Honours Differential Equations - Week 3 - Solving Nonhomogeneous Linear Systems of ODEs

Antonio León Villares

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Contents

1 Nonhomogeneous Linear Systems

- **What is a nonhomogeneous linear system?**
  - a system of ODEs in which the RHS is function of t:

$$\underline{x}' = \boldsymbol{P}(t)\underline{x} + \underline{g}(t)$$

- **What are the solutions to a nonhomoegeneous linear system?**
  - the general solution to a nonhomogeneous system is the addition of the general solution to the homogeneous system and a particular solution to the nonhomogeneous system:

$$\underline{x} = \underline{x}_h + \underline{x}_p$$

  - to see why:

$$\begin{aligned} \underline{x} &= \underline{x}_h + \underline{x}_p \quad \implies \quad \underline{x}' = \underline{x}_h' + \underline{x}_p' \\ \implies \quad \underline{x}' &= \underline{x}_h' + \underline{x}_p' \\ &= \boldsymbol{P}\underline{x}_h + \boldsymbol{P}\underline{x}_p + \underline{g} \\ &= \boldsymbol{P}(\underline{x}_h + \underline{x}_p) + \underline{g} \\ &= \boldsymbol{P}\underline{x} + \underline{g} \end{aligned}$$

  - since we can solve homogeneous systems easily, the key to solving nonhomogeneous systems is being able to determine particular solutions

1.1 Solving Nonhomogeneous Linear Systems: Diagonalisation

- **When can diagonalisation be used to solve a nonhomogeneous system?**
  - the system has constant coefficients

$$\boldsymbol{P}(t) = \boldsymbol{A}$$

  - ideally, $\boldsymbol{A}$ should be diagonalisable, but if it isn't, we will get a system with a matrix defined in Jordan form, which is also solvable
- **How can diagonalisation help find a particular solution to a nonhomogeneous system?**
  - we consider the system

$$\underline{x}' = \boldsymbol{A}\underline{x} + \underline{g}(t)$$

  - assuming $\boldsymbol{A}$ is diagonalisable, we can define a transformation matrix $\boldsymbol{T}$, with the eigenvectors of $\boldsymbol{A}$ as column vectors:

$$\boldsymbol{T} = \begin{pmatrix} \underline{\xi}^{(1)} & \underline{\xi}^{(2)} & \dots & \underline{\xi}^{(n)} \end{pmatrix}$$

  - we can define a new variable $\underline{y}$ via:

$$\underline{x} = \boldsymbol{T}\underline{y}$$

  - if we use this in the original system:

$$\begin{aligned} \underline{x}' &= \boldsymbol{A}\underline{x} + \underline{g}(t) \\ \implies \quad \boldsymbol{T}\underline{y}' &= \boldsymbol{A}\boldsymbol{T}\underline{y} + \underline{g}(t) \\ \implies \quad \underline{y}' &= \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}\underline{y} + \boldsymbol{T}^{-1}\underline{g}(t) \\ \implies \quad \underline{y}' &= \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}\underline{y} + \underline{h}(t) \end{aligned}$$

- we are guaranteed that $\boldsymbol{T}^{-1}$ exists, since its column vectors are linearly independent; moreover, $\boldsymbol{T}^{-1}\boldsymbol{AT}$ is a diagonal matrix, composed of the eigenvalues of $\boldsymbol{A}$
- this is a system that can be solved variable wise, since it is defined by a diagonal matrix (it is said to be **uncoupled)**:

$$y_i' = r_i y_i + h_i \implies y_i' - r_i y_i = h_i$$

which can be solved by using an integrating factor $e^{-r_i t}$, leading to:

$$y_i = c_i e^{r_i t} + e^{r_i t} \int_{t_0}^{t} e^{-r_i s} h_i(s) ds$$

- once we obtain $\underline{y}$, we just get $\underline{x}$ by using the transformation matrix
- notice, the term $c_i e^{r_i t}$ will lead to the general solution of the homogeneous equation, and $e^{r_i t} \int_{t_0}^{t} e^{-r_i s} h_i(s) ds$ corresponds to the particular solution

- **What if the constant coefficient matrix is not diagonalisable?**
  - we can define $\boldsymbol{T}$ using the n eigenvectors and generalised eigenvectors
  - the matrix $\boldsymbol{T}^{-1}\boldsymbol{AT}$ will be in Jordan form, which can be solved easily (solve one variable at a time)

1.2 Solving Nonhomogeneous Linear Systems: Undetermined Coefficients

- **When can undetermined coefficients be used to solve a nonhomogeneous system?**
  - the system has constant coefficients

$$\boldsymbol{P}(t) = \boldsymbol{A}$$

  - the function $g(t)$ is a linear combination of a polynomial and a complex exponential. In other words, any g which is:
    - a polynomial
    - an exponential
    - a sinusoid

    or a combination of this

- **How can undetermined coefficients help find a particular solution to a nonhomogeneous system?**
  - we assume that $\underline{x}_p$ is of the form of $\underline{g}(t)$, albeit using unspecified coefficients
  - if λ is an eigenvalue of $\boldsymbol{A}$, and $\underline{g}(t) = \underline{u}e^{\lambda t}$, then, if λ has algebraic multiplicity j, we consider a particular solution given as a product of a jth degree polynomial and $e^{\lambda t}$:

$$(\underline{a}_j x^j + \underline{a}_{j-1} x^{j-1} + \ldots + \underline{a}_0) e^{\lambda t}$$

  - typically, we consider each component of $\underline{g}(t)$ separately, such that:

$$\underline{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix} = \begin{pmatrix} g_1(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ g_2(t) \\ \vdots \\ 0 \end{pmatrix} + \ldots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g_n(t) \end{pmatrix}$$

and we solve the system for each of the components. Then $\underline{x}_p$ is taken as the sum of all the found particular solutions.

1.3 Solving Nonhomogeneous Linear Systems: Variation of Parameters

- **When can variation of parameters be used to solve a nonhomogeneous system?**
  - this is the most general method, so it assumes nothing about the nature of $\boldsymbol{P}(t)$ or $\underline{g}(t)$
- **How can variation of parameters help find a particular solution to a nonhomogeneous system?**
  - we consider a system
  $$\underline{x}' = \boldsymbol{P}(t)\underline{x} + \underline{g}(t)$$
  - moreover, we assume we have access to a fundamental matrix $\boldsymbol{\Psi}(t)$ for the homogeneous system
  - recall that the general solution to the homogeneous system can be described via:
  $$\underline{x}_h = \boldsymbol{\Psi}(t)\underline{c}$$
  where $\underline{c}$ corresponds to the constant coefficients of the general solution
  - variation of parameters assumes a particular solution of the form:
  $$\underline{x}_p = \boldsymbol{\Psi}(t)\underline{u}(t)$$
  where $\underline{u}(t)$ is a vector to be found
  - we can verify that this is indeed a solution. Firstly, recall that a fundamental matrix solves the homogeneous system, so:
  $$\boldsymbol{\Psi}(t)' = \boldsymbol{P}(t)\boldsymbol{\Psi}(t)$$
  - next, we differentiate our particular solution
  $$\begin{aligned}\underline{x}' &= (\boldsymbol{\Psi}\underline{u})' \\ &= \boldsymbol{\Psi}'\underline{u} + \boldsymbol{\Psi}\underline{u}' \\ &= \boldsymbol{P}\boldsymbol{\Psi}\underline{u} + \boldsymbol{\Psi}\underline{u}' \\ &= \boldsymbol{P}\underline{x} + \boldsymbol{\Psi}\underline{u}'\end{aligned}$$
  - in other words, for $\underline{x} = \boldsymbol{\Psi}\underline{u}$ to satisfy our system, we require:
  $$\boldsymbol{\Psi}\underline{u}' = \underline{g} \implies \underline{u}' = \boldsymbol{\Psi}^{-1}\underline{g}$$
  where we know $\boldsymbol{\Psi}^{-1}$ exists, since its a fundamental matrix
  - to find $\underline{u}$, we can integrate:
  $$\underline{u}(t) = \underline{c} + \int \boldsymbol{\Psi}^{-1}(t)\underline{g}(t)dt$$
  - to find $\underline{x}_p$, we multiply through by $\boldsymbol{\Psi}$:
  $$\underline{x}_p = \boldsymbol{\Psi}(t)\underline{c} + \boldsymbol{\Psi}(t)\int_{t^*}^{t} \boldsymbol{\Psi}^{-1}(s)\underline{g}(s)dt$$
  where t^* is any point at which $\boldsymbol{P}(t)$ and $\underline{g}(t)$ are continuous
- **Can variation of parameters be used to solve an IVP?**
  - assume the initial condition $\underline{x}(t_0) = \underline{x_0}$
  - the general solution will be:
  $$\underline{x}_p = \boldsymbol{\Psi}(t)\underline{c} + \boldsymbol{\Psi}(t)\int_{t_0}^{t} \boldsymbol{\Psi}^{-1}(s)\underline{g}(s)dt$$

– at $t = t_0$:

$$\underline{x}_0 = \mathbf{\Psi}(t_0)\underline{c} \implies \underline{c} = \mathbf{\Psi}(t_0)^{-1}\underline{x}_0$$

– so the solution to the IVP is:

$$\underline{x}_p = \mathbf{\Psi}(t)\mathbf{\Psi}(t_0)^{-1}\underline{x}_0 + \mathbf{\Psi}(t)\int_{t_0}^{t} \mathbf{\Psi}^{-1}(s)\underline{g}(s)dt$$

– if we use the special fundamental matrix $e^{\boldsymbol{A}t}$ (given constant coefficients):

$$\underline{x}_p = e^{\boldsymbol{A}t}\underline{x}_0 + e^{\boldsymbol{A}t}\int_{t_0}^{t} e^{-\boldsymbol{A}s}\underline{g}(s)dt$$

where we have used the fact that $e^{\boldsymbol{A}t} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)$

1.4 Macro Example

Solve the nonhomogeneous system:

$$\underline{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\underline{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

The first step is to find the homogeneous solution. For that, we find the eigenvalues and eigenvectors.

For the eigenvalues, we solve the system:

$$\begin{pmatrix} -2-r & 1 \\ 1 & -2-r \end{pmatrix}\underline{\xi} = \underline{0}$$

which results in:

$$\begin{vmatrix} -2-r & 1 \\ 1 & -2-r \end{vmatrix} = 0$$
$$\implies (2+r)^2 - 1 = 0$$
$$\implies r = -1, -3$$

If $r_1 = -1$, the eigenvector $\underline{\xi}_1$ is given by:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\underline{\xi}_1 = \underline{0} \implies -\xi_1 + \xi_2 = 0$$

so:

$$\underline{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If $r_2 = -3$, the eigenvector $\underline{\xi}_2$ is given by:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\underline{\xi}_2 = \underline{0} \implies \xi_1 + \xi_2 = 0$$

so:

$$\underline{\xi}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Thus, the general solution to the homogeneous system is:

$$\underline{x}_h(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

1.4.1 Diagonalisation

We define the transformation matrix:

$$\boldsymbol{T} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\boldsymbol{T}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

We define a new variable $\underline{y}$, which satisfies:

$$\underline{x} = \boldsymbol{T}\underline{y}$$

But this results in:

$$\begin{aligned}
&\underline{y}' = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}\underline{y} + \boldsymbol{T}^{-1}\underline{g}(t) \\
\implies &\underline{y}' = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} \\
\implies &\underline{y}' = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}\underline{y} + \frac{1}{2}\begin{pmatrix} 2e^{-t} + 3t \\ 2e^{-t} - 3t \end{pmatrix}
\end{aligned}$$

Thus, we need to solve:

$$\begin{aligned}
y_1' = -y_1 + e^{-t} + \frac{3}{2}t &\implies y_1' + y_1 = e^{-t} + \frac{3}{2}t \\
y_2' = -3y_2 + e^{-t} - \frac{3}{2}t &\implies y_2' + 3y_2 = e^{-t} - \frac{3}{2}t
\end{aligned}$$

These have integrating factor e^t and e^{3t} respectively:

$$\begin{aligned}
y_1' e^t + y_1 e^t = 1 + \frac{3}{2}te^t &\implies \frac{d}{dt}\left(y_1 e^t\right) = 1 + \frac{3}{2}te^t \\
y_2' e^{3t} + 3y_2 e^{3t} = e^{2t} - \frac{3}{2}te^{3t} &\implies \frac{d}{dt}\left(y_2 e^{3t}\right) = e^{2t} - \frac{3}{2}te^{3t}
\end{aligned}$$

Integrating both sides with respect to t:

$$\begin{aligned}
y_1 &= e^{-t}\int 1 + \frac{3}{2}te^t dt \\
y_2 &= e^{-3t}\int e^{2t} - \frac{3}{2}te^{3t} dt
\end{aligned}$$

We can use integration by parts to compute:

$$\int te^{nt}$$

We have:

$$u = t \implies du = 1$$
$$dv = e^{nt} \implies v = \frac{e^{nt}}{n}$$

So:

$$\begin{aligned}\int te^{nt} &= t\frac{e^{nt}}{n} - \frac{1}{n}\int e^{nt}dt \\ &= t\frac{e^{nt}}{n} - \frac{e^{nt}}{n^2} + C \\ &= e^{nt}\left(\frac{tn-1}{n^2}\right) + C\end{aligned}$$

Going back to the above:

$$\begin{aligned}y_1 &= e^{-t}\int 1 + \frac{3}{2}te^{t}dt \\ &= e^{-t}\left(t + \frac{3}{2}e^{t}(t-1) + C_1\right) \\ &= te^{-t} + \frac{3}{2}t - \frac{3}{2} + C_1e^{-t} \\ y_2 &= e^{-3t}\int e^{2t} - \frac{3}{2}te^{3t}dt \\ &= e^{-3t}\left(\frac{1}{2}e^{2t} - \frac{3}{2}e^{3t}\left(\frac{3t-1}{9}\right) + C_2\right) \\ &= \frac{1}{2}e^{-t} - \frac{3}{2}\left(\frac{3t-1}{9}\right) + C_2e^{-3t} \\ &= \frac{1}{2}e^{-t} + \frac{3-9t}{18} + C_2e^{-3t} \\ &= \frac{1}{2}e^{-t} + \frac{1}{6} - \frac{t}{2} + C_2e^{-3t}\end{aligned}$$

So:

$$\underline{y} = \begin{pmatrix} te^{-t} + \frac{3}{2}t - \frac{3}{2} + C_1e^{-t} \\ \\ \frac{1}{2}e^{-t} + \frac{1}{6} - \frac{t}{2} + C_2e^{-3t} \end{pmatrix}$$

From which it follows that:

$$\begin{aligned}
\underline{x} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \underline{y} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} te^{-t} + \frac{3}{2}t - \frac{3}{2} + C_1 e^{-t} \\ \frac{1}{2}e^{-t} + \frac{1}{6} - \frac{t}{2} + C_2 e^{-3t} \end{pmatrix} \\
&= \begin{pmatrix} te^{-t} + \frac{3}{2}t - \frac{3}{2} + C_1 e^{-t} + \frac{1}{2}e^{-t} + \frac{1}{6} - \frac{t}{2} + C_2 e^{-3t} \\ te^{-t} + \frac{3}{2}t - \frac{3}{2} + C_1 e^{-t} - \frac{1}{2}e^{-t} - \frac{1}{6} + \frac{t}{2} - C_2 e^{-3t} \end{pmatrix} \\
&= \begin{pmatrix} te^{-t} + t - \frac{4}{3} + \frac{1}{2}e^{-t} + C_1 e^{-t} + C_2 e^{-3t} \\ te^{-t} + 2t - \frac{5}{3} - \frac{1}{2}e^{-t} + C_1 e^{-t} - C_2 e^{-3t} \end{pmatrix} \\
&= C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} t + \frac{1}{2} \\ t - \frac{1}{2} \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}
\end{aligned}$$

1.4.2 Undetermined Coefficients

We can write $\underline{g}(t)$ as:

$$\underline{g}(t) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} t$$

Thus, our solution must look like the sum of an exponential, and a polynomial of degree 1. However, notice that $r = -1$ is an eigenvalue of $\boldsymbol{A}$, so the exponential will be multiplied by a polynomial of degree 1 (since $r_1 = -1$ has algebraic multiplicity 1). Thus, we expect a particular solution of the form:

$$\underline{x}_p = (\underline{a}t + \underline{b})e^{-t} + \underline{c}t + \underline{d}$$

where $(\underline{a}t + \underline{b})e^{-t}$ corresponds to the exponential part of $\underline{g}(t)$, and $\underline{c}t + \underline{d}$ corresponds to the polynomial part.

Since $\underline{x}_p$ must satisfy:

$$\underline{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \underline{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

we can substitute in. However, we can do so separately. Let:

$$\underline{x}_p^{(1)} = (\underline{a}t + \underline{b})e^{-t}$$

$$\underline{x}_p^{(2)} = \underline{c}t + \underline{d}$$

We first consider the exponential part. Differentiating:

$$\underline{x}_p^{(1)'} = -(\underline{a}t + \underline{b})e^{-t} + \underline{a}e^{-t} = (-\underline{a}t - \underline{b} + \underline{a})e^{-t}$$

If we plug in to the system (considering only the exponential part of $\underline{g}(t)$:

$$\begin{aligned}
(-\underline{a}t - \underline{b} + \underline{a})e^{-t} &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} (\underline{a}t + \underline{b})e^{-t} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t} \\
\implies \quad -\underline{a}t - \underline{b} + \underline{a} &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} (\underline{a}t + \underline{b}) + \begin{pmatrix} 2 \\ 0 \end{pmatrix}
\end{aligned}$$

From which we can extract 2 systems, separated by the power of t involved:

$$-\underline{a} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \underline{a}$$

$$\underline{a} - \underline{b} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \underline{b} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \implies \underline{a} - \underline{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \underline{b}$$

The first system we recognise as the eigenvector/eigenvalue system, from which we know that $\underline{a}$ must be a **multiple** of $\underline{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. It is important that we consider the multiple, since $\underline{\xi}_1$ was just a particular eigenvector choice, but definitely not the only possible solution to the eigenvector problem. By parametrising $\underline{a}$, we will be able to solve the second system. In other words,

$$\underline{a} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$$

For the second system, we have:

$$\left(\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} + \mathbb{I} \right) \underline{b} = \underline{a} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \underline{b} = \begin{pmatrix} \alpha - 2 \\ \alpha \end{pmatrix}$$

Notice, the row vectors of the matrix are multiples of each other (differ by factor of -1). Since the rows are linearly dependent, it follows that the system will have a solution if the RHS also has this linear dependence. Thus:

$$\alpha - 2 = -\alpha \implies \alpha = 1$$

In other words, the solution of the system must solve:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \underline{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \implies b_2 = -1 + b_1$$

So:

$$\underline{b} = \begin{pmatrix} k \\ k-1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We can just pick $k = 0$, as the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ will already be included by the general solution. From this, we get that:

$$\underline{x}_p^{(1)'} = \left[-\begin{pmatrix} 1 \\ 1 \end{pmatrix} t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] e^{-t} \implies \underline{x}_p^{(1)'} = \begin{pmatrix} 1 - t \\ 2 - t \end{pmatrix} e^{-t}$$

We now consider the polynomial part. Differentiating:

$$\underline{x}_p^{(2)'} = \underline{c}$$

If we plug in to the system (considering only the polynomial part of $g(t)$):

$$\underline{c} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} (\underline{c}t + \underline{d}) + \begin{pmatrix} 0 \\ 3 \end{pmatrix} t$$

From which we can extract 2 systems, separated by the power of t involved:

$$\underline{c} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \underline{d}$$

$$\underline{0} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \underline{c} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Which leads to the equations:

$$\begin{aligned} -2d_1 + d_2 &= c_1 \\ d_1 - 2d_2 &= c_2 \\ -2c_1 + c_2 &= 0 \\ c_1 - 2c_2 &= -3 \end{aligned}$$

We can solve for c_1, c_2 by multiplying the third equation by 2, and adding it to the fourth equation:

$$-3c_1 = -3 \implies c_1 = 1 \;\therefore\; c_2 = 2$$

Thus, we now just need to solve:

$$\begin{aligned} -2d_1 + d_2 &= 1 \\ d_1 - 2d_2 &= 2 \end{aligned}$$

Applying the same operation as for the c's:

$$-3d_1 = 4 \implies d_1 = -\frac{4}{3} \;\therefore\; d_2 = -\frac{5}{3}$$

Thus, it follows that the particular solution is:

$$\underline{x}_p = \begin{pmatrix} 1-t \\ 2-t \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Whilst this solution does not look identical to the one in diagonalisation, had we chosen $k = \frac{1}{2}$, we would've obtained the same particular solution.

1.4.3 Variation of Parameters

We first need to construct a fundamental matrix for the system.

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$$

Assuming a solution of the form $\underline{x} = \mathbf{\Psi}(t)\underline{u}(t)$, we reach the system:

$$\begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

We could solve this system by taking the inverse of the fundamental matrix. However, we can also solve via row reduction:

$$\begin{aligned} &\left(\begin{array}{cc|c} e^{-3t} & e^{-t} & 2e^{-t} \\ -e^{-3t} & e^{-t} & 3t \end{array}\right) \\ \implies &\left(\begin{array}{cc|c} e^{-3t} & e^{-t} & 2e^{-t} \\ 0 & 2e^{-t} & 3t + 2e^{-t} \end{array}\right) \\ \implies &\left(\begin{array}{cc|c} e^{-3t} & e^{-t} & 2e^{-t} \\ 0 & e^{-t} & \frac{3}{2}t + e^{-t} \end{array}\right) \\ \implies &\left(\begin{array}{cc|c} e^{-3t} & 0 & e^{-t} - \frac{3}{2}t \\ 0 & e^{-t} & \frac{3}{2}t + e^{-t} \end{array}\right) \end{aligned}$$

From which it follows that:

$$u_1' = e^{2t} - \frac{3}{2}te^{3t}$$
$$u_2' = 1 + \frac{3}{2}te^{t}$$

If we recall that $\int te^{nt} = e^{nt}\left(\frac{tn-1}{n^2}\right) + C$, then:

$$\begin{aligned} u_1 &= \int e^{2t} - \frac{3}{2}te^{3t}dt \\ &= \frac{1}{2}e^{2t} - \frac{3}{2}\left(\frac{3t-1}{9}\right)e^{3t} + C_1 \\ &= \frac{1}{2}e^{2t} - \frac{t}{2}e^{3t} + \frac{1}{6}e^{3t} + C_1 \end{aligned}$$

$$\begin{aligned} u_2 &= \int 1 + \frac{3}{2}te^{t}dt \\ &= t + \frac{3}{2}(t-1)e^{t} + C_2 \end{aligned}$$

We can now reconstruct $\underline{x}$:

$$\begin{aligned}
\underline{x} &= \mathbf{\Psi}(t)\underline{u}(t) \\
&= \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{2}e^{2t} - \frac{t}{2}e^{3t} + \frac{1}{6}e^{3t} + C_1 \\ t + \frac{3}{2}(t-1)e^{t} + C_2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}e^{-t} - \frac{t}{2} + \frac{1}{6} + C_1e^{-3t} + te^{-t} + \frac{3}{2}(t-1) + C_2e^{-t} \\ -\frac{1}{2}e^{-t} + \frac{t}{2} - \frac{1}{6} - C_1e^{-3t} + te^{-t} + \frac{3}{2}(t-1) + C_2e^{-t} \end{pmatrix} \\
&= \begin{pmatrix} te^{-t} + t - \frac{4}{3} + \frac{1}{2}e^{-t} + C_1e^{-t} + C_2e^{-3t} \\ te^{-t} + 2t - \frac{5}{3} - \frac{1}{2}e^{-t} + C_1e^{-t} - C_2e^{-3t} \end{pmatrix} \\
&= C_1e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} t + \frac{1}{2} \\ t - \frac{1}{2} \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}
\end{aligned}$$

which is identical to the solution obtain using diagonalisation.

2 Numerical Methods

2.1 Euler Method

2.2 Errors in ODEs

2.3 Higher Order Methods

2.4 Multistep Numerical Methods