\, dx \\ &= \sum_{n,m=-\infty}^{\infty} c_n\, c_m^\star\, 2L\, \delta_{mn} \text{ by orthogonality} \\ &= 2L \sum_{n=-\infty}^{\infty} |c_n|^2\,.\end{aligned}$$

To prove the last bit, we just remember

$$c_n = \frac{a_n - ib_n}{2} \quad (n > 0)\,, \quad c_{-n} = \frac{a + ib_n}{2} \quad (n < 0)$$

$$c_0 = \frac{a_0}{2}$$

Figure 3: Here $*$ is used to represent the ocmplex conjugate, and δ_{mn} is the Kronecker Delta, 1 if $m = n$, and 0 otherwise.

Proof.

Indeed,

$$\begin{aligned}2L \sum_{n=-\infty}^{\infty} |c_n|^2 &= 2L\left[|c_0|^2 + \sum_{n=1}^{\infty}\left(|c_n|^2 + |c_{-n}|^2\right)\right] \\ &= 2L\left[\frac{|a_0|^2}{4} + \sum_{n=1}^{\infty}\left(\frac{|a_n|^2 + |b_n|^2}{4} + \frac{|a_n|^2 + |b_n|^2}{4}\right)\right] \\ &= L\left[\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right)\right]\,.\end{aligned}$$

□

5 The Complex Fourier Series

Euler's Formula tells us that:

$$\cos(\omega x) + i \sin(\omega x) = e^{i\omega x}$$

Using this, we can redefine Fourier Series via the complex exponential. In particular, notice that:

$$\cos\left(\frac{n\pi x}{L}\right) = \frac{\exp\left(i\frac{n\pi x}{L}\right) + \exp\left(-i\frac{n\pi x}{L}\right)}{2}$$

$$\sin\left(\frac{n\pi x}{L}\right) = \frac{\exp\left(i\frac{n\pi x}{L}\right) - \exp\left(-i\frac{n\pi x}{L}\right)}{2i}$$

So:

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) \\ &= \frac{a_0}{2} + \sum_{m=1}^{\infty} a_n \frac{\exp\left(i\frac{n\pi x}{L}\right) + \exp\left(-i\frac{n\pi x}{L}\right)}{2} + b_n \frac{\exp\left(i\frac{n\pi x}{L}\right) - \exp\left(-i\frac{n\pi x}{L}\right)}{2i} \end{aligned}$$

If we change the limits of the sum, as to include negative natural numbers, we can compress the whole expression:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

where:

$$c_n = \begin{cases} \frac{a_n - ib_n}{2}, & n > 0 \\ \frac{a_n + ib_n}{2}, & n < 0 \\ \frac{a_0}{2}, & n = 0 \end{cases}$$

In general:

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{i\frac{n\pi x}{L}}$$

We could've derived the above, by using $e^{i\frac{n\pi x}{L}}$ as a basis for periodic functions, and extending the definition of the inner product of functions for complex valued functions in the following way:

$$(u(x), v(x)) = \int_{-L}^{L} \bar{u}(x) \times v(x) dx$$

The complex Fourier Series is clearly more convenient if f is complex valued.

More in detail view of complex Fourier; in particular, further simplifies c_n to be as a single integral

6 Worked Example (+ Bonus Identity)

1. **Consider the odd function $f(x) = x$.**
   (a) **Derive the Fourier Series for $g(x)$, the periodic version of f, with period 4, such that $f(x)$ is equal to $g(x)$ on $[-2, 2]$**

   We can visualise g as:

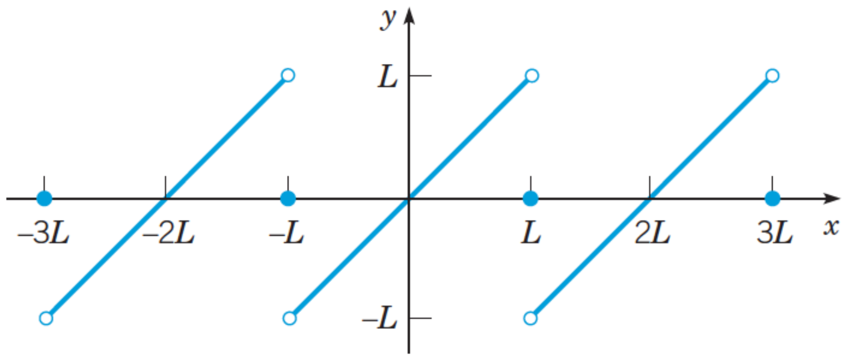


Figure 4: $g(x)$ is known as the **Sawtooth Function**

Since g is odd, we know that the coefficients a_n will vanish, so we only need to consider:

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} x \times \sin\left(\frac{n\pi x}{2}\right) dx \\ &= \int_{0}^{2} x \times \sin\left(\frac{n\pi x}{2}\right) dx \end{aligned}$$

We use integration by parts:

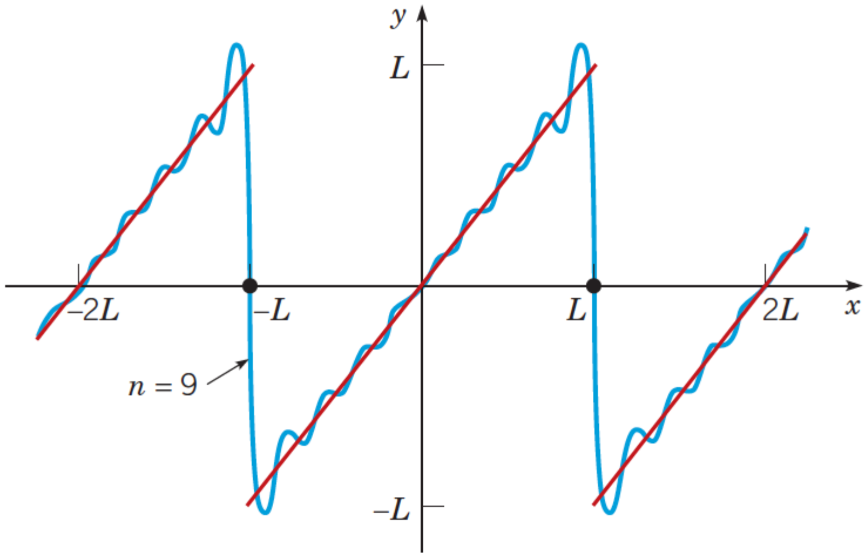
$$\begin{aligned} u = x &\implies du = 1 \\ dv = \sin\left(\frac{n\pi x}{2}\right) &\implies v = -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) \end{aligned}$$

So:

$$\begin{aligned} b_n &= \left[-x\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx \\ &= -\frac{4}{n\pi}\cos(n\pi) + \frac{4}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{2}\right)\right]_0^2 \\ &= -\frac{4}{n\pi}\cos(n\pi) + \frac{4}{n^2\pi^2}\left[\sin\left(n\pi\right)\right] \\ &= -\frac{4}{n\pi}\cos(n\pi) \\ &= \frac{4}{n\pi}(-1)^{n+1} \end{aligned}$$

Thus, it follows that:

$$g(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi x}{2}\right)$$

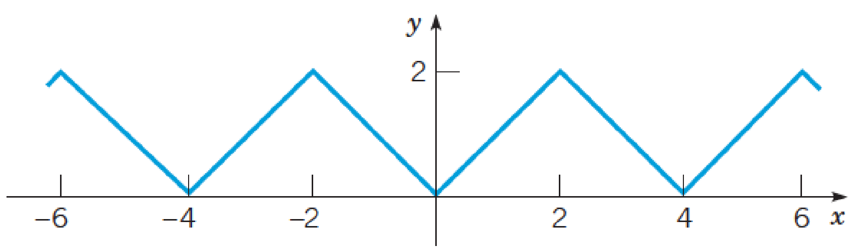


(b) **Derive the Fourier Series for $h(x)$, the periodic, even version of f, with period 4, such that $f(x)$ is equal to $g(x)$ on $[0, 2]$**

For this, we consider the even expansion of $f(x) = x$, by defining:

$$h(x) = \begin{cases} x, & 0 \leq x < 2 \\ -x, & -2 \leq x < 0 \end{cases}$$

We can visualise this:



Since h is an even function, $b_n = 0$, and we juse need to compute:

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} h(x) \times \cos\left(\frac{n\pi x}{2}\right) dx \\ &= -\frac{1}{2}\int_{-2}^{0} x \times \cos\left(\frac{n\pi x}{2}\right) dx + \frac{1}{2}\int_{0}^{2} x \times \cos\left(\frac{n\pi x}{2}\right) dx \\ &= \frac{1}{2}\left(-\int_{-2}^{0} x \times \cos\left(\frac{n\pi x}{2}\right) dx + \int_{0}^{2} x \times \cos\left(\frac{n\pi x}{2}\right) dx\right) \end{aligned}$$

We can compute $\int x \times \cos\left(\frac{n\pi x}{2}\right) dx$ again using integration by parts:

$$u = x \implies du = 1$$

$$dv = \cos\left(\frac{n\pi x}{2}\right) \implies v = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$$

So:

$$\begin{aligned}\int x \times \cos\left(\frac{n\pi x}{2}\right) dx &= x\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) - \frac{2}{n\pi}\int \sin\left(\frac{n\pi x}{2}\right) dx \\ &= x\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\end{aligned}$$

Thus:

$$\begin{aligned}a_n &= \frac{1}{2}\left(-\left[x\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0} + \left[x\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right]_{0}^{2}\right) \\ &= \frac{1}{2}\left(-\left[\frac{4}{n^2\pi^2} - (-2)\frac{2}{n\pi}\cos(-n\pi) - \frac{4}{n^2\pi^2}\cos(-n\pi)\right] + \left[\frac{4}{n\pi}\cos(n\pi) + \frac{4}{n^2\pi^2}\cos(n\pi) - \frac{4}{n^2\pi^2}\right]\right) \\ &= \frac{4\cos(n\pi) - 4}{n^2\pi^2}\end{aligned}$$

Thus, it follows that:

$$a_n = \begin{cases} 0, & n \text{ even} \\ -\frac{8}{(n\pi)^2}, & n \text{ odd}\end{cases}$$

For a_0, we can see that the integral of the function over $(-2, 2)$ has area equal to a square of side length 2, so:

$$\int_2^2 x dx = 4$$

From which it follows that:

$$a_0 = \frac{1}{2}\int_2^2 x dx = 2$$

Thus, the Cosine Fourier Series is:

$$h(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi x}{2}\right)$$

Notice, by construction, both $g(x)$ and $h(x)$ give us a series representation of $f(x) = x, x \in [0, 2]$.

(c) **Use Parseval's Theorem, alongside the above parts, to deduce that:**

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

We consider $g(x)$. We can easily compute $\|g\|^2$:

$$\|g\|^2 = \int_2^2 |x|^2 = \left[\frac{x^3}{3}\right]_{-2}^{2} = \frac{16}{3}$$

From Parseval's Theorem:

$$\begin{aligned}
\|g\|^2 &= 2\left(\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2\right) \\
&= 2\left(\sum_{n=1}^{\infty} |b_n|^2\right) \\
&= 2\left(\sum_{n=1}^{\infty} \left|\frac{4}{n\pi}(-1)^{n+1}\right|^2\right) \\
&= 2\left(\sum_{n=1}^{\infty} \frac{16}{n^2\pi^2}\right) \\
&= \frac{32}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}
\end{aligned}$$

So it follows that:

$$\frac{32}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{16}{3} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$