Honours Differential Equations - Week 4 - Phase Space and Stability

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Contents

1 The Phase Plane

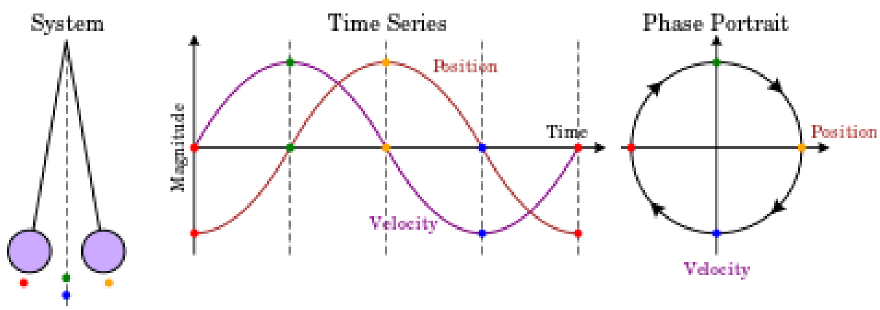
- **What is a phase plane?**
  - we consider the 2-dimension, first-order, linear homogeneous sytem of constant coefficients:

$$\underline{x}' = \boldsymbol{A}\underline{x}$$

    where:

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

  - the **phase plane** is the plane we obtain by using x_1 and x_2 as the basis (that is, we plot x_1 and x_2, as opposed to x_1 *vs* t and x_2 *vs* t



- **Why is the phase plane of interest when analysing ODEs?**
  - there are many ODEs which can't be solved analytically
  - solving said ODEs numerically is one way of visualising solutions
  - alternatively, we can use the phase plane to plot the behaviour of general solutions, without needing to solve systems
  - in other words, phase planes provide us with **qualitative understanding** of how solutions behave
  - what is most useful is that we get information about **all** possible solutions in one go
- **What is a trajectory?**
  - generally, we consider solutions of the form $\underline{x} = \underline{x}(t)$
  - if we plot $\underline{x}(t)$ in the x_1, x_2 plane, we obtain a curve/path which we call a **trajectory**
  - we can think of a trajectory as the path traversed by a particle with velocity $\frac{d\underline{x}}{dt}$
  - the number of trajectories is the number of possible initial conditions to a problem
  - if we have an autonomous ODE, we are guaranteed that trajectories will never cross each other
- **What is the phase portrait of an ODE system?**
  - a **phase portrait** is obtained by plotting a (representative) set of trajectories for a given ODE

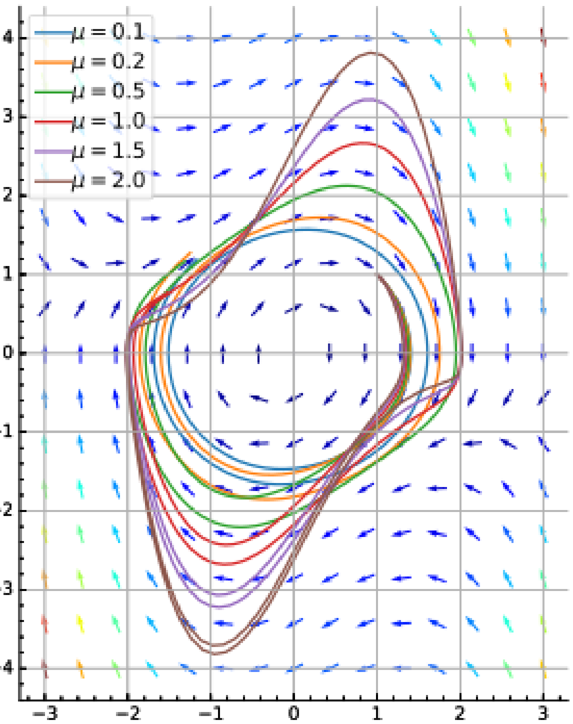


Figure 1: If we plot many different trajectories in the phase plane we get a phase portrait. Create your own here

- **What are the critical points of an ODE system?**
  - a point $\underline{x}^*$ is said to be a critical point if the vector field is $\underline{0}$ at $\underline{x}^*$
  - if we have a system:
  $$\underline{x}' = \begin{pmatrix} F(\underline{x}) \\ G(\underline{x}) \end{pmatrix}$$
  then $\underline{x}^*$ is a critical point if:
  $$F(\underline{x}^*) = 0$$
  $$G(\underline{x}^*) = 0$$
  - if we have a system with constant coefficients:
  $$\underline{x}' = \boldsymbol{A}\underline{x}$$
  if A is invertible, then the only critical point is:
  $$\underline{x}^* = \underline{0}$$
  - if we have an IVP, such that $\underline{x}(t_0) = \underline{x}^*$, then clearly $\underline{x}' = \underline{0}$, and so:
  $$\forall t \in \mathbb{R}, \underline{x}(t) = \underline{x}^*$$
- **Why are critical points important?**
  - critical points are the points in which a solution to an ODE is **constant** (known as **equilibrium solutions)**

- as we will see next, for linear, autonomous systems of the form $\underline{x}' = \boldsymbol{A}\underline{x}$, critical points basically determine the behaviour of trajectories throughout the x_1, x_2 plane
- if the system is non-linear and autonomous, such as:

$$\underline{x}' = \begin{pmatrix} F(\underline{x}) \\ G(\underline{x}) \end{pmatrix}$$

  can still be classified based on their local influence on solution behaviour

2 Classifying Critical Points: Linear, Autonomous Systems

If we have a system $\underline{x}' = \boldsymbol{A}\underline{x}$, we know that the only critical point is $\underline{x} = \underline{0}$ (given that $\boldsymbol{A}$ is non-invertible. Then, by considering the eigenvalues and eigenvectors of $\boldsymbol{A}$ we are able to not only classify the critical point, but also describe the behaviour of the trajectories near said critical point in phase space.

This video was the **most** helpful when trying to understand how to plot each case. I really recommend watching.

Paul's Online Math Notes also have all the diagrams and classifications available

Good explanations and diagrams

2.1 Real, Different, Negative Eigenvalues

- **Case**: $r_1 < r_2 < 0$
- **General Solution**: $c_1\underline{\xi}_1 e^{r_1 t} + c_2\underline{\xi}_2 e^{r_2 t}$
- **Solution Behaviour**:
  - as $t \to \infty$, $e^{r_1 t}$ goes to 0 much faster than $e^{r_2 t}$
    - $c_2 \neq 0$: $e^{r_2 t}$ is the dominant term, so $\underline{0}$ is approached along $\underline{\xi}_2$
    - $c_2 = 0$: solutions start on $\underline{\xi}_1$ and $\underline{0}$ is approached along $\underline{\xi}_1$
  - as $t \to -\infty$, $e^{r_1 t}$ is the dominant term (larger):
    - $c_1 \neq 0$: trajectories start close to $\underline{\xi}_1$
    - $c_1 = 0$: solutions start on $\underline{\xi}_2$
  - thus, unless solutions start on $\underline{\xi}_1$ or $\underline{\xi}_2$, their trajectory will begin parallel to $\underline{\xi}_1$, and eventually converge to $\underline{0}$ by being parallel to $\underline{\xi}_2$
- **Critical Point Classification**: node/nodal sink

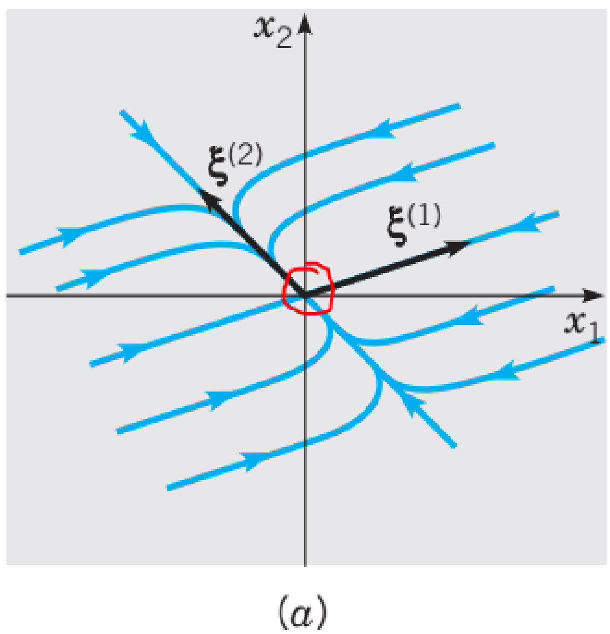


Figure 2: A nodal sink, produced when the eigenvalues are $r_1 < r_2 < 0$. Trajectories begin close to $\underline{\xi}_1$, but then get parallel to $\underline{\xi}_2$ to eventually converget at $\underline{0}$

2.2 Real, Different, Positive Eigenvalues

- **Case**: $r_1 > r_2 > 0$
- **General Solution**: $c_1\underline{\xi}_1 e^{r_1 t} + c_2\underline{\xi}_2 e^{r_2 t}$
- **Solution Behaviour**: using a similar diagnosis as before, this leads to the same trajectories as in the negative case, albeit with the direction reversed. Vectors will start parallel to $\underline{\xi}_2$, and eventually diverge parallel to $\underline{\xi}_1$. Trajectories seem to be *repelled* by the critical point
- **Critical Point Classification**: node/nodal source

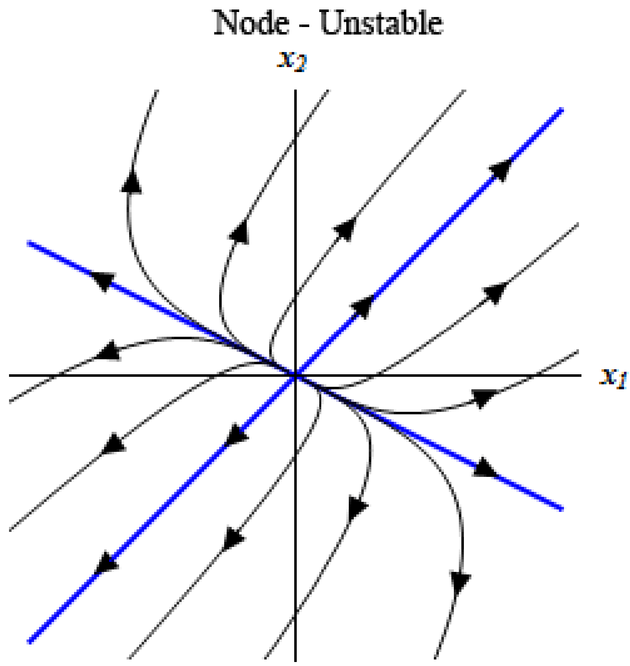


Figure 3: A nodal source, produced when the eigenvalues are $0 < r_2 < r_1$. Trajectories begin close to $\underline{\xi}_2$, but then get parallel to $\underline{\xi}_1$ to diverge

2.3 Real, Different, Opposite Signed Eigenvalues

- **Case**: $r_1 > 0, r_2 < 0$
- **General Solution**: $c_1\underline{\xi}_1 e^{r_1 t} + c_2\underline{\xi}_2 e^{r_2 t}$
- **Solution Behaviour**:
  - as $t \to \infty$, $e^{r_1 t}$ is the dominant term
    - $c_1 \neq 0$: solutions approach infinity asymptotic to $\underline{\xi}_1$ (if they start on $\underline{\xi}_1$, they approach infinity on said line)
    - $c_1 = 0$: solutions start on $\underline{\xi}_2$. $e^{r_2 t}$ will go to 0, so $\underline{0}$ is approached along $\underline{\xi}_2$
  - as $t \to -\infty$, $e^{r_2 t}$ is the dominant term:
    - $c_2 \neq 0$: solutions approach infinity asymptotic to $\underline{\xi}_2$ (if they start on $\underline{\xi}_2$, they approach infinity on said line)
    - $c_2 = 0$: solutions start on $\underline{\xi}_1$. $e^{r_1 t}$ will go to 0, so $\underline{0}$ is approached along $\underline{\xi}_1$
  - thus:
    - if trajectories start at $\underline{\xi}_1$ ($c_2 = 0$), they go to 0 as $t \to -\infty$, and to infinity as $t \to \infty$
    - if trajectories start at $\underline{\xi}_2$ ($c_1 = 0$), they go to 0 as $t \to \infty$, and to infinity as $t \to -\infty$
    - if trajectories start elsewhere, as $t \to \infty$, they they diverge to infinity parallel to $\underline{\xi}_1$
    - if trajectories start elsewhere, $t \to -\infty$, they will be parallel to $\underline{\xi}_2$
- **Critical Point Classification**: saddle

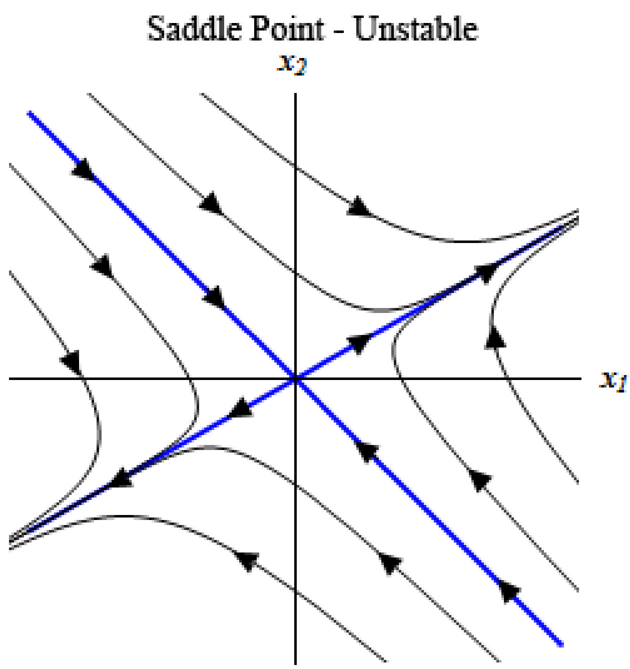


Figure 4: A saddle produced when the eigenvalues are $r_1 > 0, r_2 < 0$. Trajectories only approach the critical point along the eigenvectors.

2.4 Real, Equal Eigenvalues

2.4.1 2 Linearly Independent Eigenvectors

- **Case**: $r = r_1 = r_2$, r has geometric multiplicity 2
- **General Solution**: $e^{rt}(c_1\underline{\xi}_1 + c_2\underline{\xi}_2)$
- **Solution Behaviour**: notice that $\frac{x_2}{x_1}$ is a constant, dependent on $\underline{\xi}_1, \underline{\xi}_2, c_1, c_2$. Most importantly it is independent of t, so solutions will be straight lines passing through the critical point. Trajectories converge to $\underline{0}$ if $r < 0$, and diverge otherwise.
- **Critical Point Classification**: proper node/star point

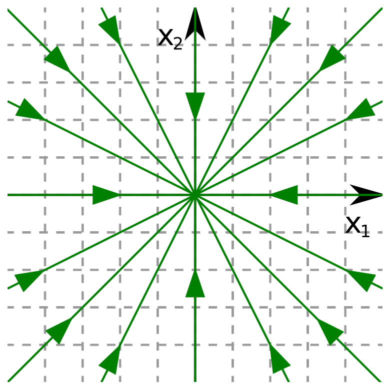


Figure 5: A proper node produced when the eigenvalues are $r < 0$. Trajectories are straight lines going through the critical point.

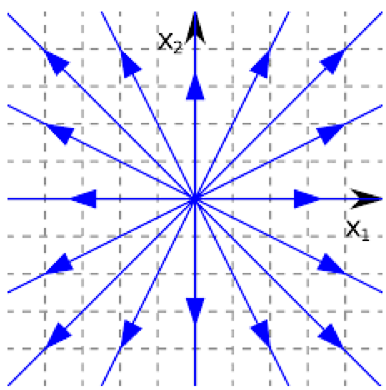


Figure 6: A proper node produced when the eigenvalues are $r > 0$. Trajectories are straight lines going through the critical point.

2.4.2 1 Linearly Independent Eigenvector

- **Case**: $r = r_1 = r_2$, r has geometric multiplicity 1
- **General Solution**: $e^{rt}(c_1\underline{\xi} + c_2(\underline{\xi}t + \underline{\eta}))$
- **Solution Behaviour**:
  - as $t \to \infty$, te^{rt} is the dominant term
    - if $r < 0$, independent of the value of c_2, the trajectory approaches $\underline{0}$ parallel to $\underline{\xi}$
    - if $r > 0$, independent of the value of c_2, the trajectory diverges to infinity parallel to $\underline{\xi}$
  - as $t \to -\infty$, te^{rt} is still the dominant term (due to the presence of t)
    - if $r > 0$, independent of the value of c_2, the trajectory approaches $\underline{0}$ parallel to $\underline{\xi}$
    - if $r < 0$, independent of the value of c_2, the trajectory diverges to infinity parallel to $\underline{\xi}$
  - the resulting trajectory is one that goes parallel to the eigenvector, and eventually curves and changes direction via a 180^{o} turn
- **Critical Point Classification**: improper/degenerate node

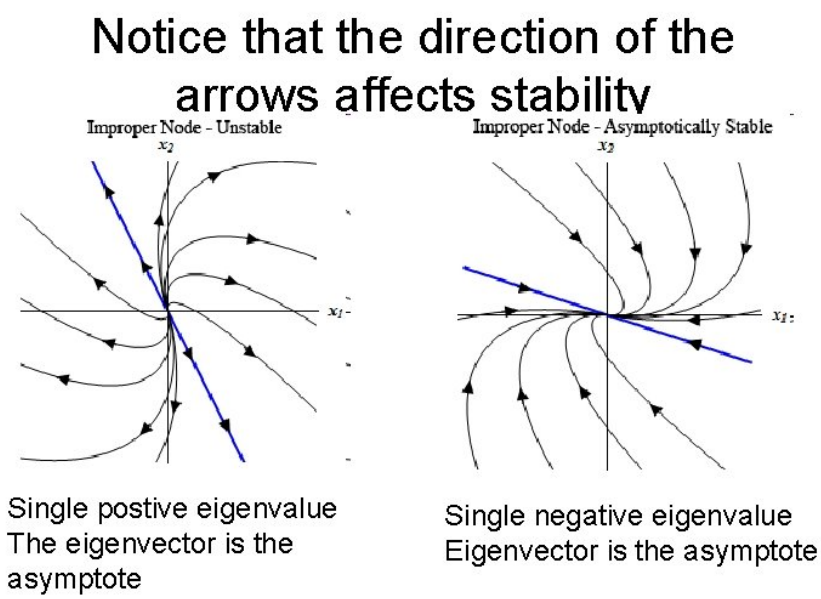


Figure 7: An improper node produced when the eigenvalues are $r < 0$ and $r > 0$. Trajectories curve, but eventually are always parallel to the eigenvector. To know the direction of the trajectory, determine the direction of the field at points like $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

2.5 Complex, Non-Zero Real Part Eigenvalues

- **Case**: $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$, $\lambda \neq 0$
- **General Solution**: $e^{\lambda t}\begin{pmatrix} c_1\cos(\mu t) + c_2\sin(\mu t) \\ -c_1\sin(\mu t) + c_2\cos(\mu t)\end{pmatrix} = Ce^{\lambda t}\begin{pmatrix}\cos(\phi - \mu t) \\ \sin(\phi - \mu t)\end{pmatrix}$. The latter uses the **amplitude-phase** form, where:

$$C = \sqrt{c_1^2 + c_2^2}$$

$$\phi = \arctan\left(\frac{c_1}{c_2}\right)$$

(see this MIT resource)

- **Solution Behaviour**: notice the general solution satisfies:

$$x_1^2 + x_2^2 = C^2 e^{2\lambda t}$$

In other words, we can think of the trajectory as a circle which as $t \to \infty$ has an ever changing radius. In other words, the trajectory draws a spiral. That is, as $t \to \infty$, if $\lambda > 0$, the spiral will go outwards, whilst if $\lambda < 0$, the spiral will converge towards the critical point.

To determine how the spiral is oriented (clockwise or anti-clockwise), we consider the vector field at $x_1 = 0, x_2 = 1$ (so we compute $\boldsymbol{A}\begin{pmatrix}0 \\ 1\end{pmatrix}$). If the x_1 component of the resulting vector is positive, we will have clockwise motion.

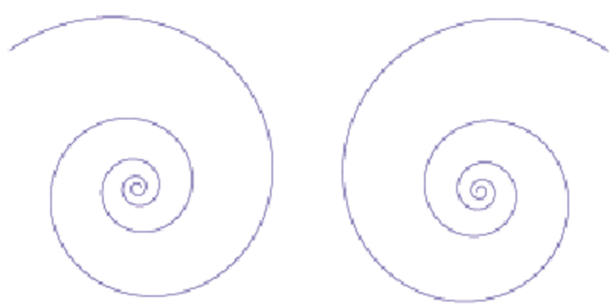


Figure 8: The left is a **clockwise** spiral, whilst the right is a **anti-clockwise** spiral.

Alternatively, we could've used polar coordinates:

An alternative way of reaching this conclusion.
Introduce polar coordinates in phase space

$$r^2 = x^2 + y^2 \,, \qquad \tan\phi = \frac{y}{x} \,.$$

(x, y) satisfy the ODE system

$$\begin{aligned} \dot{x} &= \lambda x + \mu y \\ \dot{y} &= \lambda y - \mu x \,. \end{aligned}$$

It follows

$$r\dot{r} = x\dot{x} + y\dot{y} = x(\lambda x + \mu y) + y(\lambda y - \mu x) = \lambda\, r^2 \,,$$

$$\frac{\dot{\phi}}{\cos^2\phi} = \frac{x\dot{y} - y\dot{x}}{x^2} \Rightarrow \dot{\phi} = -\mu$$

$$\begin{aligned} \dot{r} &= \lambda r \Rightarrow r = c\, e^{\lambda t} \\ \dot{\phi} &= -\mu \Rightarrow \phi = -\mu t + \phi_0 \,. \end{aligned}$$

- $\lambda > 0 \Rightarrow |\boldsymbol{x}| \to \infty$ for $t \to \infty$
- $\lambda < 0 \Rightarrow |\boldsymbol{x}| \to 0$ for $t \to \infty$
- ϕ decreases as t evolves (since $\mu > 0$) $\Rightarrow$ motion is clockwise

Conclusions are fully consistent with our spiral picture.

- **Critical Point Classification**: spiral point/spiral source/spiral sink

2.6 Imaginary Eigenvalues

- **Case**: $r_1 = i\mu$, $r_2 = -i\mu$
- **General Solution**: $\begin{pmatrix} c_1\cos(\mu t) + c_2\sin(\mu t) \\ -c_1\sin(\mu t) + c_2\cos(\mu t) \end{pmatrix} = C\begin{pmatrix} \cos(\phi - \mu t) \\ \sin(\phi - \mu t) \end{pmatrix}$. The latter uses the **amplitude-phase** form, where:

$$C = \sqrt{c_1^2 + c_2^2}$$

$$\phi = \arctan\left(\frac{c_1}{c_2}\right)$$

  (see this MIT resource)

- **Solution Behaviour**: notice the general solution satisfies:

$$x_1^2 + x_2^2 = C^2$$

  In other words, we can think of the trajectory as a circle (technically it forms an ellipse) centered at the critical point. If $\mu > 0$ the ellipse is traversed clockwise, whilst if $\mu < 0$, the ellipse is traversed anti-clockwise.

- **Critical Point Classification**: center

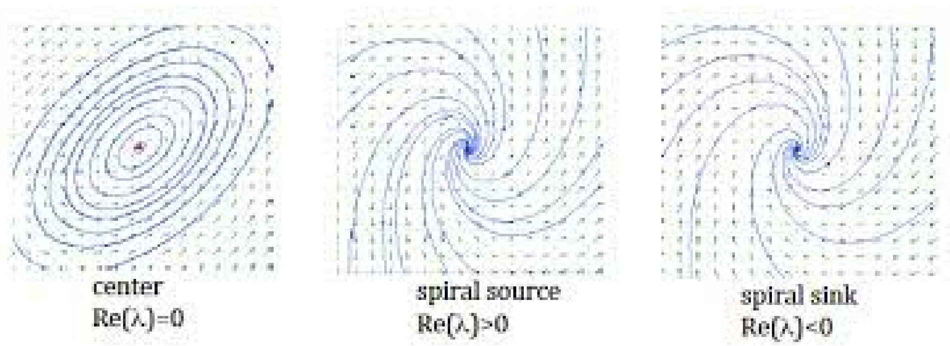


Figure 9: The cases for complex eigenvalues.

2.7 Critical Point Classification Summary

Eigenvalues	**Critical Point Type**
$r_1 > r_2 > 0$	node (source)
$r_1 < r_2 < 0$	node (sink)
$r_2 < 0 < r_1$	saddle
$r = r_1 = r_2, r > 0$	proper/improper node
$r = r_1 = r_2, r < 0$	proper/improper node
$r_1, r_2 = \lambda \pm i\mu, \lambda > 0$	spiral source
$r_1, r_2 = \lambda \pm i\mu, \lambda < 0$	spiral sink
$r_1, r_2 = \lambda \pm i\mu, \lambda = 0$	center

Table 1: It is important to recall that this is only valid for second-order, linear, autonomous systems defined by constant, invertible matrices

3 Stability of Critical Points

- **When is a critical point said to be stable?**

Definition (Stability of a Critical Point)**.** *A critical point* $\underline{x}^*$ *is said to be* ***stable*** *if* $\forall \varepsilon > 0$, $\exists \delta > 0$ *such that if* $\underline{x} = \underline{x}(t)$ *is a solution to a system, and:*

$$\|\underline{x}(0) - \underline{x}^*\| < \delta$$

then:

$$\|x(t) - \underline{x}^*\| < \varepsilon$$

What the above definition states is that if we have a solution which starts (initial condition $t = 0$) sufficiently close (δ close) to a critical point, then the whole solution will stay close (ε close) to the critical point. Diagrammatically:

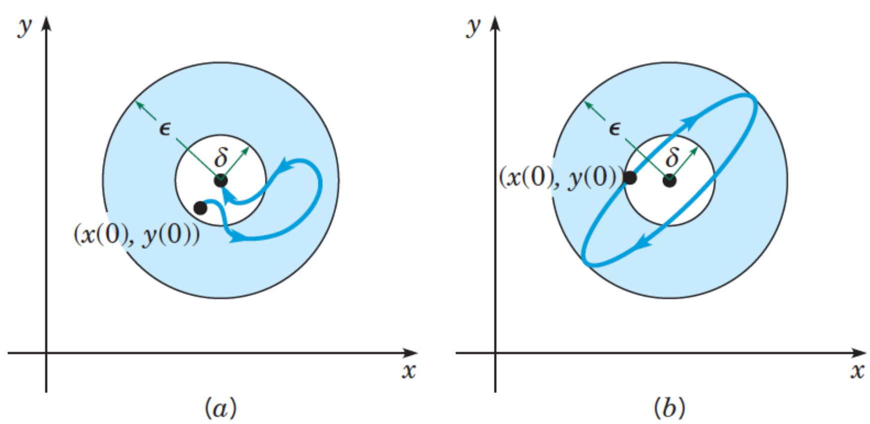


Figure 10: The sole requirement for stable critical point is that trajectories beginning close enough to a critical point will stay within ε of the critical point.

- **When is a critical point said to be unstable?**
  - whenever the requirements of stability are not satisfied
- **When is a critical point said to be asymptotically stable?**
  - informally, asymptotically stable critical points are those such that trajectories which start close enough to them will converge to the critical point at infinity
  - more formally, a critical point is **asymptotically stable** if beyond being stable, we can find some $\delta_0 > 0$ such that if:
  $$\|\underline{x}(0) - \underline{x}^*\| < \delta_0$$
  then:
  $$\lim_{t \to \infty} \underline{x}(t) = \underline{x}^*$$
  - for example, sinks are stable **and** asymptotically stable, whilst centers are stable but **not** asymptotically stable
- **How does stability correlate with the classification of a critical point?**

Eigenvalues	**Critical Point Type**	**Stability**
$r_1 > r_2 > 0$	node (source)	unstable
$r_1 < r_2 < 0$	node (sink)	asymptotically stable
$r_2 < 0 < r_1$	saddle	unstable
$r = r_1 = r_2, r > 0$	proper/improper node	unstable
$r = r_1 = r_2, r < 0$	proper/improper node	asymptotically stable
$r_1, r_2 = \lambda \pm i\mu, \lambda > 0$	spiral source	unstable
$r_1, r_2 = \lambda \pm i\mu, \lambda < 0$	spiral sink	asymptotically stable
$r_1, r_2 = \lambda \pm i\mu, \lambda = 0$	center	stable

- **What is the basin of attraction of a critical point?**
  - the **basin of attraction** is the set of all points in the x_1, x_2 plane, such that as $t \to \infty$ any trajectory passing through said points approaches a given critical point
  - it is also known as the **region of asymptotic stability** of a critical point
  - a **separatrix** is any trajectory which bounds a basin of attraction

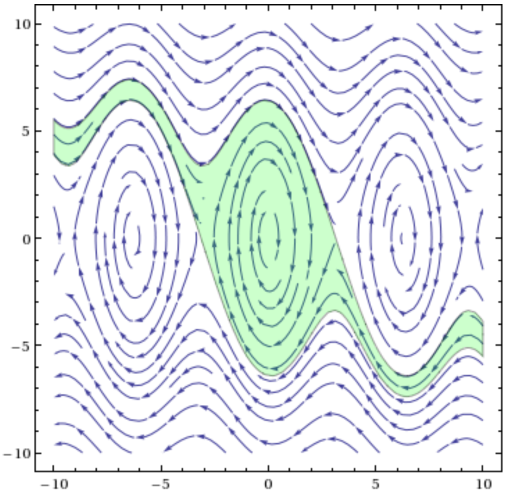


Figure 11: The basin of attraction for the central critical point is given in green

4 Classifying Critical Points: Non-Linear, Autonomous Systems

- **Can we classify critical points for non-linear systems?**
  - we consider autonomous, non-linear systems defined by:

$$x' = F(x, y)$$

$$y' = G(x, y)$$

  - in general, the techniques explained above won't work on this general system, since we don't know how to solve non-linear systems
  - however, we can exploit **local linearity** to approximate the vector field linearly

- **What is the Rectification Theorem?**

Theorem (Rectification Theorem)**.** *Consider some point $\underline{x}_*$ such that:*

$$F(\underline{x}_*) \neq 0$$

$$G(\underline{x}_*) \neq 0$$

*In other words, $\underline{x}_*$ is **not** a critical point. Then, in the neighbourhood of $\underline{x}_*$, it is possible to find some change of variables:*

$$(x, y) \longrightarrow (\bar{x}, \bar{y})$$

such that:

$$\bar{x}' = 1, \bar{y} = 0$$

In other words, for any vector field, if we take a non-critical point, we are able to change variables, such as to locally approximate the vector field as if it were a **linear** vector field

- **How can we linearise a non-linear system?**
  - we can use Taylor Expansions to show that in the neighbourhood of a critical point, the vector field is locally linear

- let $\underline{x}^*$ be a critical point, such that:

$$F(\underline{x}^*) = G(\underline{x}^*) = 0$$

- then, we can define a change of coordinates via:

$$u = x - x^*$$

$$v = y - y^*$$

  We can think of u, v as being the perturbation of x, y near a critical point

- we can then differentiate:

$$u' = F(x, y) = F(u + x^*, v + y^*)$$
$$v' = G(x, y) = G(u + x^*, v + y^*)$$

- and finally consider the Taylor Expansion (check this if you need to refresh) for u' and v':

$$u' = F(x^*, y^*) + uF_x(x^*, y^*) + vF_y(x^*, y^*) + \eta_1(x, y)$$

$$v' = G(x^*, y^*) + vG_x(x^*, y^*) + vG_y(x^*, y^*) + \eta_2(x, y)$$

- the key is that the remainder terms η_1, η_2 are **small**, in the sense that:

$$\lim_{\underline{x} \to \underline{x}^*} \frac{\eta_i}{\|\underline{x} - \underline{x}^*\|} = 0$$

  In other words, since $\langle u, v \rangle = \underline{x} - \underline{x}^*$, we say that η_1, η_2 are much smaller than u and v as these go to 0. Long story short, this means that if we consider small u, v, we can safely ignore η_1, η_2

- using this, alongside the fact that $F(\underline{x}^*) = G(\underline{x}^*) = 0$ our expansion simplifies to:

$$u' = uF_x(x^*, y^*) + vF_y(x^*, y^*)$$

$$v' = vG_x(x^*, y^*) + vG_y(x^*, y^*)$$

  which is a linear system, wince we are evaluating the partial derivatives at the critical point:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} F_x(x^*, y^*) & F_y(x^*, y^*) \\ G_x(x^*, y^*) & G_y(x^*, y^*) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

- here:

$$\boldsymbol{J} = \begin{pmatrix} F_x(x^*, y^*) & F_y(x^*, y^*) \\ G_x(x^*, y^*) & G_y(x^*, y^*) \end{pmatrix}$$

  is known as the **Jacobian Matrix**

- this system involving the Jacobian Matrix is our linear approximation to the original non-linear system. In particular, we can employ all the techniques above to determine the behaviour in the neighbouthood of $\underline{x}^*$

• **How does non-linearity affect the critical point classification and stability?**

Eigenvalues	Critical Point Type	Stability
$r_1 > r_2 > 0$	node (source)	unstable
$r_1 < r_2 < 0$	node (sink)	asymptotically stable
$r_2 < 0 < r_1$	saddle	unstable
$r = r_1 = r_2, r > 0$	proper/improper node	unstable
$r = r_1 = r_2, r < 0$	proper/improper node	asymptotically stable
$r_1, r_2 = \lambda \pm i\mu, \lambda > 0$	spiral source	unstable
$r_1, r_2 = \lambda \pm i\mu, \lambda < 0$	spiral sink	asymptotically stable
$r_1, r_2 = \lambda \pm i\mu, \lambda = 0$	center	stable

Table 2: Critical Point Classification and Stability for a Linear System

Eigenvalues	Critical Point Type	Stability
$r_1 > r_2 > 0$	node (source)	unstable
$r_1 < r_2 < 0$	node (sink)	asymptotically stable
$r_2 < 0 < r_1$	saddle	unstable
$r = r_1 = r_2, r > 0$	node/spiral point	unstable
$r = r_1 = r_2, r < 0$	node/spiral point	asymptotically stable
$r_1, r_2 = \lambda \pm i\mu, \lambda > 0$	spiral source	unstable
$r_1, r_2 = \lambda \pm i\mu, \lambda < 0$	spiral sink	asymptotically stable
$r_1, r_2 = \lambda \pm i\mu, \lambda = 0$	center/spiral point	indeterminate

Table 3: Critical Point Classification and Stability for a Locally Linear System