Honours Differential Equations - Week 1 - Systems of ODEs

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- Systems of ODEs can be expressed by using vectors
- Solutions for linear, homogeneous systems of ODEs can be derived by using linear combinations of linearly independent solutions to the system
- Given a set of n linearly independent solutions to a linear, homogeneous system of n ODEs, if their Wronskian is non-zero, then the set of solutions form a fundamental set of solutions
- By Abel's Theorem, the Wronskian needs to be only evaluated at a single point in an open interval to determine whether the solutions are linearly independent on the interval

2.1 General ODEs and Linear Algebra

- **When do systems of ODEs arise?**
  - whenever we have to model a system with **many dependent variables**, and each of these dependent variables are functions of the **same independent variable**
  - ***Example***: a system of 2 sprints, at positions x_1 and x_2, which depend on time t
- **What is the most general system of ODEs?**
  - let $i \in [1, n]$. Then a system of n ODEs can be described by:

    $$x_i'(t) = F_i(x_1, x_2, \ldots, x_n, t)$$

    where F_i is just a function.
  - ***Example***:

    $$x_1' = 2x_1^2 - 2te^{x_2}$$
    $$x_2' = \frac{x_1}{2t} - t^2 x_2$$
- **Why are first order ODEs desirable?**
  - recall, the **order** of an ODE is the highest derivative which appears in the system
  - any high order ODE can be converted into an **equivalent** system of **first order** ODEs
  - ***Example***: a general n^{th} order ODE can be described by:

    $$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)})$$

    In order to turn this into a system, we define n new variables $x_1, x_2, \ldots, x_n$, such that:

    $$x_1 = y \qquad x_2 = y' \qquad \ldots \qquad x_n = y^{(n-1)}$$

from which we obtain the system:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_2 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= F(t, x_1, x_2, \dots, x_n) \end{aligned}$$

More generally, the derived system can be:

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \dots, x_n) \\ x_2' &= F_2(t, x_1, x_2, \dots, x_n) \\ &\vdots \\ x_n' &= F(t, x_1, x_2, \dots, x_n) \end{aligned}$$

- **How can a system of ODEs be expressed in terms of vectors?**
  - it is somewhat inconvenient to have to write equations at all times. We can use vectors to encode all the information:
  $$\underline{x}(t) = \langle x_1(t), x_2(t), \dots, x_n(t) \rangle$$
  $$\underline{F} = \langle F_1, F_2, \dots, F_n \rangle$$
  $$\underline{x}'(t) = \underline{F}(\underline{x}(t), t)$$
  - if $\frac{\partial F_i}{\partial t} = 0$, then the system is autonomous
- **How can we geometrically interpret a system of ODEs?**
  - viewing systems of ODEs in terms of vectors allow us to obtain intuition as to what they "represent"
  - if t and x are **constant**, $\underline{F}(\underline{x}(t), t)$ is just a vector
  - if t is **constant**, $\underline{F}(\underline{x}(t), t)$ is a **vector field**. Then, $\underline{x}'(t) = \underline{F}(\underline{x}(t), t)$ tells us that the solution $\underline{x}(t)$ represents a curve in space which is tangential to the vector field $\underline{F}(\underline{x}(t)$

2.2 The Existence and Uniqueness Theorem: General Systems of ODEs

Theorem (The Existence and Uniqueness Theorem)**.** *Consider a system of ODEs:*

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \dots, x_n) \\ x_2' &= F_2(t, x_1, x_2, \dots, x_n) \\ &\vdots \\ x_n' &= F(t, x_1, x_2, \dots, x_n) \end{aligned}$$

and consider the set of n initial conditions:

$$x_1(t_0) = x_1^0 \qquad x_2(t_0) = x_2^0 \qquad \dots \qquad x_n(t_0) = x_n^0$$

*The ODEs and the inital conditions define an **initial value problem**.*

Let each of the n functions $F_1, \dots, F_n$ and the n^2 partial derivatives:

$$\frac{\partial F_1}{\partial x_1}, \dots \frac{\partial F_1}{\partial x_n}, \dots, \frac{\partial F_n}{\partial x_1}, \dots, \frac{\partial F_n}{\partial x_n}$$

be continuous in a region R of $(tx_1x_2\dots x_n)$-space, defined by:

$$\alpha < t < \beta$$
$$\alpha_1 < x_1 < \beta_1$$
$$\vdots$$
$$\alpha_n < x_n < \beta_n$$

and let $(t_0, x_1^0, x_2^0, \dots, x_n^0) \in R$. Then there is an interval $t - t_0| < h$ in which there exists a unique solution:

$$x_1 = \phi_1(t)$$
$$\vdots$$
$$x_n = \phi_n(t)$$

which satisfies the initial value problem.

2.3 Linear Systems of ODEs

- **What is a linear system of ODEs?**
  - a system of ODEs in which each of the F_i are linear functions of $x_1, x_2, \dots, x_n$
- **How can a system of linear ODEs be represented?**
  - since F_i will be linear, we can consider a summation, in which each of the x_i are multiplied by a function of t, and we also have an additional term, independent of $x_1, x_2, \dots, x_n$, so:

$$x_1' = g_1(t) + \sum_{i=1}^{n} P_{1i}(t)x_i(t)$$
$$x_2' = g_2(t) + \sum_{i=1}^{n} P_{2i}(t)x_i(t)$$
$$\vdots$$
$$x_n' = g_n(t) + \sum_{i=1}^{n} P_{ni}(t)x_i(t)$$

  - more generally, if we define:

$$\underline{x}(t) = \langle x_1(t), x_2(t), \dots, x_n(t)\rangle$$
$$\boldsymbol{P}(t) = \begin{pmatrix} P_{11}(t) & P_{12}(t) & \cdots & P_{1n}(t) \\ P_{21}(t) & P_{22}(t) & \cdots & P_{2n}(t) \\ \vdots & \dots & \ddots & \vdots \\ P_{n1}(t) & P_{n2}(t) & \cdots & P_{nn}(t) \end{pmatrix}$$
$$\underline{g}(t) = \langle g_1(t), g_2(t), \dots, g_n(t)\rangle$$

    Then our system can be succintly described as:

$$\underline{x}'(t) = \boldsymbol{P}(t)\underline{x}(t) + \underline{g}(t)$$

- **When is a system of linear ODEs homogeneous?**
  - the system $\underline{x}'(t) = \boldsymbol{P}(t)\underline{x}(t) + \underline{g}(t)$ is homogeneous if $\underline{g}(t) = \underline{0}$

2.4 The Existence and Uniqueness Theorem: Linear Systems of ODEs

Theorem (The Existence and Uniqueness Theorem (Linear Systems))**.** *If the functions* $P_{11}, P_{12}, \ldots, P_{nn}, g_1, \ldots, g_n$ *are* ***continuous*** *on an* ***open*** *interval* $I = (\alpha, \beta)$*, then there exists a unique solution:*

$$x_1 = \phi_1(t)$$
$$\vdots$$
$$x_n = \phi_n(t)$$

of the system $\underline{x}'(t) = \boldsymbol{P}(t)\underline{x}(t) + \underline{g}(t)$ *which also satisfies the initial conditions*

$$x_1(t_0) = x_1^0 \qquad x_2(t_0) = x_2^0 \qquad \ldots \qquad x_n(t_0) = x_n^0$$

where $t_0 \in I$*, and* $x_1^0, \ldots, x_n^0$ *are any prescribed numbers. Moreover, the solution exists throughout the interval.*

Notice that we get a lot more (in terms of the interval in which the solution is defined, and how the initial values are arbitrary) than in the general case.

2.5 The Principle of Superposition

Theorem (The Principle of Superposition)**.** *Consider a linear, homogeneous system of ODEs* $\underline{x}'(t) = \boldsymbol{P}(t)\underline{x}(t)$*. If the vector functions:*

$$\{\underline{x}^{(j)}(t)\} = \underline{x}^{(1)}, \underline{x}^{(1)}, \ldots, \underline{x}^{(1)}$$

are ***linearly independent*** *solutions of the system* $\forall t \in (\alpha, \beta)$*, then each solution* $\underline{x}(t)$ *of the linear, homogeneous ODE can be expressed as a linear combination of* $\{\underline{x}^{(j)}(t)\}$*:*

$$\underline{x}(t) = \sum_{i=1}^{n} c_i \underline{x}^{(i)}(t) = c_1\underline{x}^{(1)}(t) + \ldots + c_n\underline{x}^{(n)}(t)$$

in ***exactly one way****.*

Proof. We show that the linear combination $\underline{x}(t) = \sum_{i=1}^{n} c_i \underline{x}^{(i)}(t)$ satisfies its linear, homogeneous ODE:

$$\begin{aligned}
\frac{d}{dt}\left(\sum_{i=1}^{n} c_i \underline{x}^{(i)}(t)\right) &= \sum_{i=1}^{n} c_i \frac{d}{dt}(\underline{x}^{(i)}(t)) \\
&= \sum_{i=1}^{n} c_i \boldsymbol{P}(t)\underline{x}^{(i)}(t) \qquad \textit{since } x^{(j)} \textit{ is a solution, it follows that } \underline{x}^{(j)\prime} = \boldsymbol{P}(t)\underline{x}^{(j)} \\
&= \boldsymbol{P}(t)\left(\sum_{i=1}^{n} c_i \underline{x}^{(i)}(t)\right) \qquad \textit{since matrices are linear operators}
\end{aligned}$$

□

- **What is the general solution of a linear, homogeneous ODE?**
  - a solution which encompasses all possible solutions to a given system:

$$\underline{x}(t) = \sum_{i=1}^{n} c_i \underline{x}^{(i)}(t) = c_1\underline{x}^{(1)}(t) + \ldots + c_n\underline{x}^{(n)}(t)$$

2.6 The Wronskian

- **Since all solutions to a linear, homogeneous system of ODEs can be expressed as a linear combination of linearly independent solutions, what is the interpretation of the set of all solutions?**
  - the solution space, given by all possible solutions, is precisely a vector subspace of dimension n, where n is the dimension of $\underline{x}(t)$
- **How can we know if a set of n solutions is linearly independent?**
  - we need to calculate the **Wronskian**
  - if $\{\underline{x}^{(j)}\}$ is the set of all solutions, consider a matrix M:

    $$\boldsymbol{M} = \langle \underline{x}^{(1)}, \underline{x}^{(2)}, \ldots, \underline{x}^{(n)} \rangle$$

    then, the Wronskian is defined as the **determinant** of $\boldsymbol{M}$, and we denote it as:

    $$W[\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots, \underline{x}^{(n)}](t)$$

  - the columns of a matrix are linearly independent **if and only if** the determinant of the matrix is non-zero. Thus, if at a point t we have that $W[\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots, \underline{x}^{(n)}](t) \neq 0$, then $\{\underline{x}^{(j)}\}$ is a set of linearly independent solutions

  Proof: A Matrix $\boldsymbol{M}$ has Linearly Dependent Columns if det$\boldsymbol{M} = 0$. Let $\boldsymbol{M}$ be a $n \times n$ matrix. Apply elementary row operations, until we reach row-echelon form. Call the new matrix $\boldsymbol{M'}$ (notice it will be an upper triangular matrix). If the columns of $\boldsymbol{M}$ are linearly dependent, then there must be a row in $\boldsymbol{M'}$ that is all 0s. But the determinant of an upper triangular matrix is the product of the elements in its diagonal entries. Since one of these elements is 0, it follows that $det = 0$. Since elementary row operations preserve properties of the matrix, it follows that $det = 0$.

  The result follows directly from the Fundamental Theorem of Invertible Matrices □

- **What is a fundamental set of solutions?**
  - for a system of n linear, homogeneous ODEs, the fundamental set of solutions is any set of n linearly independent solutions at each point of an interval $\alpha < t < \beta$
  - so $\{\underline{x}^{(j)}(t)\}$ forms the fundamental set of solutions **iff** $W[\{\underline{x}^{(j)}(t)\}] \neq 0$
  - any linear, homogeneous system of ODEs ($\underline{x}^{(j)\prime} = \boldsymbol{P}(t)\underline{x}^{(j)}$) will always have at least one fundamental set of solutions

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Proof: The Principle of Superposition Gives All Solutions. We know by the Principle of Superposition that $\underline{x}(t) = \sum_{i=1}^{n} c_i \underline{x}^{(i)}(t)$ is a solution to the system $\underline{x}^{(j)\prime} = \boldsymbol{P}(t)\underline{x}^{(j)}$. Let $t_0 \in (\alpha, \beta)$. Is there any solution of the form $\underline{x}(t) = \sum_{i=1}^{n} c_i \underline{x}^{(i)}(t)$ that satisfies the initial condition $\underline{x}(t_0) = \underline{y}$? In other words, we seek $c_1, c_2, \ldots, c_n$, such that:

$$c_1 \underline{x}^{(1)}(t_0) + \ldots + c_n \underline{x}^{(n)}(t_0) = \underline{y}$$

But this is just a linear system:

$$\begin{pmatrix} \underline{x}^{(1)}(t_0) & \ldots & \underline{x}^{(n)}(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \underline{y}$$

But by the Fundamental Theorem of Invertible Matrices, such a system will have a unique solution if and only if the determinant of the matrix is non-zero. And the determinant of the matrix is just $W[\{x^{(j)}\}](t_0)$. Since the vectors are linearly independent, we are guaranteed that the Wronskian is non-zero, and so, a unique solution exists to the linear system, which satisfies the initial conditions. □

2.7 Abel's Theorem

Theorem (Abel's Theorem)**.** *If $\underline{x}^{(1)}, \ldots, \underline{x}^{(n)}$ are solutions of the linear, homogeneous system $\underline{x}'(t) = \boldsymbol{P}(t)\underline{x}(t)$ on the interval $I = (\alpha, \beta)$, then if $t_0 \in I$, and $W[\underline{x}^{(1)}, \ldots, \underline{x}^{(n)}](t_0) \neq 0$, then $\forall t \in I$, $W[\underline{x}^{(1)}, \ldots, \underline{x}^{(n)}](t) \neq 0$.*

Abel's Theorem means that for any interval it is sufficent for us to check the value of the Wronskian at a single value, in order to confirm lindear independence of solutions. To prove Abel's Theorem, we can us **Abel's Formula**:

$$\frac{dW}{dt} = (tr\boldsymbol{P}(t))W$$
$$\implies W = c\exp\left(\int(tr\boldsymbol{P}(s))ds\right)$$

Alternatively, proving that if n solutions are linearly dependent at a point, they must be linearly dependent at all points, which implies that linear independence at a point implies linear independence at all points.

2.8 Real Valued Systems and Complex Solutions

Theorem. *Consider the system $\underline{x}' = \boldsymbol{P}(t)\underline{x}$. Moreover, let $P_{ij}(t)$ be a real-valued, continuous function. If $\underline{x} = \underline{u}(t) + i\underline{v}t$ is a complex-valued solution of the linear, homogeneous system of ODEs, then $\underline{u}(t)$ and $\underline{v}(t)$ are also solutions.*

Proof. Substituting $\underline{x} = \underline{u}(t) + i\underline{v}t$ into $\underline{x}' = \boldsymbol{P}(t)\underline{x}$:

$$\begin{aligned}
\underline{x}' - \boldsymbol{P}(t)\underline{x} &= (\underline{u}(t) + i\underline{v}(t))' - \boldsymbol{P}(t)(\underline{u}(t) + i\underline{v}(t)) \\
&= \underline{u}'(t) + i\underline{v}'(t) - \boldsymbol{P}(t)\underline{u}(t) - i\boldsymbol{P}(t)\underline{v}(t) \\
&= (\underline{u}'(t) - \boldsymbol{P}(t)\underline{u}(t)) + i(\underline{v}'(t) - \boldsymbol{P}(t)\underline{v}(t)) \\
&= 0
\end{aligned}$$

Now, a complex number is 0 **if and only if** its real and imaginary parts are both 0. Thus, we require:

$$\underline{u}'(t) - \boldsymbol{P}(t)\underline{u}(t) = 0$$

$$\underline{v}'(t) - i\boldsymbol{P}(t)\underline{v}(t)$$

which implies that $\underline{u}(t)$ and $\underline{v}(t)$ are solutions of $\underline{x}' = \boldsymbol{P}(t)\underline{x}$. □

3 Solving Homogeneous Systems of Linear ODEs With Constant Coefficients

For this section, we consider the following system:

$$\underline{x}' = \boldsymbol{A}\underline{x}$$

where $\boldsymbol{A}$ is a constant, $n \times n$ matrix.

Furthermore, here are some slides that also discuss the topic, from University of Pennsylvania.

3.1 Deriving Solutions

To derive solutions to $\underline{x}' = \boldsymbol{A}\underline{x}$, we can extend what we know for linear, first-order, homogeneous ODEs such as:

$$\frac{dx}{dt} = ax$$

which has an exponential solution e^{at}. Thus, we consider solutions of the form:

$$\underline{x} = \underline{\xi} e^{rt}$$

where $r, \underline{\xi}$ are constants to be determined.

Substituting into the equation:

$$\begin{aligned}
&\underline{x}' = \boldsymbol{A}\underline{x} \\
\implies &\frac{d}{dt}\left(\underline{\xi} e^{rt}\right) = \boldsymbol{A}\underline{\xi} e^{rt} \\
\implies &r\underline{\xi} e^{rt} = \boldsymbol{A}\underline{\xi} e^{rt} \\
\implies &r\underline{\xi} = \boldsymbol{A}\underline{\xi} \\
\implies &\boldsymbol{A}\underline{\xi} = r\underline{\xi}
\end{aligned}$$

But then this means that r is the **eigenvalue** of the **eigenvector** $\underline{\xi}$ of $\boldsymbol{A}$. We obtain the system:

$$(\boldsymbol{A} - r\boldsymbol{I})\underline{\xi} = \underline{0}$$

from which we know that r solves $det(\boldsymbol{A} - r\boldsymbol{I}) = 0$, and then $\underline{\xi}$ can be found algebraically!

If all the eigenvalues are different (either they are real, or complex conjugates), then we are guaranteed that the corresponding eigenvectors are **linearly independent**. If we evaluate the Wronskian:

$$W[\underline{x}^{(1)}, \ldots, \underline{x}^{(n)}](t) = |\langle \underline{\xi}^{(1)} e^{r_1 t}, \ldots, \underline{\xi}^{(n)} e^{r_n t} \rangle| = e^{(r_1 + \ldots + r_n)t} |\langle \underline{\xi}^{(1)}, \ldots, \underline{\xi}^{(n)} \rangle|$$

[1] $e^{(r_1+\ldots+r_n)t}$ is always non-zero, and each of the $\underline{\xi}^{(i)}$ are linearly independent eigenvectors, so the determinant is non-zero. Thus, since the Wronskian is non-zero, each of the $e^{r_i t}\underline{\xi}^{(i)}$ are part of the **fundamental set of solutions**.

In general, if the **algebraic multiplicity** (number of times an eigenvalue appears as a root in the characteristic equation) is the same as the **geometric multiplicity** (number of eigenvectors associated with an eigenvalue) are equal, the solutions will form a fundamental set of solutions.

Using the superposition theorem, the general solution to $\underline{x}' = \boldsymbol{A}\underline{x}$ will be:

$$\underline{x} = \sum_{i=1}^{n} c_i \underline{\xi} e^{rt}$$

Proof: Eigenvectors of Distinct Eigenvalues Are Linearly Independent. This proof is from Dr. Peyam.

We proceed by induction. We claim that for a given matrix $\boldsymbol{T}$, given a set of non-zero n eigenvectors with n distinct eigenvalues, the eigenvectors are linearly independent.

[1] To see why we can extract the constant factor from each of the columns, see this article

If $n = 1$ we have a single vector, and it is clearly linearly independent, as:

$$c_1\underline{x}_i = \underline{0}$$

if and only if $c_1 = 0$.

Now, assume that $n = k$ is true: there exist a set of k distinct eigenvalues $r_1, r_2, \ldots, r_k$ with corresponding eigenvectors $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k$, and these are all linearly independent.

Now consider the case $n = k+1$. Lets first assume that $\exists c_1, c_2, \ldots, c_k, c_{k+1}$:

$$c_1x_1 + \ldots + c_kx_k + c_{k+1}x_{k+1} = 0$$

If we can show that $\forall i \in [1, k+1],\ c_i = 0$, then we have proven that the eigenvectors are linearly independent.

Apply the transformation T to the expression above to obtain:

$$T(c_1x_1 + \ldots + c_kx_k + c_{k+1}x_{k+1}) = T(0) \implies c_1r_1x_1 + \ldots + c_kr_kx_k + c_{k+1}r_{k+1}x_{k+1} = 0$$

Now multiply the linear combination by the eigenvalue r_{k+1}, getting:

$$c_1r_{k+1}x_1 + \ldots + c_kr_{k+1}x_k + c_{k+1}r_{k+1}x_{k+1} = 0$$

If we substract $c_1r_1x_1 + \ldots + c_kr_kx_k + c_{k+1}r_{k+1}x_{k+1}$ from the above, we get:

$$c_1(r_1 - r_{k+1})\underline{x_1} + \ldots + c_k(r_k - r_{k+1})\underline{x_k} = 0$$

Now, the eigenvectors $\underline{x}_i,\ i \in [1, k]$ are linearly independent by the inductive hypothesis, so it must be the case that:

$$\forall i \in [1, k],\ c_i(r_i - r_{k+1}) = 0$$

But since all the eigenvalues are distinct, $r_i - r_{k+1} \neq 0$, so it follows that $\forall i \in [1, k],\ c_i = 0$. But then our original linear combination reduces to:

$$c_1x_1 + \ldots + c_kx_k + c_{k+1}x_{k+1} = 0 \implies c_{k+1}x_{k+1} = 0$$

and since $\underline{x}_{k+1}$ is non-zero, we must have that $c_{k+1} = 0$, thus showing that $\forall i \in [1, k+1],\ c_i = 0$, and thus, the set of $k+1$ eigenvectors are linearly independent. □

- **Further Pointers**
  - if the eigenvalues are repeated, as long as $AM = GM$, the system can be easily solved; otherwise, further techniques need to be applied
  - if $\boldsymbol{A}$ is complex, then complex eigenvalues need not occur as complex conjugate pairs; this is not a problem, so long as the eigenvectors are linearly independent

3.2 Solving Linear, Homogeneous, First Order Systems of ODEs

3.2.1 Real Eigenvalues

1. **Consider the system:**

$$x_1' = x_1 + x_2$$
$$x_2' = 4x_1 + x_2$$

Compute the general solution to the system.

This is just the system $\underline{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \underline{x}$

Assuming a solution of the form $\underline{x} = \underline{\xi} e^{rt}$, we obtain the following system:

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \underline{0}$$

Thus, the eigenvalues are given by:

$$\begin{aligned} \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} &= (1-r)^2 - 4 \\ \implies \quad (1-r)^2 - 4 &= 0 \\ \therefore \quad r &= 1 \pm 2 \\ \therefore \quad r &= -1, 3 \end{aligned}$$

Let $r_1 = -1$ with corresponding eigenvector $\underline{\xi}_1$, and let $r_2 = 3$ with corresponding eigenvector $\underline{\xi}_2$.

We first consider $r_1 = -1$. Our system of equations then becomes:

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \underline{0}$$

which implies:

$$2\xi_1 + \xi_2 = 0 \implies \xi_2 = -2\xi_1$$

so it follows that:

$$\underline{\xi}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Now consider $r_r = 3$. Our system of equations then becomes:

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \underline{0}$$

which implies:

$$-2\xi_1 + \xi_2 = 0 \implies \xi_2 = 2\xi_1$$

so it follows that:

$$\underline{\xi}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Thus, since the eigenvectors are clearly linearly independent, they conform the fundamental set of solutions, and thus, the general solution is:

$$\underline{x}(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(We can check that the Wronskian is indeed never 0:

$$\begin{vmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{vmatrix} = 4e^{2t} \neq 0$$

)

2. **Find the general solution of:**

$$\underset{\sim}{x} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \underset{\sim}{x}$$

Assuming a solution of the form $\underline{\xi} e^{rt}$, we must solve the system:

$$\begin{pmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{pmatrix} \underline{\xi} = \underline{0}$$

We first find the eigenvalues:

$$\begin{aligned}
\begin{vmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{vmatrix} &= -r \begin{vmatrix} -r & 1 \\ 1 & -r \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -r \end{vmatrix} + \begin{vmatrix} 1 & -r \\ 1 & 1 \end{vmatrix} \\
&= -r(r^2 - 1) - (-r - 1) + (1 + r) \\
&= -r^3 + 3r + 2 \\
\implies & r^3 - 3r - 2 = 0 \\
\implies & (r+1)(r^2 - r - 2) = 0 \\
\implies & (r+1)^2(r-2) = 0 \\
\therefore & \; r = -1, 2
\end{aligned}$$

Letting $r_1 = -1$ and $r_2 = 2$, notice that r_1 has algebraic multiplicity 2, so we require geometric multiplicity 2 in order to obtain a solution.

Solving the system with $r = r_1$, we get:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \underline{0}$$

which implies:

$$\xi_1 + \xi_2 + \xi_3 = 0$$

We can parametrise. Letting $s = \xi_1, t = \xi_2$, ξ_3 can be written as $\xi_3 = -s - t$; thus, the eigenvector is:

$$\begin{pmatrix} s \\ t \\ -s - t \end{pmatrix} = s \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Thus, the linearly independent eigenvectors associated with $r_1 = -1$ are:

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Since the algebraic multiplicity is the same as the geometric multiplicity for r_1, they lead to valid solutions.

We now consider $r = r_2 = 2$:

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \underline{0}$$

$$\implies \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 1 & 3 & -3 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)$$

which implies:

$$-2\xi_1 + \xi_2 + \xi_3 = 0$$

$$\xi_2 - \xi_3 = 0$$

Again, parametrising $s = \xi_2$, we have $\xi_3 = s$, so:

$$-2\xi_1 + \xi_2 + \xi_3 = 0 \implies -2\xi_1 + 2s = 0$$

so $\xi_1 = s$, and the eigenvector associated with $r_2 = 2$ is:

$$\underline{\xi}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Overall, the general solution thus becomes:

$$\underline{x}(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

3.2.2 Complex Eigenvalues

As discussed previously, if the system has complex solutions, then the real and imaginary parts are also solutions to the system. We show further why this is the case in particular when we have constant coefficients.

Proof: Constructing A Fundmanetal Set From One Complex Solution. Assume that our system $\underline{x}' = \boldsymbol{A}\underline{x}$ has a solution $\underline{x}_1 = e^{rt}\underline{\xi}$, derived from a complex eigenvalue $r \in \mathbb{C}$, and its associated eigenvector $\underline{\xi}$. Recall that:

$$z \in \mathbb{C},\ z = a + ib \implies z^* = a - ib$$

Then, we must have that:

$$(\boldsymbol{A} - r\boldsymbol{I})\underline{\xi} = \underline{0}$$

But if we take the complex conjugate:

$$(\boldsymbol{A} - r\boldsymbol{I})^* \underline{\xi}^* = \underline{0}^* \implies (\boldsymbol{A} - r^*\boldsymbol{I})\underline{\xi}^* = \underline{0}$$

so $r^*, \underline{\xi}^*$, the complex conjugates, also lead to the solution $\underline{x}_2 = e^{r^* t}\underline{\xi}^*$.

Now, recall that:

$$Re(z) = \frac{z + z^*}{2}$$

$$Im(z) = \frac{z - z^*}{2i}$$

So define:

$$\underset{\sim}{\widetilde{x}}_1 = Re(\underline{x}_1(t))$$

$$\underset{\sim}{\widetilde{x}}_2 = Im(\underline{x}_1(t))$$

Then, notice that $\widetilde{\underline{x}}_1, \widetilde{\underline{x}}_2$ are given as linear combinations of known solutions, namely $\underline{x}_1$ and $\underline{x}_2$, so they are both solutions. Moreover, it can be shown that they are also linearly independent.

□

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1. **Find a fundamental set of real-valued solutions of the system:**

$$\underline{x}' = \begin{pmatrix} \frac{-1}{2} & 1 \\ -1 & \frac{-1}{2} \end{pmatrix} \underline{x}$$

Assuming solutions of the form $\underline{x} = e^{rt}\underline{\xi}$, we must solve:

$$\begin{pmatrix} \frac{-1}{2} - r & 1 \\ -1 & \frac{-1}{2} - r \end{pmatrix} \underline{\xi} = 0$$

First, we find the eigenvalues r:

$$\begin{vmatrix} \frac{-1}{2} - r & 1 \\ -1 & \frac{-1}{2} - r \end{vmatrix} = \left(\frac{-1}{2} - r\right)^2 + 1$$

$$\implies \left(\frac{-1}{2} - r\right)^2 + 1 = 0$$

$$\implies r = -\frac{1}{2} \pm i$$

Notice, since we obtain complex conjugate eigenvalues, their eigenvectors will also be complex conjugates, so we only need to compute the eigenvector for one of the eigenvalues, say $r_1 = -\frac{1}{2} + i$. Then, we solve:

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \underline{\xi} = 0$$

which implies:

$$-i\xi_1 + \xi_2 = 0 \implies \xi_2 = i\xi_1$$

so the eigenvector is:

$$\underline{\xi}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Thus, we obtain 2 complex solutions:

$$\underline{x_1}(t) = e^{(-\frac{1}{2}+i)t} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\underline{x_2}(t) = e^{(-\frac{1}{2}-i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Since we want a real solution, we consider only:

$$\underline{x_1}(t) = e^{(-\frac{1}{2}+i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-\frac{1}{2}t} \begin{pmatrix} cos(t) + isin(t) \\ icos(t) - sin(t) \end{pmatrix}$$

and define the new solutions:

$$\underset{\sim}{\widetilde{x}}_1(t) = Re(\underline{x}_1(t)) = e^{-\frac{1}{2}t}\begin{pmatrix} cos(t) \\ -sin(t) \end{pmatrix}$$

$$\underset{\sim}{\widetilde{x}}_2(t) = Im(\underline{x}_1(t)) = e^{-\frac{1}{2}t}\begin{pmatrix} sin(t) \\ cos(t) \end{pmatrix}$$

So the general solution is:

$$\underline{x} = e^{-\frac{1}{2}t}\left(C_1\begin{pmatrix} cos(t) \\ -sin(t) \end{pmatrix} + C_2\begin{pmatrix} sin(t) \\ cos(t) \end{pmatrix}\right)$$