Group Theory - Weeks 9 - Jordan-Hölder Theorem

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Contents

Based on the notes by Susan J. Sierra, Chapter 7

1 Composition Series

1.1 Definition: Composition Series

*Let G be a **group**. A **composition series** for G is a **chain of normal subgroups**:*

$$\{e\} = G_0 \lhd G_1 \lhd \ldots \lhd G_{s-1} \lhd G_s = G$$

satisfying:

- $G_i \neq G_{i+1}$
- G_{i+1}/G_i *is **simple** for any* $i \in [0, s]$

*Notice, this does **not** mean that $G_i \lhd G_j, j > i+1$; we only have normality for immediately adjacent subgroups.*

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*We call s the **length** of the **composition series**.*
(Definition 7.1.2)

1.2 Definition: Composition Factors

*Let G be a **group** with **composition series**:*

$$\{e\} = G_0 \lhd G_1 \lhd \ldots \lhd G_{s-1} \lhd G_s = G$$

*A **composition factor** is the **simple** group obtained by taking the quotient G_{i+1}/G_i.*

1.3 Composition Series as Prime Factorisation

We can think of composition series as a "factorisation". For example:

$$\{0\} \lhd 4\mathbb{Z}/12\mathbb{Z} \lhd 2\mathbb{Z}/12\mathbb{Z} \lhd \mathbb{Z}/12\mathbb{Z}$$

$$\{0\} \lhd 6\mathbb{Z}/12\mathbb{Z} \lhd 3\mathbb{Z}/12\mathbb{Z} \lhd \mathbb{Z}/12\mathbb{Z}$$

(Here for example $2\mathbb{Z}/12\mathbb{Z}$ is a group of 6 elements isomorphic to C_6:

$$\bar{z} = \{kz + 12 \mid k \in \mathbb{Z}\} \in 2\mathbb{Z}/12\mathbb{Z}, \qquad z \in 2\mathbb{Z}$$

) If we apply the Third and First Isomorphism Theorems:

$$4\mathbb{Z}/12\mathbb{Z} \cong C_3 \qquad \frac{2\mathbb{Z}/12\mathbb{Z}}{4\mathbb{Z}/12\mathbb{Z}} \cong 2\mathbb{Z}/4\mathbb{Z} \cong C_2 \qquad \frac{\mathbb{Z}/12\mathbb{Z}}{2\mathbb{Z}/12\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z} \cong C_2$$

$$6\mathbb{Z}/12\mathbb{Z} \cong C_2 \qquad \frac{3\mathbb{Z}/12\mathbb{Z}}{6\mathbb{Z}/12\mathbb{Z}} \cong 3\mathbb{Z}/6\mathbb{Z} \cong C_2 \qquad \frac{\mathbb{Z}/12\mathbb{Z}}{3\mathbb{Z}/12\mathbb{Z}} \cong \mathbb{Z}/3\mathbb{Z} \cong C_3$$

In the same way as we decompose 12 into $2 \times 2 \times 3$, we decompose $\mathbb{Z}_{12}$ using C_2, C_2, C_3. Notice, the length of the ***composition series*** *are the* ***same*** *,and the* ***composition factors*** *are also the same (albeit with a different ordering).*

2 The Jordan-Hölder Theorem

2.1 Theorem: Classification of Finite Simple Groups

Let G be a ***finite, simple*** *group. Then, G is isomorphic to an element in* ***one*** *of the following families:*

1. ***Family 1***: *C_p, where p is prime*
2. ***Family 2***: *A_n for $n \geq 5$*
3. ***...16 other infinite families***
4. ***26 sporadic groups*** *(this includes the* ***Monster*** *and* ***Baby Monster*** *groups)*

(Theorem 7.1.4)

2.2 Theorem: The Jordan-Hölder Theorem

Let G be a ***finite*** *group. Then:*

- *G has a* ***composition series***
- *any 2* ***composition series*** *have:*
  - *the* ***same composition length***
  - *the* ***same composition factors*** *(up to* ***isomorphism*** *of groups, and* ***ordering*** *of the factors)*

(Theorem 7.1.3)

The ***Jordan-Hölder Theorem****, alongside the* ***Classification of Finite Simple Groups*** *tells us that we can "understand" any finite group, since it can be decomposed into a* ***composition series****, whose* ***composition factors*** *are* ***simple groups****, and we can classify all simple groups.*
However, this doesn't mean all of group theory is "solved": these theorems don't touch ***infinite groups****. Moreover, if we think of* ***composition factors*** *as* ***bricks****, and the group as a* ***house****:*

- *combining bricks to construct a house is very non-trivial*

- *the bricks don't determine a house*

For example:

$$\{e\} \triangleleft C_2 \triangleleft C_6$$

$$\{e\} \triangleleft C_3 \triangleleft C_6$$

So C_6 has C_2, C_3 as composition factors.
On the other hand, S_3 only has A_3 as a normal subgroup:

$$\{e\} \triangleleft A_3 \triangleleft S_3$$

$|A_3| = 3$, and the only group of order 3 is C_3. Moreover:

$$|S_3/A_3| = 2$$

and C_2 is the only group of order 2. Hence, S_3 and C_6 have the ***same composition series*** *and the* ***same composition factors****, but they're clearly not isomorphic (one is abelian, the other isn't).*

2.2.1 Lemma: Combining Composition Series

Let G be a group, with $N \triangleleft G$.
Let:

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_s = N$$

be a ***composition series*** *for N, and:*

$$N = H_0 \triangleleft H_1 \triangleleft \ldots \triangleleft H_r = G/N$$

be a ***composition series*** *for G/N.*

Then, there is a ***composition series*** *for G of length $s + r$, whose* ***composition factors*** *are:*

$$G_1, G_2/G_1, \ldots, G_s/G_{s-1}, H_1, H_2/H_1, \ldots, H_r/H_{r-1}$$

(Sublemma 7.2.2)

Proof. Recall the **Correspondence Theorem**:

Let G be a group, $N \triangleleft G$ and let:

$$can : G \to G/N$$

be the ***canonical map****.*
The map:

$$H \mapsto can(H)$$

is a ***bijection*** *between* ***subgroups*** *of G containing N, and* ***subgroups*** *of G/N.*
Under this bijection, ***normal subgroups*** *match with* ***normal subgroups****.*
Further, if $N \subseteq A, B$ are subgroups of G, then:

$$can(A) \subseteq can(B) \iff A \subseteq B$$

(Theorem 2.3.3)

Thus, for $i \in [0, r]$, define:

$$G_{i+s} = can^{-1}(H_i)$$

(these shall be the r additional groups which we add to $G_0, G_1, \ldots, G_s$ to create the composition series for G)

Since $H_i \triangleleft H_{i+1}$, the Correspondence Theorem ensures that $G_{s+i} \triangleleft G_{s+i+1}$.

Now, recall the Third Isomorphism Theorem:

If $N \leq H \leq G$, with $N, H \triangleleft G$, then:

$$(G/N)/(H/N) \cong G/H$$

(Theorem 2.3.5)

Hence:

$$G_{s+i+1}/G_{s+i} \cong \frac{G_{s+i+1}/N}{G_{s+i}/N} = H_{i+1}/H_i$$

In particular, since H_{i+1}/H_i is simple, then G_{s+i+1}/G_{s+i} is also simple.

Thus, we have a composition series for G:

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_s \triangleleft G_{s+1} \triangleleft \ldots \triangleleft G_{s+r} = can^{-1}(G/N) = G$$

□

2.2.2 Proposition: Existence of Composition Series

If G is a ***finite*** *group, then* ***G*** *has a* ***composition series****.*
(Proposition 7.2.1)

Proof. We induct on $|G| = n$.

(1) $|G| = 2$

If $|G| = 2$, then G will be simple, so $e \triangleleft G$ is the only possible composition chain.

(2) $|G| = k$

Assume that if $2 \leq |G| \leq k$, then G has a composition series.

(3) $|G| = k + 1$

Now, consider a group G, such that $|G| = k + 1$. Then, either G is simple or not. If G is simple, then $e \triangleleft G$ is the only possible composition chain. Otherwise, $\exists N \triangleleft G$ such that $N \subset G$. Then, since $|N| \leq k$ and $|G/N| \leq k$, by the inductive hypothesis N and G/N have a composition chain. By the lemma above, we can combine these composition chains into a composition chain for G.

Hence, we have shown that for any G, there exists a composition chain, as required.

□

2.2.3 Theorem: Uniqueness of Composition Series

*Let G be a **finite group**.*
*Then, any 2 **composition series** have the **same length** and the **same composition factors** (up to isomorphism and ordering).*
More precisely, if:

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_{s-1} \triangleleft G_s = G$$

$$\{e\} = H_0 \triangleleft H_1 \triangleleft \ldots \triangleleft H_{r-1} \triangleleft H_r = G$$

*are 2 **composition series** for G, then:*

- $s = r$
- *there is a **permutation** σ **of** $\{0, \ldots, s-1\}$ **such that:***

$$\forall i \in [0, s-1], \quad H_{i+1}/H_i \cong G_{\sigma(i)+1}/G_{\sigma(i)}$$

Proof. Again, we induct on $|G| = n$

(1) $|G| = 1, 2$

Either G is the trivial group, or G is simple, so there is a unique composition series.

(2) $|G| = k$

Assume that the composition factors are unique for all groups $1 \leq |G| \leq k$, up to isomorphism and ordering.

(3) $|G| = k + 1$

We consider 2 cases.

1. $G_{s-1} = H_{r-1}$
   This implies that there are (at least) 2 composition series for G_{s-1}:
   $$\{e\} = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_{s-1}$$
   $$\{e\} = H_0 \triangleleft H_1 \triangleleft \ldots \triangleleft H_{r-1} = G_{s-1}$$
   By the inductive hypothesis, since $|G_{s-1}| < |G| = k+1$, it follows that any composition factor of G_{s-1} will be unique, so in particular $s - 1 = r - 1$ and there is a permutation σ on the set $\{0, 1, \ldots, s - 1\}$, such that:
   $$\forall i \in [0, s-2], \quad H_{i+1}/H_i \cong G_{\sigma(i)+1}/G_{\sigma(i)}$$
   But then, since $s-1 = r-1$, then $s = r$. Moreover, we can extend σ to a permutation on $\{1, 2, \ldots, s-1\}$, via $\sigma(s - 1) = s - 1$. Then:
   $$H_s/H_{s-1} = G/H_{s-1} = G_s/G_{s-1} = G_{\sigma(s-1)+1}/G_{\sigma(s-1)}$$
   as required.
2. $G_{s-1} \neq H_{r-1}$
   We first note that this implies:
   $$G_{s-1} \not\subseteq H_{r-1} \qquad H_{r-1} \not\subseteq G_{s-1}$$
   To see why, assume that $G_{s-1} \subseteq H_{r-1}$. Note that $H_{r-1} \triangleleft G$. Moreover, we have that G/G_{s-1} will be simple, so its only normal subgroups are G_{s-1} and G/G_{s-1}. By the Correspondence Theorem, there must exist a bijection between H_{r-1} and one of the normal subgroups of G/G_{s-1} (since $G_{s-1} \subseteq H_{r-1}$ and H_{r-1} is normal). But since by assumption $H_{r-1} \neq G$, we must have that $H_{r-1} = G_{s-1}$, which is a contradiction. The same argument shows that $H_{r-1} \not\subseteq G_{s-1}$.

   Now, define:
   $$K = G_{s-1} \cap H_{r-1}$$
   Recall the Second Isomorphism Theorem:

Let $N \lhd G$ and $H \leq G$. Then:

(a) $HN \leq G$

(b) $N \lhd HN$

(c) $H \cap N \lhd H$

(d) there exists an isomorphism:

$$H/(H \cap N) \cong HN/N$$

(Theorem 2.3.7)

This tells us that:

$$K \lhd G_{s-1} \qquad K \lhd H_{r-1}$$

Now, consider the factor group G_{s-1}/K. Again, by the Second Isomorphism Theorem:

$$G_{s-1}/K = G_{s-1}/(G_{s-1} \cap H_{r-1}) \cong G_{s-1}H_{r-1}/H_{r-1}$$

We now prove that:

$$G_{s-1}H_{r-1} = G$$

To this end, we begin by showing it is a normal subgroup:

$$G_{s-1}H_{r-1} \lhd G$$

Indeed, let $a \in G$ and $gh \in G_{s-1}H_{r-1}$ where $g \in G_{s-1} \lhd G$ and $h \in H_{r-1} \lhd G$. Then:

$$a(gh)a^{-1} = agh(a^{-1}a)ha^{1} = (aga^{-1})(aha^{-1}) = gh$$

Thus:

$$\forall a \in G, aG_{s-1}H_{r-1}a^{-1} \subseteq G_{s-1}H_{r-1}$$

so $G_{s-1}H_{r-1} \lhd G$. Again, by the Correspondence Theorem, the simplicity of G/G_{s-1} means that since $G_{s-1} \subseteq G_{s-1}H_{r-1}$, either $G_{s-1}H_{r-1} = G_{s-1}$ or $G_{s-1}H_{r-1} = G$. But since $H_{r-1} \not\subseteq G_{s-1}$, we can't have that $G_{s-1}H_{r-1} = G_{s-1}$. Thus, the only possibility is that:

$$G_{s-1}H_{r-1} = G$$

Hence, this tells us that:

$$G_{s-1}/K = G_{s-1}/(G_{s-1} \cap H_{r-1}) \cong G_{s-1}H_{r-1}/H_{r-1} = G/H_{r-1}$$

Using identical logic:

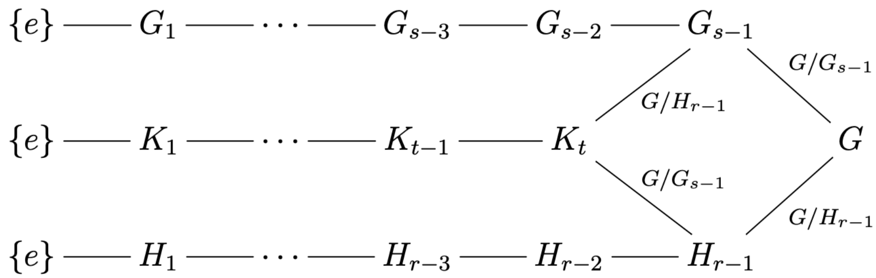
$$H_{r-1}/K = H_{r-1}/(G_{s-1} \cap H_{r-1}) \cong G_{s-1}H_{r-1}/G_{s-1} = G/G_{s-1}$$

This also tells us that G_{s-1}/K and H_{r-1}/K are simple, since G/H_{r-1} and G/G_{s-1} are simple.

Now, by the proposition on existence of composition series, we know that K will have a composition series:

$$\{e\} = K_0 \lhd K_1 \lhd \ldots \lhd K_t = K$$

We can group together all our composition series into a diagram:



In particular, this allows us to write a new composition series for G_{s-1}:

$$\{e\} = K_0 \triangleleft K_1 \triangleleft \ldots \triangleleft K_t = K \triangleleft G_{s-1}$$

which alongside:

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_{s-1}$$

and the inductive hypothesis tells us that $s-1 = t+1$. Similarly with H, we will have that $r-1 = t+1$, which means that $s-1 = r-1$, and so, $s = r$.

Overall, we now have 4 composition series for G:

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_{s-1} \triangleleft G_s = G$$
$$\{e\} = K_0 \triangleleft K_1 \triangleleft \ldots \triangleleft K_t \triangleleft G_{s-1} \triangleleft G_s = G$$
$$\{e\} = K_0 \triangleleft K_1 \triangleleft \ldots \triangleleft K_t \triangleleft H_{r-1} \triangleleft H_r = G$$
$$\{e\} = H_0 \triangleleft H_1 \triangleleft \ldots \triangleleft H_{r-1} \triangleleft H_r = G$$

By Case I, we know that the first 2 composition series produce the same composition factors (up to isomorphism and ordering):

$$(G_1/G_0, \ldots, G_{s-1}/G_{s-3}, G_{s-1}/G_{s-2}, G/G_{s-1}) = (K_1/K_0, \ldots, K/K_{t-1}, G_{s-1}/K, G/G_{s-1})$$

Likewise, the last 2 composition series product the same composition factors (up to isomorphism and ordering):

$$(H_1/H_0, \ldots, H_{s-1}/H_{s-3}, H_{s-1}/H_{s-2}, G/H_{s-1}) = (K_1/K_0, \ldots, K/K_{t-1}, H_{r-1}/K, G/H_{r-1})$$

Notice, looking at the composition factors involving K, the 2 sets of composition factors only differ in the last 2 terms:

$$(G_{s-1}/K, G/G_{s-1}) \qquad vs \qquad (H_{r-1}/K, G/H_{r-1})$$

But recall, we showed above that:

$$G_{s-1}/K \cong G/H_{r-1} \qquad H_{r-1}/K \cong G/G_{s-1}$$

Hence:

$$(G_{s-1}/K, G/G_{s-1}) \cong (G/H_{r-1}, G/G_{s-1})$$
$$(H_{r-1}/K, G/H_{r-1}) \cong (G/G_{s-1}, G/H_{r-1})$$

Thus, the composition factors across the composition series are identical, up to isomorphism and ordering of composition factors, which is what we required.

□