Honours Differential Equations - Week 5 - Implicit Trajectories and Global Analysis of Non-Linear Systems: Lyapunov Theory and the Poincare-Bendixson Theorem

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Contents

Last week we focused on the behaviour surrounding critical points in linear systems. We also approximated non-linear systems linearly, but could only do so locally. We now try and develop techniques that allows us to analyse the global behaviour of nonlinear systems.

1 Implicit Trajectories

1.1 Defining Implicit Trajectories

We consider a system of the form:

$$x'(t) = F(x, y)$$
$$y'(t) = G(x, y)$$

- **What is an implicit trajectory?**
  - a trajectory in the xy plane which can be described by:

    $$H(x, y) = c$$

    where for each c we get a different trajectory
- **How can we find an implicit trajectory?**
  - if we have an autonomous system, and plot its trajectories in the xy plane, we can sometimes think of plotting trajectories as plotting the function $y(x)$
  - to find $y(x)$, we can exploit the fact that the system is autonomous, so:

    $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'} = \frac{F(x, y)}{G(x, y)}$$

  - in general, this differential equation won't be solvable, but if it is, we can write the solution implicitly as above:

    $$H(x, y) = c$$

1.2 Worked Example

Consider the nonlinear system:

$$x' = 4 - 2y$$
$$y' = 12 - 3x^2$$

We can do a *local* (via linearisation) and a *global* (via implicit trajectories) analysis.

1.2.1 Local Analysis

The critical points are $(2, 2)$ and $(-2, 2)$. We then obtain the Jacobian Matrix:

$$\boldsymbol{J} = \begin{pmatrix} 0 & -2 \\ -6x & 0 \end{pmatrix}$$

The eigenvalues of the matrix satisfy:

$$r^2 - 12x = 0 \implies r = \pm\sqrt{12x}$$

Thus, at the critical point $(2, 2)$:

$$r = \pm\sqrt{24}$$

Thus, we have 2 real valued eigenvalues of opposite signs. This means that $(2, 2)$ is a **saddle point**.

For the critical point $(-2, 2)$:

$$r = \pm i\sqrt{24}$$

Thus, we get purely imaginary eigenvalues, so $(-2, 2)$ is a **centre** *or* a **spiral point**. This is because we have a nonlinear system

1.2.2 Global Analysis

To determine the global behaviour, we notice that:

$$\frac{dy}{dx} = \frac{12 - 3x^2}{4 - 2y}$$

which is a separable differential equation:

$$\int 4 - 2y dy = \int 12 - 3x^2 dx$$

which leads to the implicit solution:

$$x^3 - y^2 + 4y - 12x = C$$

If we plot the solution for different C:

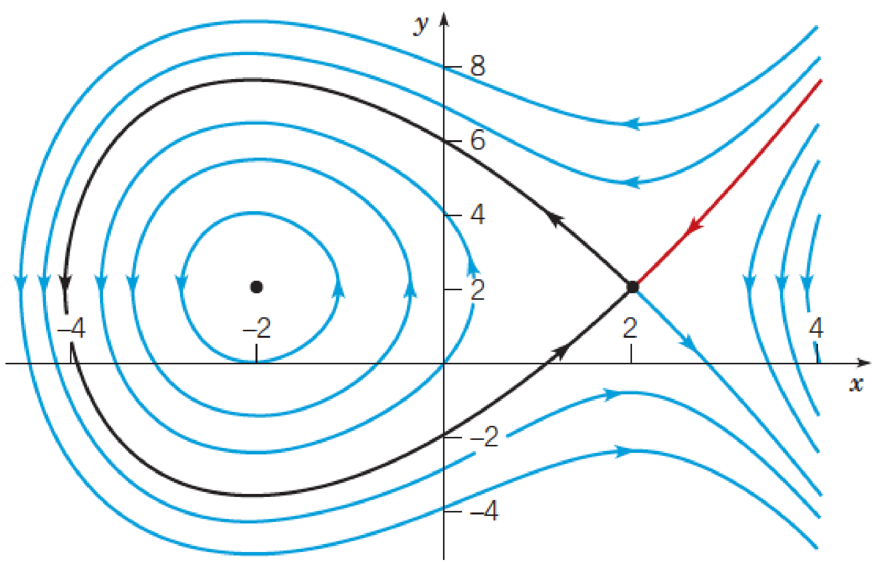


Figure 1: The black line corresponds to the separatrix. Beyond this curve, solutions get "attracted" by the saddle at $(2, 2)$, and shoot off to infinity (except those which are along the attracting eigenvector). Within the separatrix, we get periodic solutions about $(-2, 2)$.

2 The Lotka Volterra Prey-Predator Model

2.1 Defining the Lotka-Volterra Model

We consider:

- $x(t)$, the size of the prey population

- $y(t)$, the size of the predator population

Then for $a, c, \alpha, \gamma > 0$:

$$\frac{dx}{dt} = ax - \alpha xy$$
$$\frac{dy}{dt} = -cy + \gamma xy$$

where xy is an interaction term. For example, for the prey, as the number of prey or predator increases, we expect the prey population to decrease.

2.2 Local Analysis

We find that the critical points are $(0, 0)$ and $(\frac{c}{\gamma}, \frac{a}{\alpha})$. $(0, 0)$ is obvious: if there are no predators or prey in the system, we don't expect any to spontaneously appear.

The Jacobian Matrix is:

$$\boldsymbol{J} = \begin{pmatrix} a - \alpha y & -\alpha x \\ \gamma y & -c + \gamma x \end{pmatrix}$$

For the critical point $(0, 0)$:

$$\boldsymbol{J}(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$$

which has eigenvalues:

$$r = a, -c$$

Thus, $(0, 0)$ is a **saddle point**.

If we look at the eigenvectors, these are clearly:

$$\underline{\xi}_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \underline{\xi}_{-c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In other words, the y and x axes act as "asymptotes" of the solutions, with trajectories moving parallel to $\underline{\xi}_{-c}$ as $t \to -\infty$, and moving parallel to $\underline{\xi}_a$ as $t \to \infty$. The only solutions which actually reach the origin are those beginning in the y axis (corresponding to a non-zero predator population, and a zero prey population). Similarly, any solution starting in the x axis will blow up (corresponding to uncontrolled growth of prey in the absence of predators).

For the more interesting critical point $(\frac{c}{\gamma}, \frac{a}{\alpha})$:

$$\boldsymbol{J}\left(\frac{c}{\gamma}, \frac{a}{\alpha}\right) = \begin{pmatrix} 0 & -\frac{\alpha c}{\gamma} \\ \frac{\gamma a}{\alpha} & 0 \end{pmatrix}$$

which has eigenvalues satisfying $r^2 = ac = 0$. In other words:

$$r = \pm i\sqrt{ac}$$

so $(\frac{c}{\gamma}, \frac{a}{\alpha})$ is a **centre**

2.3 Global Analysis

We now look at a global analysis, in particular to see the trajectories in the neighbourhood of the second critical point. We can easily get an implicit equation for the trajectories:

$$\frac{dy}{dx} = \frac{y(-c+\gamma x)}{x(a-\alpha y}$$

Which is a separable ODE:

$$\int \frac{a}{y} - \alpha dy = \int -\frac{c}{x} + \gamma dx$$

with implicit solution:

$$a \ln y - \alpha y + c \ln x - \gamma x = C$$

However, this is not too useful without plotting tools. To gain further intuition, we can return to the linear approximation:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\alpha c}{\gamma} \\ \frac{\gamma a}{\alpha} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

which results in:

$$u' = -\frac{\alpha c}{\gamma} v$$

$$v' = \frac{\gamma a}{\alpha} u$$

If we differentiate the first ODE:

$$u'' = -\frac{\alpha c}{\gamma} v' = (-ac)u$$

If we let $\omega = \sqrt{ac}$, we have a standard, second order, linear, homogeneous ODE:

$$u'' + \omega^2 u = 0$$

with corresponding characteristic polynomial:

$$r^2 + \omega^2 = 0$$

which has complex roots $r = i\omega$. Thus, the solutions are:

$$u = A\cos(\omega t) + B\cos(\omega t) = K\cos(\omega t - \phi)$$

where we have used the amplitude phase form of the sinusoidal.

To get v, we can just differentiate the expression above:

$$u' = -\omega K \cos(\omega t - \phi) \implies v = \frac{\gamma \omega K}{\alpha c} \sin(\omega t - \phi)$$

We can then use the fact that x close to a critical point is given by $x^* + u$:

$$x = \frac{c}{\gamma} + K\cos(\omega t - \phi)$$

$$x = \frac{a}{\alpha} + \frac{\gamma \omega K}{\alpha c} \sin(\omega t - \phi)$$

which tells us that close to the critical point we should indeed expect periodic behaviour, with period $\frac{2\pi}{\omega}$.

Overall, we get:

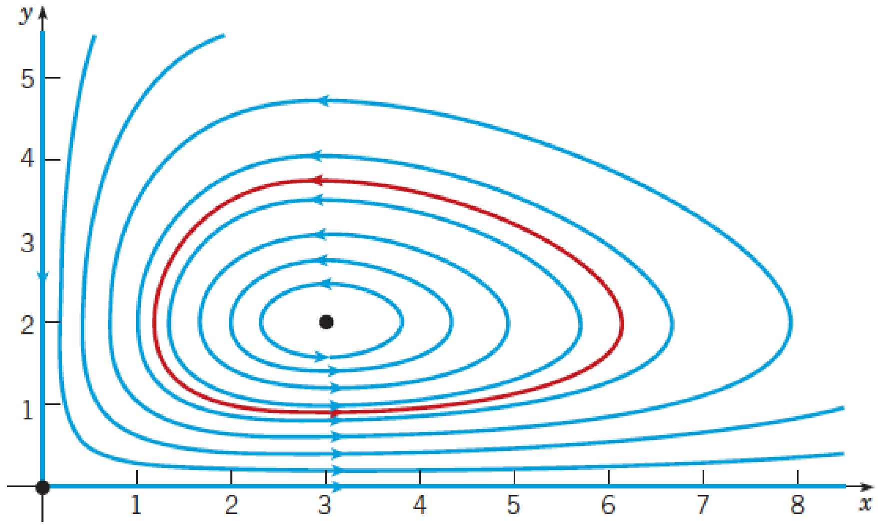


Figure 2: The phase portrait for the Lotka-Volterra Equations. Notice the periodicity of trajectories close to the interesting critical point.

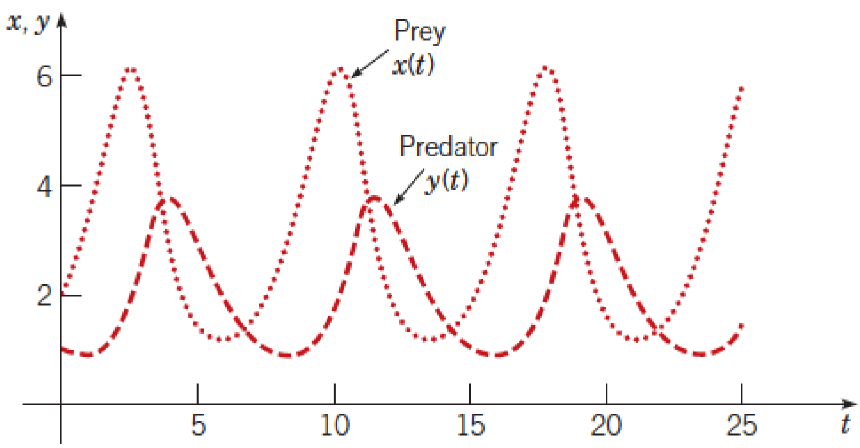


Figure 3: One possible solution for the Lotka-Volterra Equations. The predator and prey populations are out of phase by one quarter of a cycle, which makes sense. Interestingly the average population of predator and prey in one complete cycle are precisely $\frac{c}{\gamma}$ and $\frac{a}{\alpha}$.

3 The Lyapunov Methods

3.1 Gaining Intuition: Oscillating Pendulum

We can consider a system described by an oscillating pendulum. This will be focused on *intuition*; for the mathematics see the textbook (pg 436).

The Potential Energy
The potential energy of a pendulum is associated with the height of the mass. It is easy to see that:

- the potential energy is a minimum when the pendulum is **stable** (i.e at rest)

- the potential energy is a maximum when the pendulum is at an angle of 180°; in other words, at its most unstable state

This can be confirmed by linearising, and seeing that we get a **saddle** in the first case, and a **centre** in the second case.

The Total Energy: Independent of Time

Using the equations of the system, we can derive a formula for the energy of the system. If we differentiate with respect to time, we get (as expected) that **energy doesn't change**. Most importantly, this is **independent of the pendulum's trajectory**.

Behaviour Close to a Critical Point

It can be shown that if we are close to the stable critical point, the trajectories will be ellipses enclosing said critical point. This makes sense: if we are close the stable configuration, an undamped pendulum will just oscillate continuously. Moreover, the closer to a stable critical point, the less potential energy we expect to have.

Damped Pendulum

If the pendulum is damped, the energy will be nonincreasing along any trajectory, except possibly if the pendulum just falls from the unstable to the stable position. Otherwise, the energy will be constantly decreasing as the pendulum oscillates and settles into equilibrium.

3.2 Positive Definite Functions (and Others)

Definition (Positive Definite Functions)**.** *Let V be a function defined on some domain D containing the origin. Then V is **positive definite** on D if:*

- $V(0,0) = 0$
- $\forall x, y \in D, x, y \neq 0, \quad V(x, y) > 0$

Definition (Positive Semidefinite Functions)**.** *Let V be a function defined on some domain D containing the origin. Then V is **positive semidefinite** on D if:*

- $V(0,0) = 0$
- $\forall x, y \in D, x, y \neq 0, \quad V(x, y) \geq 0$

Definition (Negative Definite Functions)**.** *Let V be a function defined on some domain D containing the origin. Then V is **negative definite** on D if:*

- $V(0,0) = 0$
- $\forall x, y \in D, x, y \neq 0, \quad V(x, y) < 0$

Definition (Negative Semidefinite Functions)**.** *Let V be a function defined on some domain D containing the origin. Then V is **negative semidefinite** on D if:*

- $V(0,0) = 0$
- $\forall x, y \in D, x, y \neq 0, \quad V(x, y) \leq 0$

3.3 Lyapunov Stability and Instability Theorems

- **What is the objective of the Lyapunov Theory?**
  - to construct an auxiliary function, in order to derive conclusions about the stability or instability of a critical point
- **What are the benefits of using Lyapunov Theory?**
  - we don't need to compute the solution to the ODE in order to derive conclusions about critical points
  - it allows us to investigate the basin of attraction of a critical point, which linear analysis doesn't help with
  - given a locally linear system, no conclusion can be drawn about the stability of a center; Lyapunov solves this
- **How is the intuition above related to Lyapunov Theory?**
  - say we have an autonomous system:
  $$\frac{dx}{dt} = F(x, y)$$
  $$\frac{dy}{dt} = G(x, y)$$
  and assume that $(0, 0)$ is an asymptotically stable critical point
  - we expect that there exists a region D containing $(0, 0)$, such that any trajectory in D approaches $(0, 0)$ as $t \to \infty$
  - if we have a positive definite/semidefinite "energy" function $V(x, y)$, then across any trajectory in D, we'd expect that $V \to 0$ as $t \to \infty$ (in other words, V is non-increasing on D)
  - we'd be able to prove stability if the converse were true: if finding a function V which goes to 0 as $t \to \infty$ on a trajectory within a region implied that all trajectories must approach the origin as $t \to \infty$
  - this is in essence what Lyapunov Theory asserts

Theorem (Lyapunov Stability Theorem)**.** *Suppose the autonomous system:*

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y)$$

has an isolated critical point at the origin. If $\exists V$ *that:*

- *is* ***continuous***
- *has* ***continuous first partial derivatives***
- *is* ***positive definite***
- $\dot{V}$ *is* ***negative definite*** *on some domain* D *in the xy plane containing* $(0, 0)$ *(where* $\dot{V}$ *is the derivative of* V *with respect to the autonomous system, defined as:*

  $$\frac{dV(x(t), y(t))}{dt} = V_x(x, y)F(x, y) + V_y(x, y)G(x, y)$$

)

*then the origin is an **asymptotically stable critical point**.*

*Alternatively, if $\dot{V}$ is **negative semidefinite**, the origin is a **stable critical point***

Theorem (Lyapunov Instability Theorem)**.** *Suppose the autonomous system:*

$$\frac{dx}{dt} = F(x, y)$$

$$\frac{dy}{dt} = G(x, y)$$

has an isolated critical point at the origin. If $\exists V$ that:

- *is **continuous***
- *has **continuous first partial derivatives***
- $V(0, 0) = 0$
- *in every neighbourhood of the origin, there is **at least** one point at which V is positive/negative*

*Then, if there exists a domain D containing the origin, such that $\dot{V}$ is **positive definite/negative definite** on D, then the origin is an **unstable critical point**.*

- **What is a Lyapunov Function?**
  - any function V which applies in the Stability/Instability Theorem
- **Is there a systematic way of determining Lyapunov Functions?**
  - no; it mainly relies on intuition, or whether we are handling a particular case
  - for example, in physical systems, V can be defined as the total energy of the system
  - another useful potential function is:

    $$V(x, y) = ax^2 + bxy + cy^2$$

    which is positive definite if and only if:

    $$a > 0 \qquad 4ac - b^2 > 0$$

    and negative definite if and only if:

    $$a < 0 \qquad 4ac - b^2 > 0$$

- **How can Lyapunov Theory lead to estimates of the basin of attraction?**
  - if we can find some bounded domain D_k, such that $V(x, y) < K$ for some K, where V is positive definite, and $\dot{V}$ is negative definite, then every solution to the system within D_k must approach the origin as $t \to \infty$
  - thus, D_k will define a subset of the basin of attraction
- **What is the intuition behind the Lyapunov Theorem?**

- the details are more extensively explained in the book
- in essence, we can consider a curve given by $V(x, y) = c$ (surrounding the origin), and a trajectory
- when they intersect, at some point (x_1, y_1) we get that:

$$\dot{V}(x_1, y_1) = \nabla V(x_1, y_1) \cdot \underline{T}(t_1)$$

  where $\underline{T}$ denotes the tangent vector of the trajectory, and ∇V is the gradient vector of V
- if $\dot{V}(x_1, y_1) \leq 0$ this can only be the case if the cosine of the angle between $\nabla V(x_1, y_1)$ and $\underline{T}(t_1)$ is less than or equal to 0; in particular the angle must be within the interval:

$$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

- thus, the trajectory must be heading towards the origin (or at most must be tangent to the curve)!

$$\begin{aligned}\dot{V}(x_1, y_1) &= V_x(x_1, y_1)x'(t_1) + V_y(x_1, y_1)y'(t_1) \\ &= \big(V_x(x_1, y_1)\mathbf{i} + V_y(x_1, y_1)\mathbf{j}\big) \cdot \big(x'(t_1)\mathbf{i} + y'(t_1)\mathbf{j}\big) \\ &= \nabla V(x_1, y_1) \cdot \mathbf{T}(t_1). \qquad (10)\end{aligned}$$

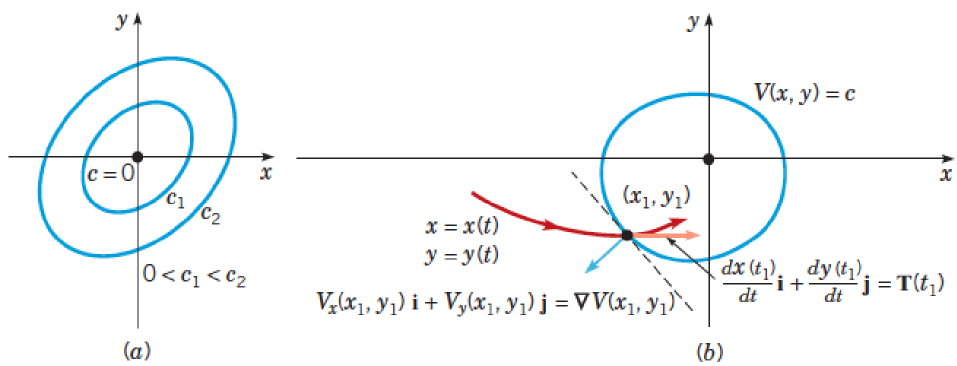


FIGURE 9.6.1 (a) The point $V(x, y) = 0$ and the level curves $V(x, y) = c_1$ and $V(x, y) = c_2$ with $0 < c_1 < c_2$. (b) Geometric interpretation of Liapunov's second method.

4 The Poincare-Bendixson Theorem

A very nice video on PBT, with example and nice animations

The motivation behind this section is that linearisation only gives us an idea of stability *locally*. The global picture can be different: unstable nodes might only be unstable on a certain neighbourhood, but away from that neighbourhood converge. In particular, we focus on those cases in which the convergence is done on **periodic trajectories**, which are known as **limit cycles**.

4.1 Limit Cycles

Definition (Limit Cycle)**.** *A* ***limit cycle*** *is a periodic (closed) trajectory in phase space, such that as $t \to \infty$ or $t \to -\infty$, at least one other non-closed trajectory asymptotes to them.*

4.1.1 Limit Cycles: Worked Example

We characterise the existence and properties of limit cycles by using an example.

We discuss the qualitative properties of the solution to the system:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y - x(x^2 + y^2 - a^2) \\ -x - y(x^2 + y^2 - a^2) \end{pmatrix}$$

Since this is a non-linear system, we begin by linearising:

$$\boldsymbol{J} = \begin{pmatrix} -3x^2 - y^2 + a^2 & 1 - 2yx \\ -1 - 2xy & -x^2 - 3y^2 + a^2 \end{pmatrix}$$

The only critical point is $(0, 0)$:

$$\boldsymbol{J}(0, 0) = \begin{pmatrix} a^2 & 1 \\ -1 & a^2 \end{pmatrix}$$

which has eigenvalues satisfying:

$$(a^2 - r)^2 + 1 = 0 \implies r = a^2 \pm i$$

This indicates that at $(0, 0)$ we have an **unstable spiral**, which we expect means that all trajectories end up at infinity.

However, this is not the case. To see why, we can use polar coordinates:

$$x = r \cos \theta$$
$$y = r \sin \theta$$

It follows that:

$$r^2 = x^2 + y^2$$

If we differentiate this with respect to t:

$$2rr' = 2xx' + 2yy'$$

and using our system:

$$\begin{aligned} 2rr' &= 2x(y - x(x^2 + y^2 - a^2)) + 2y(-x - y(x^2 + y^2 - a^2)) \\ &= 2xy - 2x^2(r^2 - a^2) - 2xy - 2y^2(r^2 - a^2) \\ &= -2r^2(r^2 - a^2) \\ \implies \quad r' &= -r(r^2 - a^2) \end{aligned}$$

To find an expression for θ, we notice that:

$$x' = r' \cos \theta - r\theta' \sin \theta$$

We can use the expression above for r':

$$\begin{aligned} & y - x(x^2 + y^2 - a^2) = -r(r^2 - a^2) \cos \theta - r\theta' \sin \theta \\ \implies \quad & r\theta' \sin \theta = -r(r^2 - a^2) \cos \theta - y + x(r^2 - a^2) \\ \implies \quad & r\theta' \sin \theta = -r(r^2 - a^2) \cos \theta - r \sin \theta + r \cos \theta(r^2 - a^2) \\ \implies \quad & \theta' = -(r^2 - a^2) \cot \theta - 1 + \cot \theta(r^2 - a^2) \\ \implies \quad & \theta' = -1 \end{aligned}$$

Thus, we end up witht he fact that our system can be defined by:

$$r' = -r(r^2 - a^2)$$

$$\theta' = -1$$

Whilst we can solve for r explicitly, we can derive its behaviour by consider the plot r' vs r. Since $r \geq 0$, we see that 2 "roots" of r' occur when $r = 0$ or $r = a$. When $r \in (0, a)$, we see that $r' > 0$, and when $r > a$, $r' < 0$. What this means is that:

- when $r = a$, r remains constant, since $r' = 0$ (this corresponds to a circle, since θ' is constant)
- when $0 < r < a$, r will be increasing since $r' > 0$ (this corresponds to a spiral, moving outwards and clockwise, since $\theta' = -1$)
- when $r > a$, r will be decreasing, since $r' < 0$ (this corresponds to a spiral, moving outwards and clockwise, since $\theta' = -1$)

Thus, we can see that eventually all trajectories will converge towards a circle of radius a. A phase portrait of the system can be seen below:

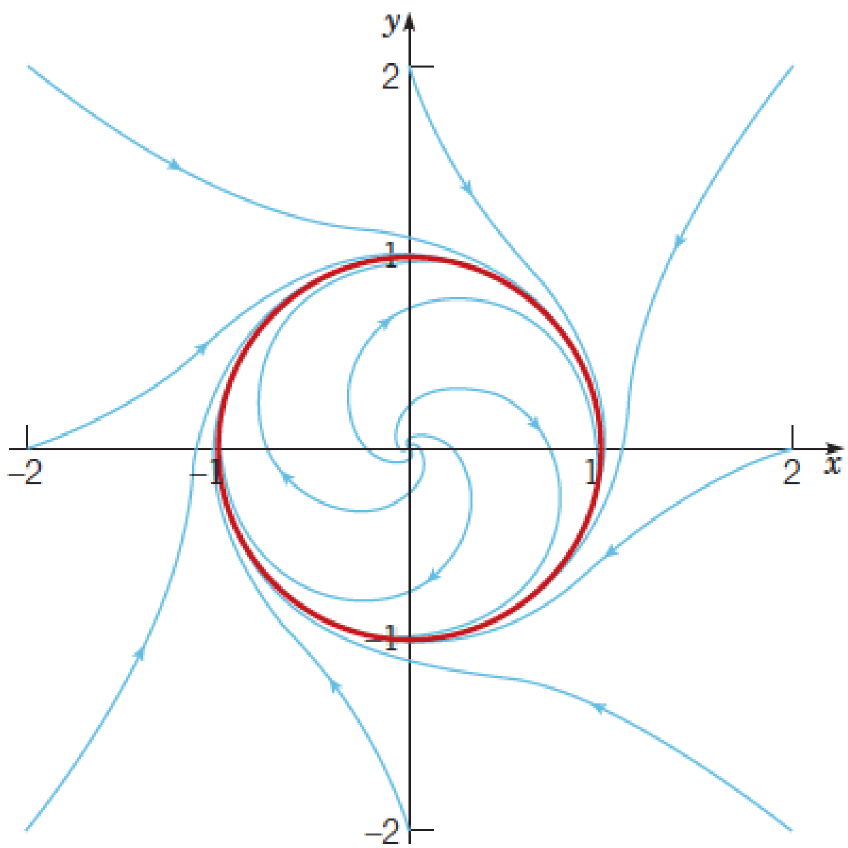


Figure 4: Whilst local analysis indicated the presence of a spiral, we had assumed it would be unbounded. However, further analysis of the system reveals the existence of a closed, periodic solution, which all trajectories eventually approach.

- **Are all closed trajectories periodic?**
  - a trajectory is periodic if $\exists T$ such that:

$$x(t+T) = x(t)$$

    so a periodic trajectory is one that eventually repeats itself, so in particular it must form a closed curve in phase space

- **Aren't all periodic trajectories centres?**
  - whilst centres are periodic trajectories (in particular a *family* of periodic trajectories), limit cycles are an example of an isolated periodic trajectory

- **What are the types of orbital stability that a limit cycle can exhibit?**
  - **asymptotically stable**: all trajectories close to a closed trajectory spiral towards the close trajectory
  - **semistable**: all trajectories on one side of the closed trajectory spiral toward the closed trajectory, whilst trajectories on the other side spiral away
  - **unstable**: trajectories on both sides of a closed trajectory spiral away
  - **stable**: closed trajectories which are neither approached nor departed from (for example, Lotka Volterra)

- **How can we determine whether a limit cycle exists for a given system?**
  - being able to find such cycles analytically is typically not possible
  - thus, we develop 3 Theorems that allow us to find limit cycles, by just exploring the topology of the space surrounding critical points

4.2 Theorem 1: Critical Points and Closed Trajectories

Theorem 1. *Let the functions F and G have* ***continuous first partial derivatives*** *in a domain D of the xy-plane. A* ***closed trajectory*** *of the system:*

$$\frac{dx}{dt} = F(x, y)$$

$$\frac{dy}{dt} = G(x, y)$$

must ***necessarily*** *enclose* ***at least one critical point****.*

If it enclosed ***only one*** *critical point, the critical point can* ***not*** *be a saddle.*

4.2.1 Implications

- closed trajectories always enclose at least 1 critical pint
- the presence of a saddle implies that a closed trajectory must enclose at least 2 critical points. Clearly, if there is a saddle, and a closed trajectory encloses it, this isn't possible, as the saddle shoots off all trajectories to infinity. Thus, there must be other critical points which attract the ends of the trajectories. Diagrammatically:

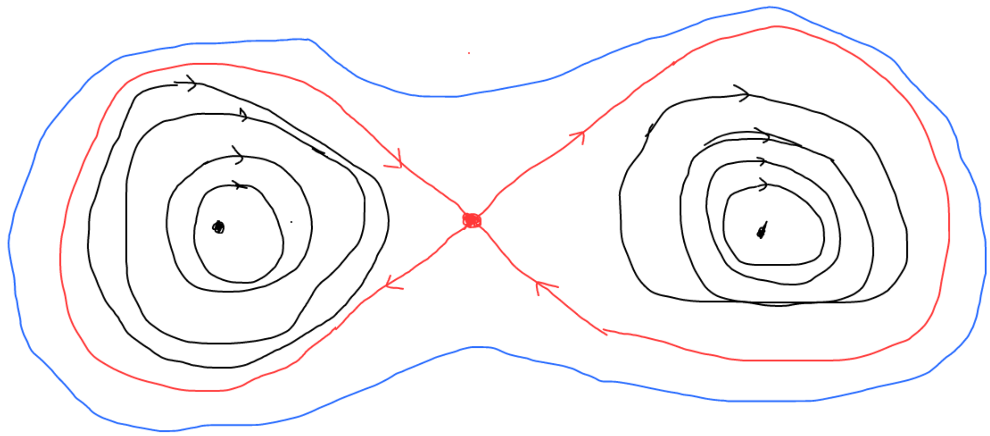


Figure 5: The red spot is a saddle critical point, whilst the black points are centres. The blue closed curve encloses the 3 critical points, but no closed trajectory would be able to enclose the red critical point.

- if a region has no critical points, then there can't be any closed trajectory in the region
- if a region only has a saddle point, then there can't be any closed trajectory in the region

4.3 Theorem 2: Divergence and Closed Trajectories

Theorem. *Let the functions F and G have continuous first partial derivatives in a* ***simply connected domain*** *D of the xy-plane. If $F_x + G_y$ has the* ***same*** *sign throughout D, then there is* ***no closed trajectory*** *for the system:*

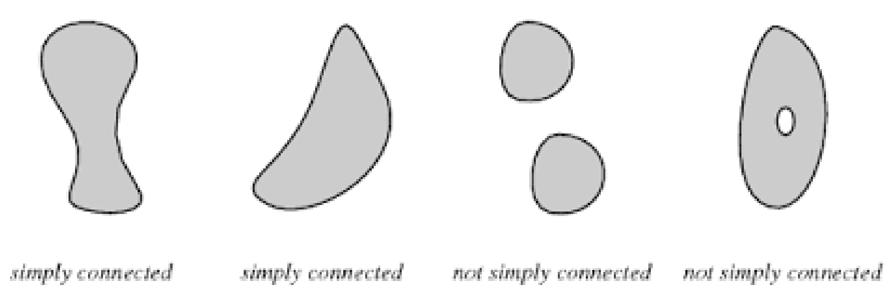
$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y)$$

lying ***entirely*** *in D.*

Remark (Simply Connected Region)**.** *Recall that a* ***simply connected region*** *is a region of space such that:*

- *the region is* ***connected*** *(any 2 points within the region can be connected by a line within the region)*
- *any* ***simple, closed*** *curve (a curve which is closed and not self-intersecting) in the region only encloses points in the region*

In other words, we require a region of space which has no holes, or is not split into parts.



4.3.1 Implications

- $F_x + G_y$ is the divergence of the vector field defined by the equation; in other words, it denotes the rate (magnitude) and direction (sign) of "flow".
  - if $F_x + G_y > 0$, we expect outwards flow
  - if $F_x + G_y < 0$, we expect inwards flow
- intuitively, if within a region the divergence is constant, we expect any point within the region to either contract (if divergence is negative), or to expand (if divergence is positive); in other words, no closed curve can exist in the region, as said closed curve would not be invariant under the ODE
- if the sign of the divergence doesn't change, a closed trajectory could or could not exist. For example, recall the system:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y - x(x^2 + y^2 - a^2) \\ -x - y(x^2 + y^2 - a^2) \end{pmatrix}$$

  The divergence of its vector field is:

$$F_x + G_y = -3x^2 - y^2 + a^2 - x^2 - 3y^2 + a^2 = -4r^2 + 2a^2$$

  In other words, if $r < \frac{a}{\sqrt{2}}$ then $F_x + G_y < 0$, so no closed trajectory exists in this region (as we saw, no closed trajectory existed when $0 < r < a$).

4.4 Poincare-Bendixson Theorem

Theorem. *Let the functions F and G have* ***continuous first partial derivatives*** *in a domain D of the xy-plane. Let D_1 be a bounded subdomain in D, and let $R = \partial D_1 \cup D_1$. Suppose that R contains* ***no critical point*** *of the system:*

$$\frac{dx}{dt} = F(x, y)$$

$$\frac{dy}{dt} = G(x, y)$$

If there is a solution of the above system, and said solution stays in R for $t \geq t_0$, then either:

- *the solution is* ***periodic*** *(closed trajectory)*
- *the solution spirals towards a closed trajectory as $t \to \infty$*

In either case, the system has a periodic solution in R.

4.4.1 Implication

- the above Theorem states that it is sufficient to prove that a solutions remains within a bounded region which contains no critical points, in order to prove that in such a region there must be a closed, periodic solution
- from Theorem 1 (4.2), since there is a closed trajectory in R, it must enclose a critical point. But R can't contain a critical point, so it must be the case that R is **not simply connected** (i.e it has a hole, where the critical point will be)
- intuitively, since trajectories can't self intersect, and a trajectory is trapped in R, it must do something: either it eventually wraps around itself (periodic solution), or it tends to some other trajectory (limit cycle)

- in practice, we typically don't find specific trajectories, but rather we find **trapping regions**: regions in which the vector field points towards the region at all points:

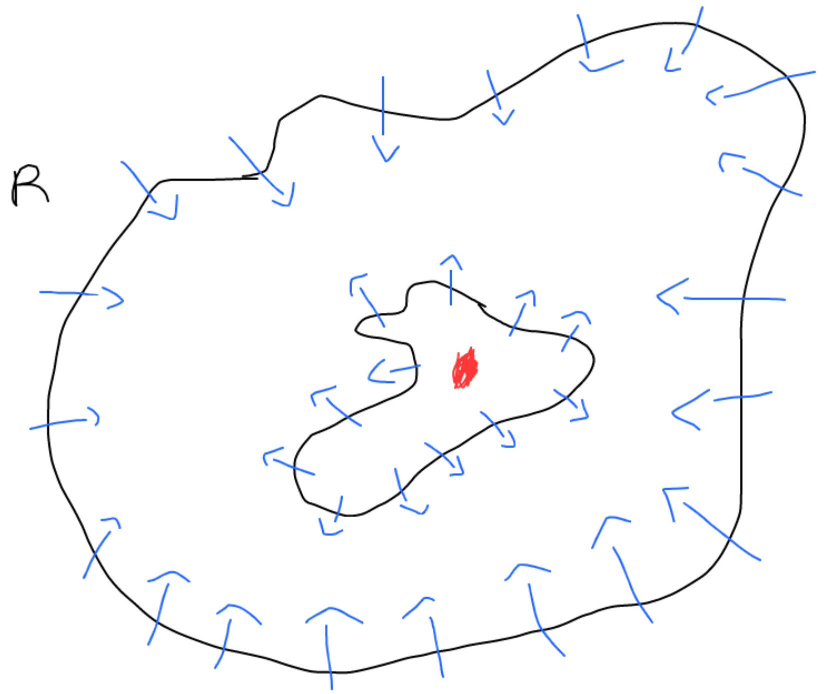


Figure 6: The black region is a **trapping region**, since the vector field always points towards its interior. This means that any trajectory that enters the region won't be able to escape the region.

For example, in the system defined in (4.1), we converted to polar coordinates, and obtained the ODE:

$$r' = -r(r^2 - a^2)$$

If we consider the region $\frac{a}{2} < r < 2r$:

- at $r = \frac{a}{2}$, $r' > 0$
- at $r = 2a$, $r' < 0$

Thus, the region defined above is a trapping region. Moreover, the critical point is at $(0, 0)$, so it isn't in the region. It thus follows by the Poincare-Bendixson Theorem that there is a periodic trajectory within $\frac{a}{2} < r < 2r$.

5 Exercises

1. **Determine the stability of the critical point at the origin for the system:**

$$x' = -2x$$

$$y' = x - y$$

using the Lyapunov Function:

$$V(x, y) = ax^2 + bx^2, \qquad a, b > 0$$

It is clear that $V(x, y)$ is positive definite, since it is only 0 at the origin. We compute the derivative with respect to the system:

$$\dot{V} = 2axx' + 2byy' = -4ax^2 + 2bxy - 2by^2$$

The terms $-4ax^2$ and $-2by^2$ are great, as they are always negative, so they ensure that we are negative definite. The issue is the term $2bxy$, as it can also be positive. To solve this, we can pick a, b so that the expression for $\dot{V}$ becomes a perfect square with a negative term. Notice:

$$\dot{V} = -2b\left(\frac{2a}{b}x^2 - xy + y^2\right)$$

So if we let $\frac{2a}{b} = \frac{1}{4}$, we can rearrange, and we get:

$$\dot{V} = -2b\left(\frac{x}{2} - y\right)^2 \leq 0$$

So, $\dot{V}$ is **negative semidefinite**, so $(0, 0)$ corresponds to a stable critical point.

In fact, $(0, 0)$ is **asymptotically stable**. The equations can be easily solved to get:

$$x = e^{-2x} + C_1$$

$$y = e^{-2x} + C_1 + C_2e^{-x}$$

Clearly as $t \to \infty$ these solutions go towards the origin.

2. **Show that the system:**

$$x' = y$$

$$y' = -x + y(1 - 3x^2 - 2y^2)$$

has a periodic solution.

Again we make use of polar coordinates. Using $r^2 = x^2 + y^2$, we know that:

$$rr' = xx' + yy' \implies rr' = xy - yx - y^2(-1 + 3x^2 + 2y^2)$$

Thus:

$$r' = \frac{-y^2(-1 + 3x^2 + 2y^2)}{r}$$

Now, notice that:

$$r' \leq \frac{-y^2(-1 + 2x^2 + 2y^2)}{r} = \frac{-y^2(2r^2 - 1)}{r}$$

So if $r > \frac{1}{\sqrt{2}}$, it follows that $r' < 0$.

Moreover:

$$r' \geq \frac{-y^2(-1 + 3x^2 + 3y^2)}{r} = \frac{-y^2(3r^2 - 1)}{r}$$

So if $r < \frac{1}{\sqrt{3}}$, it follows that $r' > 0$.

So we can define the region R as:

$$\frac{1}{\sqrt{2}} < r < \frac{1}{\sqrt{3}}$$

this is a trapping region. The critical point is at $(0, 0)$, so it follows by Poincare-Bendixson that there must exist some periodic trajectory on the region.