Honours Differential Equations - Week 2 - Fundamental Matrices and Repeated Eigenvalues

Antonio León Villares

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Contents

1 Fundamental Matrices

1.1 Properties of Fundamental Matrices

- **If we have a system $\underline{x}' = \boldsymbol{A}\underline{x}$, what is a fundamental matrix for this system?**
  - a system of ODEs can have many fundamental matrices
  - if a system has a fundamental set of solutions $\underline{x}^{(1)}(t), \underline{x}^{(2)}(t), \ldots, \underline{x}^{(n)}(t)$, then a fundamental matrix is:
$$\boldsymbol{\Psi}(t) = \left(\underline{x}^{(1)}(t) \ \underline{x}^{(2)}(t) \ \ldots, \underline{x}^{(n)}(t)\right)$$
- **What are the properties of the fundamental matrix?**
  1. it is invertible, since $det\boldsymbol{\Psi}(t) = W(t) \neq 0$
  2. if the general solution of $\underline{x}' = \boldsymbol{A}\underline{x}$ is:
$$\underline{x}(t) = c_1\underline{x}^{(1)}(t) + c_2\underline{x}^{(2)}(t) + \ldots + c_n\underline{x}^{(n)}(t)$$
     then:
$$\underline{x}(t) = \boldsymbol{\Psi}(t)\underline{c}$$
     where
$$\underline{c} = \begin{pmatrix} c_! \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$
  3. the fundamental matrix also satisfies the system:
$$\boldsymbol{\Psi}'(t) = \boldsymbol{A}\boldsymbol{\Psi}(t)$$
     This is because each column vector in $\boldsymbol{\Psi}(t)$ satisfies the system, so it follows that:
$$\boldsymbol{\Psi}'(t)_{ij} = \boldsymbol{A}\boldsymbol{\Psi}(t)_{ij}$$
- **How can we use a fundamental matrix to specify a particular solution to the system?**
  - if we have an initial value problem, and we know that $\underline{x}(t) = \boldsymbol{\Psi}(t)\underline{c}$ is a general solution, if we need to satisfy $\underline{x}(t_0) = \underline{x}_0$, then:
$$\begin{aligned} \underline{x}(t_0) = \boldsymbol{\Psi}(t_0)\underline{c} = \underline{x_0} \quad &\implies \underline{c} = \boldsymbol{\Psi}^{-1}(t_0)\underline{x}_0 \\ \therefore \quad &\underline{x} = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)\underline{x}_0 \end{aligned}$$
    satisfies the initial value problem

1.2 The Special Fundamental Matrix

- **What other types of fundamental matrices exist/are relevant?**
  - another very relevant fundamental matrix is $\boldsymbol{\Phi}(t)$, constructed by using vectors, such that for the initial condition:
$$\boldsymbol{\Phi}(t_0) = \mathbb{I}$$
  - this matrix is particularly useful when finding particular solutions, as $\boldsymbol{\Phi}^{-1}(t_0) = \mathbb{I}$, so it follows that if $\underline{x}(t_0) = \underline{x}_0$ is our initial condition, then the particular solution is:
$$\underline{x} = \boldsymbol{\Phi}(t)\underline{x}_0$$

- thus, solving an initial value problem gets reduced to simple matrix multiplication, whilst if we use $\mathbf{\Psi}(t)$ we would need to compute a matrix inverse
- one can think of $\mathbf{\Phi}(t)$ as a linear transformation from the initial conditions to the value of the solution at time t

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For example, if we have the system

$$\underline{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \underline{x}$$

with general solution:

$$\underline{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

We have:

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

If we want the special fundamental matrix, we need to satisfy:

$$\underline{x}^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\underline{x}^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Notice, for $x^{(1)}(t)$, we can define $c_1 = c_2 = \frac{1}{2}$, such that:

$$\underline{x}^{(1)}(t) = \frac{1}{2} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + \frac{1}{2} \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t} \end{pmatrix}$$

So indeed:

$$\underline{x}^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Similarly, for $x^{(2)}(t)$, we can define $c_1 = \frac{1}{4}, c_2 = -\frac{1}{4}$, such that:

$$\underline{x}^{(1)}(t) = \frac{1}{4} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} - \frac{1}{4} \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} = \begin{pmatrix} \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}$$

So indeed:

$$\underline{x}^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus, our special fundamental matrix is:

$$\mathbf{\Phi}(t) = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}$$

Indeed, $\mathbf{\Phi}(t)$ is much harder to express than $\mathbf{\Psi}(t)$, but it is easier to use when computing solutions for initial value problems.

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2 The Matrix Exponential

- **How is the exponential of a matrix defined?**
  - recall, the Taylor Expansion of the exponential function is:

$$e^{at} = \sum_{n=0}^{\infty} \frac{(at)^n}{n!}$$

    and this power series converges for all t
  - we can use this expansion to *define* the matrix exponential:

$$e^{\boldsymbol{A}t} = \sum_{n=0}^{\infty} \frac{(\boldsymbol{A}t)^n}{n!}$$

    and each element of $e^{\boldsymbol{A}t}$ converges for all t
  - alternatively, it can be defined as the matrix which satisfies the differential equation:

$$\frac{d}{dt}\left(e^{\boldsymbol{A}t}\right) = \boldsymbol{A}e^{\boldsymbol{A}t}$$

  - alternatively, define it as a limit:

$$e^{\boldsymbol{A}t} = \lim_{n\to\infty}\left(\mathbb{I} + \frac{1}{n}\boldsymbol{A}\right)^n$$

- **How is the matrix exponential related to fundamental matrices?**
  - notice, from the differential definition of the matrix exponential, it follows that the matrix exponential satisfies the system:

$$e^{\boldsymbol{A}t\prime} = \boldsymbol{A}e^{\boldsymbol{A}t}$$

  - thus, it follows that the matrix exponential must be a **fundamental matrix**
  - moreover, notice that:

$$e^{\boldsymbol{A}0} = \sum_{n=0}^{\infty} \frac{(\boldsymbol{A}0)^n}{n!} = \mathbb{I}$$

  - in other words, not only is $e^{\boldsymbol{A}t}$ a fundamental matrix, but it also evaluates to the identity when $t = 0$. Thus, by the existence and uniqueness theorem, it must be the case that:

$$\boldsymbol{\Phi}(t) = e^{\boldsymbol{A}t}$$

  - thus, a solution to an IVP can be given by:

$$\underline{x} = e^{\boldsymbol{A}t}\underline{x}_0$$

- **How can a matrix exponential be computed from a Fundamental Matrix?**
  - we can express a particular solution to our system via:

$$\underline{x} = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0)\underline{x}_0$$

    or via:

$$\underline{x} = e^{\boldsymbol{A}t}\underline{x}_0$$

    From which it follows that:

$$e^{\boldsymbol{A}t} = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0)$$

– notice that if $\boldsymbol{B}$ is a matrix, then $\boldsymbol{\Psi}(t)\boldsymbol{B}$ will also be a fundamental matrix:

$$\begin{aligned}\frac{d}{dt}\left(\boldsymbol{\Psi}(t)\boldsymbol{B}\right) &= \boldsymbol{\Psi}'(t)\boldsymbol{B} \\ &= \boldsymbol{A}\boldsymbol{\Psi}(t)\boldsymbol{B} \\ &= \boldsymbol{A}(\boldsymbol{\Psi}(t)\boldsymbol{B})\end{aligned}$$

3 Matrix Diagonalisation

- **Why are diagonal matrices desirable?**
  1. the determinant of a diagonal matrix is the product of its entries in the main diagonal
  2. systems defined by a diagonal matrix are immediately solvable (can just "read off" the eigenvalues from the diagonal)
  3. if:

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

     then:

$$\boldsymbol{A}^k = \begin{pmatrix} a_{11}^k & 0 & 0 & \dots & 0 \\ 0 & a_{22}^k & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \dots & a_{nn}^k \end{pmatrix}$$

  4. it $\boldsymbol{A}$ is a diagonal matrix, its matrix exponential is just:

$$diag(e^{A_{11}}, e^{A_{22}}, \dots, e^{A_{nn}})$$

     To see why, we can use (3), alongside the Taylor series expansion, and notice that at the diagonal entries we will just have the Taylor expansion of the exponential.

- **How can one turn a matrix $\boldsymbol{A}$ into a diagonal matrix?**
  – a matrix is *diagonalisable* if and only if for each eigenvalue, its algebraic multiplicity is equal to its geometric multiplicity
  – if the above holds, then we can define 2 matrices $\boldsymbol{T}$ and $\boldsymbol{D}$, such that:

$$\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} = \boldsymbol{D}$$

    where $\boldsymbol{T}$ is an invertible matrix, and $\boldsymbol{D}$ is a diagonal matrix
  – in particular:
    * $\boldsymbol{T}$ is the matrix we obtain by using the n linearly independent eigenvectors of $\boldsymbol{A}$ as columns
    * $\boldsymbol{D}$ is the matrix we obtain by using the eigenvalues as the entries of the diagonal
  – to see how the formula works, let:

$$\boldsymbol{T} = \begin{pmatrix} \underline{v}_1 & \underline{v}_2 & \dots & \underline{v}_n \end{pmatrix}$$

$$\boldsymbol{D} = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

But then:

$$\boldsymbol{AT} = \boldsymbol{A}\begin{pmatrix}\underline{v}_1 & \underline{v}_2 & \dots & \underline{v}_n\end{pmatrix} = \begin{pmatrix}\boldsymbol{A}\underline{v}_1 & \boldsymbol{A}\underline{v}_2 & \dots & \boldsymbol{A}\underline{v}_n\end{pmatrix} = \begin{pmatrix}\lambda_1\underline{v}_1 & \lambda_1\underline{v}_2 & \dots & \lambda_1\underline{v}_n\end{pmatrix}$$

$$\boldsymbol{TD} = \begin{pmatrix}\underline{v}_1 & \underline{v}_2 & \dots & \underline{v}_n\end{pmatrix}\begin{pmatrix}\lambda_1 & 0 & 0 & \dots & 0\\ 0 & \lambda_1 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & 0\\ 0 & 0 & 0 & \dots & \lambda_n\end{pmatrix} = \begin{pmatrix}\lambda_1\underline{v}_1 & \lambda_1\underline{v}_2 & \dots & \lambda_1\underline{v}_n\end{pmatrix}$$

Thus, it follows that:

$$\boldsymbol{AT} = \boldsymbol{TD} \implies \boldsymbol{T^{-1}AT} = \boldsymbol{D}$$

- **How can we use diagonalisation to compute the matrix exponential?**
  - if $\boldsymbol{A}$ is diagonalisable, then:

    $$\boldsymbol{T^{-1}AT} = \boldsymbol{D} \implies \boldsymbol{A} = \boldsymbol{TDT^{-1}}$$

    from which it follows that:

    $$e^{\boldsymbol{A}t} = \boldsymbol{T}e^{\boldsymbol{D}t}\boldsymbol{T^{-1}} = \boldsymbol{T}diag(e^{r_1t}, e^{r_2t}, \dots, e^{r_nt})\boldsymbol{T^{-1}}$$

  - to see why $e^{\boldsymbol{A}t} = \boldsymbol{T}e^{\boldsymbol{D}t}\boldsymbol{T^{-1}}$, we use the fact that:

    $$(\boldsymbol{TDT^{-1}})^n = \boldsymbol{TDT^{-1}TDT^{-1}TDT^{-1}}\dots\boldsymbol{TDT^{-1}} = \boldsymbol{TD^nT^{-1}}$$

    so in particular:

    $$\begin{aligned} e^{\boldsymbol{A}t} &= e^{\boldsymbol{TDT^{-1}}t} \\ &= \sum_{n=0}^{\infty}\frac{(\boldsymbol{TDT^{-1}}t)^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{\boldsymbol{TD^nT^{-1}}t^n}{n!} \\ &= \boldsymbol{T}\sum_{n=0}^{\infty}\frac{\boldsymbol{D^n}t^n}{n!}\boldsymbol{T^{-1}} \\ &= \boldsymbol{T}e^{\boldsymbol{D}t}\boldsymbol{T^{-1}} \end{aligned}$$

- **How can we use diagonalisation to find a fundamental matrix for a system of ODEs?**
  - if we have the system $\underline{x}' = \boldsymbol{A}\underline{x}$, one way of solving it is by assuming the existence of a solution of the form $\underline{x} = \underline{\xi}e^{rt}$
  - we can reach solutions by using **diagonalisation**
  - lets assume that $\boldsymbol{A}$ has n linearly independent eigenvectors
  - define $\underline{x} = \boldsymbol{T}\underline{y}$, where $\boldsymbol{T}$ is the matrix of eigenvectors (we can think of $\boldsymbol{T}$ as leading to a change of basis)
  - using this in the system:

    $$\begin{aligned} & \underline{x}' = \boldsymbol{A}\underline{x} \\ \implies\ & \boldsymbol{T}\underline{y}' = \boldsymbol{AT}\underline{y} \\ \implies\ & \underline{y}' = \boldsymbol{T^{-1}AT}\underline{y} \\ \implies\ & \underline{y}' = \boldsymbol{D}\underline{y} \end{aligned}$$

But this is very easy to solve, as we have a system of n equations of the form:

$$y_i' = r_i y_i$$

which has solution $y_i = e^{r_i t}$

- thus, a fundamental matrix for the system $\underline{y}' = \boldsymbol{D}\underline{y}$ is just $diag(e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}) = e^{\boldsymbol{D}t}$
- since $\underline{x} = \boldsymbol{T}\underline{y}$ the fundamental matrix for the system $\underline{x}' = \boldsymbol{A}\underline{x}$ will be the product of $\boldsymbol{T}$ and the fundamental matrix for the system with $\underline{y}$, so:

$$\boldsymbol{\Psi}(t) = \boldsymbol{T}e^{\boldsymbol{D}t} = \begin{pmatrix} \underline{\xi_1} e^{r_1 t} & \underline{\xi_2} e^{r_2 t} & \ldots & \underline{\xi_n} e^{r_n t} \end{pmatrix}$$

which is precisely what we expected

4 Solving Systems With Repeated Eigenvalues

- **Can we use methods from linear ODEs with repeated roots to solve systems with repeated eigenvalues?**
  - when we had a differential equation:

$$a\ddot{y} + b\dot{y} + cy = 0$$

we considered the characteristic polynomial:

$$ar^2 + br + c = 0$$

If this had a repeated root r, then a basis for solutions was given by:

$$e^{rt} \qquad te^{rt}$$

  - this does **not** work for systems of ODEs. Consider the system $\underline{x}' = \boldsymbol{A}\underline{x}$. Assume we have found a solution $\underline{x}^{(2)} = \underline{\xi}e^{rt}$. Consider $\underline{x}^{(2)} = \underline{\xi}te^{rt}$. Plugging in to the equation:

$$\underline{\xi}(e^{rt} + rte^{rt}) = \boldsymbol{A}\underline{\xi}te^{rt}$$

But then, using t as a "coefficient", and matching coefficients, we must satisfy:

$$t\boldsymbol{A}\underline{\xi} = rt\underline{\xi}$$

$$\underline{\xi}e^{rt} = 0$$

The latter implies $\underline{\xi} = \underline{0}$, but we require non-zero eigenvectors. Thus, $\underline{\xi}te^{rt}$ doesn't solve the system.

- **What is the general form for a system of ODEs with repeated eigenvalues?**
  - assume $\underline{x}' = \boldsymbol{A}\underline{x}$ has repeated eigenvalue r, and we have found that $\underline{x}^{(1)} = \underline{\xi}e^{rt}$ is a solution
  - a second solution can be found by considering:

$$\underline{x}^{(2)}(t) = \underline{\xi}te^{rt} + \underline{\eta}e^{rt}$$

  - we can confirm this by plugging it into the system:

$$\begin{aligned} & \underline{\xi}e^{rt} + \underline{\xi}rte^{rt} + \underline{\eta}re^{rt} = \boldsymbol{A}(\underline{\xi}te^{rt} + \underline{\eta}e^{rt}) \\ \Longrightarrow \quad & \underline{\xi} + \underline{\xi}rt + \underline{\eta}r = \boldsymbol{A}(\underline{\xi}t + \underline{\eta}) \\ \Longrightarrow \quad & (\underline{\xi} + \underline{\eta}r) + \underline{\xi}rt = t\boldsymbol{A}\underline{\xi} + \boldsymbol{A}\underline{\eta} \end{aligned}$$

Thus, we must satisfy:

$$\boldsymbol{A}\underline{\xi} = \underline{\xi}r$$

$$\boldsymbol{A}\underline{\eta} = \underline{\xi} + \underline{\eta}r \implies (\boldsymbol{A} - r\mathbb{I})\underline{\eta} = \underline{\xi}$$

The first is our standard eigenvalue equation, which we know has a solution ($\underline{x}^{(1)}$). The second equation is one that we are **guaranteed** to have a solution for.

– the vector $\underline{\eta}$ is known as the **generalised eigenvector** of matrix $\boldsymbol{A}$

5 Jordan Forms

- **Do repeated eigenvalues affect Fundamental Matrices?**
  - no, Fundamental Matrices can be calculated in the same way, as we are capable of producing n linearly independent solutions to the system
- **Can we diagonalise the matrices with repeated eigenvalues?**
  - only if the algebraic multiplicity is the same as the geometric multiplicity
  - if we have more eigenvalues than eigenvectors, the next best thing after diagonalisation is to put it in **Jordan Form**, with eigenvalues in the main diagonal, ones in positions above the main diagonal, and zeros elsewhere
- **How can Jordan Form matrices be constructed?**
  - for the transformation matrix $\boldsymbol{T}$, since we no longer have n linearly independent eigenvectors, we must make use of the generalised eigenvectors, $\underline{\eta}$
  - then, the transformation will be:

$$\boldsymbol{T^{-1}AT} = \boldsymbol{J}$$

  where $\boldsymbol{J}$ is our matrix in jordan form. For example,

$$\boldsymbol{J} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

  is the Jordan Form for a matrix $\boldsymbol{A}$ with repeated eigenvalue $r = 2$
  - from linear algebra, every matrix can be transformed into a diagonal matrix, or into a Jordan form
  - as in the case above, we can also use the transformation $\underline{x} = \boldsymbol{T}\underline{y}$
- **How useful are Jordan forms?**
  - matrix systems with Jordan Form are particularly simple to solve. For example:

$$\underline{y}' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \underline{y}$$

  is equivalent to the system:

$$y_2' = \lambda y_2$$

$$y_1' = \lambda y_1 + y_2$$

  The first equation is a standard first order ODE which can be solved. Then, the solution can be used in the second equation to find y_1.
- **What is the format of a exponential matrix in Jordan Form?**

- let:

$$\boldsymbol{J}_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

  Then:

$$e^{\boldsymbol{J}_\lambda t} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

- to see why, notice that:

$$\boldsymbol{J}_\lambda = diag(\lambda, \lambda) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- most importantly, these 2 components commute. For matrix exponentials,

$$e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}} e^{\boldsymbol{B}} \iff \boldsymbol{AB} = \boldsymbol{BA}$$

- thus, it follows that:

$$e^{\boldsymbol{J}_\lambda t} = e^{diag(\lambda t, \lambda t)} e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t}$$

- lastly, notice:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

  so from the series expansion, it follows that:

$$e^{\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}} = \mathbb{I} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

- so finally:

$$e^{\boldsymbol{J}_\lambda t} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

- **How can we generalise for cases in which $n > 2$?**
  - if we have a 3×3 system, with an eigenvalue r with algebraic multiplicity 3 and geometric multiplicity 1, then:

$$\begin{aligned} \underline{x}^{(1)}(t) &= \underline{\xi} e^{rt} \\ \underline{x}^{(2)}(t) &= t\underline{\xi} e^{rt} + \underline{\eta} e^{rt}, \qquad \underline{\xi} = (\boldsymbol{A} - r\mathbb{I})\underline{\eta} \\ \underline{x}^{(3)}(t) &= \frac{t^2}{2}\underline{\xi} e^{rt} + t\underline{\eta} e^{rt} + \underline{\zeta} e^{rt}, \qquad \underline{\eta} = (\boldsymbol{A} - r\mathbb{I})\underline{\zeta} \end{aligned}$$

  - if we have a 3×3 system, with an eigenvalue r with algebraic multiplicity 3 and geometric multiplicity 2, then:

$$\begin{aligned} \underline{x}^{(1)}(t) &= \underline{\xi}_1 e^{rt} \\ \underline{x}^{(2)}(t) &= \underline{\xi}_2 e^{rt} \\ \underline{x}^{(3)}(t) &= t(a\underline{\xi}_1 + b\underline{\xi}_2) e^{rt} + \underline{\eta} e^{rt}, \qquad a\underline{\xi}_1 + b\underline{\xi}_2 = (\boldsymbol{A} - r\mathbb{I})\underline{\eta} \end{aligned}$$

6 Exercises

1. **Find a fundamental matrix for the system:**

$$\underline{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \underline{x}$$

Assuming a solution of the form $\underline{x} = \underline{\xi} e^{rt}$, we consider the system:

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} \underline{\xi} = \underline{0}$$

The determinant is:

$$\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)^2 - 4$$
$$\implies \quad r = -1, 3$$

Let $r_1 = -1$, then:

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \underline{\xi}_1 = \underline{0}$$

leads to:

$$2\xi_1 + \xi_2 = 0 \quad \implies \quad \underline{\xi} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Let $r_1 = 3$, then:

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \underline{\xi}_2 = \underline{0}$$

leads to:

$$-2\xi_1 + \xi_2 = 0 \quad \implies \quad \underline{\xi} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Thus, we get 2 solutions:

$$\underline{x}^{(1)}(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$
$$\underline{x}^{(2)}(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$$

So a fundamental matrix for the system is:

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{pmatrix}$$

Notice, the special fundamental matrix for this system was already discussed above in Section 1.2.

2. **Consider the matrix:**

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

Find the similarity transformation matrix $\boldsymbol{T}$ and show that $\boldsymbol{A}$ can be diagonalised

From the exercise above, we know that:

$$r_1 = -1 \implies \underline{\xi}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$r_2 = 3 \implies \underline{\xi}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Thus, the transformation matrix is:

$$\boldsymbol{T} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$$

Recall, the inverse of a 2×2 matrix $\boldsymbol{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is:

$$\frac{1}{det\boldsymbol{X}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

So it follows that:

$$\boldsymbol{T}^{-1} = -\frac{1}{4} \begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix}$$

Since the algebraic multiplicity is the same as the geometric multiplicity, and $\boldsymbol{T}$ has an inverse, we can diagonalise $\boldsymbol{A}$ (otherwise explicitly comput $\boldsymbol{T^{-1}AT}$ and ensure it gives a diagonal matrix).

3. **Consider once again the system of differential equations $\underline{x}' = \boldsymbol{A}\underline{x}$, where:**

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

Using the transformation $\underline{x} = \boldsymbol{T}\underline{y}$ where:

$$\boldsymbol{T} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$$

the above system reduces to the diagonal system:

$$\underline{y}' = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \underline{y}$$

Obtain a fundamental matrix for the system involving $\underline{y}$, and then transform it to obtain a fundamental matrix for the original system.

The exponential of the diagonal matrix gives:

$$\boldsymbol{Q} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}$$

Moreover, we know that $\boldsymbol{\Psi}(t) = \boldsymbol{TQ}$, so:

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

which corresponds with the fundamental matrix we obtained above.

4. **Find a fundamental set of solutions of:**

$$\underline{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \underline{x}$$

We compute the eigenvalues of the system:

$$\begin{aligned} \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} &= (1-r)(3-r)+1 \\ \implies \quad & r^2 - 4r + 4 = 0 \\ \implies \quad & (r-2)^2 = 0 \\ \therefore \quad & r = 2 \end{aligned}$$

Thus, we have an eigenvalue with algebraic multiplicity 2. To compute its corresponding eigenvector:

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \underline{\xi} = 0 \implies \underline{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Since $r = 2$ has geometric multiplicity 1, we seek an additional solution of the form:

$$\underline{x}^{(2)}(t) = \underline{\xi} t e^{2t} + \underline{\eta} e^{2t}$$

For this, we seek to solve:

$$(\boldsymbol{A} - 2\mathbb{I})\underline{\eta} = \underline{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This requires $-\eta_1 - \eta_2 = 1$, so if we parametrise via $\eta_1 = s$, then $\eta_2 = -s - 1$, so:

$$\underline{\eta} = \begin{pmatrix} s \\ -s-1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Thus, the general solution is:

$$\underline{x} = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \left(t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$$

But notice the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ appears twice, so the second one can be ignored by setting $s = 0$:

$$\underline{x} = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \left(t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right)$$

5. **Derive a Fundamental Matrix, the Special Fundamental Matrix and a Jordan Form for the system:**

$$\underline{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \underline{x}$$

Verify that we can re-derive a Fundamental Matrix from the Jordan Form

We computed the fundamental set of solution above, so:

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 & t \\ -1 & -t-1 \end{pmatrix}$$

For the special fundamental matrix, we recall that $\mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t)$, so:

$$\mathbf{\Psi}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \implies \mathbf{\Psi}^{-1}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

So:

$$\mathbf{\Phi}(t) = e^{2t}\begin{pmatrix} 1 & t \\ -1 & -t-1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} = e^{2t}\begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix}$$

Alternatively, we can use the exponential definition. Define:

$$\boldsymbol{T} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

$$\boldsymbol{T}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

$$\boldsymbol{J} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Then:

$$\begin{aligned} e^{\boldsymbol{A}t} &= \boldsymbol{T}e^{\boldsymbol{J}t}\boldsymbol{T}^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \\ &= e^{2t}\begin{pmatrix} 1 & t \\ -1 & -t-1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \\ &= e^{2t}\begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix} \\ &= \mathbf{\Phi}(t) \end{aligned}$$

as expected.

Moreover, notice that we have confirmed $\mathbf{\Psi}(t) = \boldsymbol{T}e^{\boldsymbol{J}t}$, in the calculation above, and thus, we confirm that we can go to $\mathbf{\Psi}(t)$ using the Jordan Form $\boldsymbol{J}$.

6. **Derive the Special Fundamental Matrix for the system:**

$$\underline{x}' = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\underline{x}$$

There are 2 ways to do this: either compute a fundamental matrix, or compute the matrix exponential. Either way, we require the eigenvalues and eigenvectors of the matrix

For the eigenvalues:

$$\begin{aligned} \begin{vmatrix} 1-r & 2 \\ 0 & 3-r \end{vmatrix} &= (1-r)(3-r) = 0 \\ \implies \quad & r = 1, 3 \end{aligned}$$

If $r_1 = 1$, then consider:

$$\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}\underline{\xi} = \underline{0}$$

from which it follows that:

$$\underline{\xi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

If $r_1 = 3$, then consider:

$$\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \underline{\xi} = \underline{0}$$

from which it follows that:

$$\underline{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus, the solutions are:

$$\underline{x}^{(1)}(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$$

$$\underline{x}^{(2)}(t) = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$$

Hence, a fundamental matrix will be:

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^t & e^{3t} \\ 0 & e^{3t} \end{pmatrix}$$

Recall, $\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0)$. Thus:

$$\boldsymbol{\Psi}(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \implies \boldsymbol{\Psi}^{-1}(0) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

So:

$$\boldsymbol{\Phi}(t) = \begin{pmatrix} e^t & e^{3t} \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & e^{3t} - e^t \\ 0 & e^{3t} \end{pmatrix}$$

Alternatively, we can use **diagonalisation**. Let:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

$$\boldsymbol{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{T}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

Thus:

$$\begin{aligned}
\boldsymbol{\Phi}(t) &= e^{\boldsymbol{A}t} \\
&= \boldsymbol{T} e^{\boldsymbol{D}t} \boldsymbol{T}^{-1} \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} e^t & e^{3t} \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} e^t & e^{3t} - e^t \\ 0 & e^{3t} \end{pmatrix}
\end{aligned}$$

as required.

7. **Solve the system:**

$$\underline{x}' = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \underline{x}$$

We first find the eigenvalues:

$$\begin{vmatrix} 3-r & 1 \\ -4 & -1-r \end{vmatrix} = -(3-r)(1+r)+4$$
$$\implies -(3+2r-r^2)+4=0$$
$$\implies r^2-2r+1=0$$
$$\implies (r-1)^2=0$$
$$\therefore r=1$$

Thus, the eigenvalue 1 has algebraic multiplicity 2. We determine its corresponding eigenvector:

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \underline{\xi} = \underline{0} \implies 2\xi_1 + \xi_2 = 0$$

Thus, it follows that:

$$\underline{\xi} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

So a solution to the system:

$$\underline{x}^{(1)}(t) = e^t \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Since $r = 1$ has geometric multiplicity 1, we require another solution. This shall be of the form:

$$\underline{x}^{(2)} = e^t \left(t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \underline{\eta} \right)$$

where $\underline{\eta}$ satisfies:

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \underline{\eta} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \implies 2\eta_1 + \eta_2 = 1$$

So then:

$$\underline{\eta} = \begin{pmatrix} s \\ 1-2s \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

We can choose $s = 0$, so that:

$$\underline{\eta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

And so, the general solution becomes:

$$\underline{x} = e^t \left(c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \left(t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right)$$

8. **Solve the system:**

$$\underline{x}' = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \underline{x}$$

The eigenvalues are given by:

$$\begin{vmatrix} 5-r & -3 & -2 \\ 8 & -5-r & -4 \\ -4 & 3 & 3-r \end{vmatrix} = (5-r)\begin{vmatrix} -5-r & -4 \\ 3 & 3-r \end{vmatrix} + 3\begin{vmatrix} 8 & -4 \\ -4 & 3-r \end{vmatrix} - 2\begin{vmatrix} 8 & -5-r \\ -4 & 3 \end{vmatrix}$$
$$= (5-r)\left[-(5+r)(3-r)+12\right] + 3\left[24-8r-16\right] - 2\left[24-4(5+r)\right]$$
$$= r^3 - 3r^2 + 3r - 1$$
$$\therefore \; (r-1)^3 = 0 \implies r = 1$$

$r = 1$ has algebraic multiplicity 3. We now compute the corresponding eigenvector:

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \underline{\xi} = \underline{0} \implies 4\xi_1 - 3\xi_2 - 2\xi_3 = 0$$

If we parametrise via $s = \xi_1$ and $t = \xi_2$, it follows that:

$$\underline{\xi} = \begin{pmatrix} s \\ t \\ 2s - \frac{3}{2}t \end{pmatrix}$$

From which we obtain 2 eigenvectors:

$$\underline{\xi}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\underline{\xi}_2 = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$$

Since the geometric multiplicity is 2, we are missing one solution. In particular, we seek a solution of the form:

$$\underline{x}^{(3)}(t) = e^t \left[t \left(a \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + b \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} \right) + \underline{\eta} \right]$$

where $\underline{\eta}$ satsifies:

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \underline{\eta} = a \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + b \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ 2a - 3b \end{pmatrix}$$

Notice, the rows of the matrix are multiples of each other ($\times 2$, $\times -1$), so the elements of the right vector must also be related in this way. Thus:

$$2a = 2b \implies a = b$$

$$a = b \quad \& \quad a = -(2a - 3b) \implies a = a$$

Thus, the system is always satisfied, so long as $a = b$, so we can pick $a = b = 1$, which leads to:

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \underline{\eta} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \implies 4\eta_1 - 3\eta_2 - 2\eta_3 = 1$$

(Note that we could have reached values of a, b by using EROs) Thus, we get:

$$\underline{\eta} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Hence, the general solution is:

$$\underline{x}(t) = e^t \left[c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} + c_3 t \left(\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) \right]$$