Group Theory - Weeks 8 - Alternating Groups

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Contents

Based on the notes by Susan J. Sierra, Chapter 6

1 Symmetric Groups

1.1 Definition: Symmetric Group

*The **symmetric group** S_n is the **group** containing all **bijections** (also known as **permutations**):*

$$\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$$

*We often represent **permutations** via **cycle notation**:*

$$(1243) \iff \begin{Bmatrix} 1 \mapsto 2 \\ 2 \mapsto 4 \\ 4 \mapsto 3 \\ 3 \mapsto 1 \end{Bmatrix}$$

1.2 Lemma: Permutations as Products of Disjoint Cycles

*Every **permutation** can be written as a **product** of **disjoint cycles**.*
*This product is **unique** up to re-ordering of the factors.*
(Lemma 6.1.1)

1.3 Transpositions

1.3.1 Definition: Transposition

*A **transposition** is a **2-cycle**. That is, a cycle containing only 2 elements.*
*An **adjacent transposition** is a **transposition** of the form $(i\ i+1)$.*

1.3.2 Lemma: Permutations as Products of Transpositions

*S_n is **generated** by **transpositions**.*
*That is, **every** permutation can be written as a **product** of **transpositions**.*

For instance:

$$(1243) = (13)(14)(12)$$

(Lemma 6.1.2)

1.3.3 Definition: Even and Odd Permutations

*A **permutation** is **even** if it can be written as a **product** of an **even** number of **transpositions**.*
*If it is **odd** otherwise.*
*The **identity** permutation is **even** (it is written as a product of 0 transpositions)*

For instance:

$$(1243) = (13)(14)(12)$$

*is **even**, whilst:*

$$(154)(23) = (14)(15)(23)$$

*is **odd**.*

---

*In particular, this means that if σ has **odd** cycle length, it is an **even** permutation, whilst if the cycle length is **even**, σ is **odd**.*

1.3.4 Lemma: Parity of Product of Permutations

The product of:

- *2 **even** permutations*
- *2 **odd** permutations*

*is **even**.*
*The product of an **odd** and an **even** permutation is **odd**.*

1.4 Conjugate Permutations

1.4.1 Definition: Cycle Type

Let:

$$\sigma = c_1 c_2 \dots c_k$$

be a ***product*** *of* k ***disjoint*** *cycles of length:*

$$l_1, l_2, \dots, l_k, \qquad l_1 \geq l_2 \geq \dots \geq l_k$$

Then, the ***cycle type*** *of* σ *is the k-tuple:*

$$(l_1, l_2, \dots, l_k)$$

(Definition 6.1.3)

1.4.2 Lemma: Effect of Conjugating Permutations

Let:

$$\sigma = (a_1 \ a_2 \ \dots \ a_k) \in S_n$$

and $\tau \in S_n$*.*
Then:

$$\tau \sigma \tau^{-1} = (\tau(a_1) \ \tau(a_2) \ \dots \ \tau(a_k))$$

(Lemma 6.1.7)

1.4.3 Theorem: Conjugate Permutations Have the Same Cycle Type

Two ***permutations*** *in* S_n *are* ***conjugate if and only if*** *they have the* ***same cycle type****.*
(Theorem 6.1.8)

2 Alternating Groups

2.1 Definition: The Alternating Group (1)

The ***alternating group*** *is a the* ***subgroup*** $A_n \leq S_n$ *containing all the* ***even*** *permutations of* S_n*.*

2.2 Definition: The Alternating Group (2)

Let $x_1, \ldots, x_n$ be variables, and define an object:

$$P = \prod_{1 \leq i \leq j \leq n} (x_i - x_j)$$

If $X = \{P, -P\}$, S_n acts on X via:

$$\sigma \cdot P = \prod_{1 \leq i \leq j \leq n} (x_{\sigma(i)} - x_{\sigma(i)})$$

*In particular, if σ is **even**:*

$$\sigma \cdot P = P$$

*and if σ is **odd**:*

$$\sigma \cdot P = -P$$

---

*Hence, A_n is the **stabiliser** of the **action** of S_n on X.*

2.3 Theorem: The Alternating Group is a Normal Subgroup

Let $n \geq 2$. Then $A_n \triangleleft S_n$, and:

$$|S_n| = 2|A_n| \implies |A_n| = \frac{n!}{2}$$

(Theorem 6.2.3)

Proof. By definition, A_n is the stabiliser of the action of S_n on X, so by the Orbit-Stabilizer Theorem:

$$|S_n| = |Stab_{S_n}(P)||Orb_{S_n}(P)|$$

The orbit of P over S_n is clearly X (since $n \geq 2$, S_n contains the identity (**even** permutation) and a transposition (**odd** permutation).

Hence:

$$|S_n| = 2|A_n|$$

as required.

We now show that A_n is a normal subgroup. Define a homomorphism:

$$sgn : S_n \to C_2$$

$$sgn(\sigma) = \begin{cases} 1, & \sigma \text{ is even} \\ -1, & \sigma \text{ is odd} \end{cases}$$

If σ, τ have the same parity, their product is even, so:

$$sgn(\sigma\tau) = 1 = sgn(\sigma)sgn(\tau)$$

Otherwise, their product is odd:

$$sgn(\sigma\tau) = -1 = sgn(\sigma)sgn(\tau)$$

so sgn is indeed a homomorphism.

Moreover, by definition of A_n:

$$ker(sgn) = A_n$$

Since the kernel of a homomorphism is a **normal subgroup**, it follows that A_n is a normal subgroup of S_n.

□

2.4 The Alternating Groups A_4 and A_5

We choose to focus on A_4 and A_5 because if $n \leq 3$, the alternating groups are rather uninteresting:

$$S_1 = \{\iota\} \implies A_1 = \{\iota\}$$

$$S_2 = \{\iota, (12)\} \implies A_2 = \{\iota\}$$

$$S_3 = \{\iota, (12), (23), (13), (123) = (13)(12), (132) = (12)(13)\} \implies A_3 = \{\iota, (123), (132)\}$$

It is once we look at A_4 and A_5 that we see interesting behaviour. In fact, it is this behaviour which leads to proving that polynomials of degree ≤ 4 can be solved by radicals, whilst polynomials with degree ≥ 5 can't!

2.4.1 Recap: Conjugacy Classes and Centralizers

If we let G act on itself via conjugation:

$$g \cdot a = gag^{-1}$$

we can define the ***conjugacy classes*** *of a* ***group****, alongside the* ***centralisers****.*

Let G be a group. The ***conjugacy class*** *of $a \in G$ is the* ***orbit*** *of a:*

$$Cl(a) = Orb_G(a) = \{gag^{-1} \mid g \in G\}$$

Let G be a group. The ***centralizer*** *of $a \in G$ is the* ***stabilizer*** *of a:*

$$C_G(a) = Stab_G(a) = \{g \mid gag^{-1} = a, g \in G\} = \{g \mid ga = ag, g \in G\}$$

That is, the ***centralizer*** *of a is the set of all elements in G which* ***commute*** *with a.*

Let G be a finite group. By the Orbit-Stabilizer Theorem:

$$\forall a \in G, \qquad |G| = |C_G(a)||Cl(a)|$$

(Lemma 4.2.7)

2.4.2 Lemma: Normal Groups are Unions of Conjugacy Classes

*Let G be a **finite group**, and suppose that:*

$$H \triangleleft G$$

Then, $\exists h_1, \ldots, h_k \in H$ such that:

$$H = \sqcup_{i=1}^{k} Cl_G(h_i)$$

*That is, a **normal subgroup** can be described as the **disjoint union** of **conjugacy classes**.*
(Lemma 6.2.5)

Proof. Let $H \triangleleft G$. Then:

$$\forall g \in G, \qquad gHg^{-1} = H$$

Now, let $h_i \in H$. The **conjugacy class** of h_i is the set of all elements in g which are conjugate to h_i:

$$Cl(h_i) = \{gh_ig^{-1} \mid g \in G\} \subseteq H$$

But now, conjugacy classes are disjoint (they are equivalence classes under the equivalence relation of conjugation), so there must exist representatives in H, such that the disjoint union of their conjugacy classes create H.

□

2.4.3 Proposition: Properties of A_4

The following are properties of A_4:

1. $$|A_4| = 12$$
2. *A_4 has a **unique** subgroup N of order 4*
3. $$N \triangleleft S_4 \quad \text{and} \quad N \triangleleft A_4$$
4. $$A_4/N \cong C_3 \quad \text{and} \quad S_4/N \cong S_3$$

Proof. 1. $|A_4| = 12$

This follows immediately:

$$|A_4| = \frac{4!}{2} = 12$$

2. The cycle types of elements in S_4 are:

- 4 (odd)
- 3,1 (even)
- 2,2 (even)
- 2,1,1 (odd)
- 1,1,1,1 (even)

Moreover, by Sylow I, $|A_4| = 12 = 4 \times 3$ has a Sylow 2-subgroup of order 4, call it N. Furthermore, by Lagrange's Theorem, elements in N must have an order which divides 4, so $\sigma \in N \implies o(\sigma) \in \{1, 2, 4\}$. Finally, the order of a permutation is the lcm of cycle lengths of its disjoint cycle decomposition. Hence, this tells us that elements in N must have cycle type $(2, 2)$ or 4 (the identity is obviously in N). However, notice that the elements with cycle length 4 are odd permutations, so they aren't even part of A_4. Hence, there is a **unique** subgroup of order 4, and it must be formed by the elements of S_4 with cycle shape $(2, 2)$. That is:

$$N = \{\iota, (12)(34), (13)(24), (14)(23)\}$$

which indeed has 4 elements, as expected.

3. $N \triangleleft S_4, N \triangleleft A_4$

Clearly, N must be normal, since it is the unique Sylow 2-subgroup of A_4. It is also a subgroup of S_4. It will be a normal subgroup of S_4 because N contains all elements of cycle type $(2, 2)$, so in particular, they are all conjugate. Since a normal subgroup is a disjoint union of conjugacy classes, N must still be a normal subgroup of S_4.

4. $A_4/N \cong C_3, S_4/N \cong S_3$

By Lagrange's Theorem:

$$|A_4/N| = \frac{12}{4} = 3$$

Hence, A_4/N is a group of order 3. 3 is prime, so by Lagrange's Theorem, A_4/N is a cyclic group of order 3, so $A_4/N \cong C_3$.

Similarly,

$$|S_4/N| = \frac{24}{4} = 6$$

so S_4/N is a group of order 6. There are only 2 groups of order 6: C_6 and S_3. However, since S_4 isn't abelian, S_4/N won't be abelian. For instance, if we pick $\alpha = (12)$ and $\beta = (14)$, then:

$$\alpha N * \beta N = (\alpha \circ \beta)N = (142)N$$

$$\beta N * \alpha N = (\beta \circ \alpha)N = (124)N$$

Hence, the $S_4/N \cong S_3$.

□

2.4.4 Theorem: A_5 is Simple

> *The **alternating group** A_5 is a **simple** group.*
> *(Theorem 6.3.1)*

Proof. We will make use of the following table of cycle types in S_5:

Cycle Type	**Number of Permutations**	**Even/Odd**
5	24	E
4,1	-	O
3,2	-	O
3,1,1	20	E
2,2,1	15	E
2,1,1,1,1	-	O
1,1,1,1,1	1	E

Counting the number of permutations of a certain cycle type is a combinatorial problem. For example, for the cycle type $(3,1,1)$, the 3 cycle has a total of $5 \times 4 \times 3$ possibilities. However, we are overcounting:

$$(123) = (312) = (231)$$

so the possibilities for the 3 cycle are:

$$\frac{5 \times 4 \times 3}{3} = 20$$

Moreover, once we have chosen the 3-cycle, the whole permutation is decided (since the other 2 elements are fixed). For the cycle type $(2,2,1)$, the first 2-cycle has $\frac{5\times4}{2} = 10$ possibilities. The second 2 cycle has $\frac{3\times2}{2} = 3$ possibilities. Hence, there are $10 \times 3 = 30$ total cycles with type $(2,2,1)$. However, notice once again we are overcounting: since the 2 cycles are disjoint, it doesn't matter which comes first, so we have $\frac{30}{2} = 15$ cycles of type $(2,2,1)$.

To show that A_5 is a simple group, we need to show that any normal subgroup will either be the trivial subgroup, or A_5. To do this, it is sufficient to compute the conjugacy classes in A_5, since their union will determine any potential normal subgroup. We need to be careful though: a conjugacy class in S_5 is simply determined by the cycle type; however, in A_5, there might be permutations which won't appear.

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(1) **Cycle Type** 5

We first consider the conjugacy classes for permutations of cycle type 5. To do this, we exploit the Orbit-Stabilizer Theorem:

$$\forall a \in G, \qquad |G| = |C_G(a)||Cl(a)|$$

In particular, let $\sigma = (1\ 2\ 3\ 4\ 5)$. Using the above Theorem, alongside the fact that there are 24 elements of cycle type 5 in S_5:

$$C_{S_5}(\sigma) = |S_5|/|Cl(\sigma)| = 120/24 = 5$$

Hence, the number of elements in S_5 which commute with σ is 5. Moreover, consider they cyclic subgroup generated by σ:

$$\langle\sigma\rangle = \{\sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5 = \iota\}$$

Notice, these will be the only permutations which commute with σ (any other permutation $\tau \in S_5$ won't be disjoint with σ, and thus, won't commute). Hence:

$$\langle\sigma\rangle = C_{S_5}(\sigma)$$

Since the powers of σ are even permutations, they are contained in A_5, and so:

$$\langle\sigma\rangle = C_{S_5}(\sigma) = C_{A_5}(\sigma)$$

By the Orbit-Stabilizer Theorem:

$$|Cl(\sigma)| = |A_5|/|C_{A_5}(\sigma)| = 60/5 = 12$$

Now, pick $\sigma' \in S_5$, such that $\sigma' \notin \langle\sigma\rangle$. Then, by similar logic $Cl(\sigma') = 12$. Hence, the 2 conjugacy classes, $Cl(\sigma), Cl(\sigma')$ will partition the conjugacy class containing elements of order 5 in A_5 (since there are 24 such elements).

---

(2) **Cycle Type** $(3, 1, 1)$

With this cycle type, finding the centraliser is a bit harder. We can still use the Orbit-Stabilizer Theorem. Let σ have cycle type $(3, 1, 1)$. Then:

$$|C_{S_5}(\sigma)| = |S_5|/|Cl(\sigma)| = 120/20 = 6$$

So we expect 6 elements of S_5 to commute with out σ. To be more concrete, lets pick a specific permutation, $\sigma = (1\ 2\ 3)$. Trivially, we know that $\tau = (4\ 5)$ commutes (since they are disjoint), and σ also commutes. Similarly:

$$\sigma^2 = (1\ 2\ 3)(1\ 2\ 3) = (1\ 3\ 2)$$

and σ^3 will just be the identity. Since the centraliser is a subgroup, we must (at least) have that:

$$C_{S_5}(\sigma) = \{\iota, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau\}$$

This already contains 6 commuting elements, so it must be the centraliser in S_5. But notice, some of these elements are not in A_5: τ is odd, and so, any product contraining τ will be odd (since σ is even). Thus, in A_5 the centraliser becomes:

$$C_{A_5}(\sigma) = \{\iota, \sigma, \sigma^2\}$$

so by the Orbit-Stabilizer Theorem:

$$Cl(\sigma) = |A_5|/|C_{A_5}(\sigma)| = 60/3 = 20$$

Hence, the conjugacy class for cycle type 3 permutations in A_5 is the same as in S_5.

---

(3) **Cycle Type** $(2, 2, 1)$

We reach the last non-trivial conjugacy class. In S_5, there are 15 elements with this cycle-type, so if σ has cycle type $(2,2,1)$:

$$|C_{S_5}(\sigma)| = |S_5|/|Cl(\sigma)| = 120/15 = 8$$

so we expect 8 elements to commute with σ. Lets consider $\sigma = (1\ 2)(3\ 4)$. σ has order 2 (since it is composed of 2 cycles of this order). Finding the remaining elements isn't as easy as above. A systematic way of checking is to use the fact that:

$$\sigma\tau = \tau\sigma \implies \tau\sigma\tau^{-1} = \sigma$$

Alongside:

> *Let:*
>
> $$\sigma = (a_1\ a_2\ \dots\ a_k) \in S_n$$
>
> *and* $\tau \in S_n$.
> *Then:*
>
> $$\tau\sigma\tau^{-1} = (\tau(a_1)\ \tau(a_2)\ \dots\ \tau(a_k))$$
>
> *(Lemma 6.1.7)*

This tells us that τ commutes with σ if upon conjugating we obtain σ. One can then see that:

$$C_{S_5}(\sigma) = \{\iota, (1\ 2), (3\ 4), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3), (1\ 3\ 2\ 4), (1\ 4\ 2\ 3)\}$$

Again, not all of these are even, which leads to:

$$C_{A_5}(\sigma) = \{\iota, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

so:

$$|Cl(\sigma)| = |A_5|/|C_{S_5}(\sigma)| = 60/4 = 15$$

so once again, conjugacy classes of elements of cycle type $(2,2,1,1)$ will be the same in A_5 as in S_5.

All in all, we get the following table relating conjugacy classes in S_5 and A_5:

Cycle Type	$\|Cl_{S_5}(\sigma)\|$	$\|C_{S_5}(\sigma)\|$	$\|Cl_{A_5}(\sigma)\|$	$\|C_{A_5}(\sigma)\|$
5	24	5	12, 12	5
3,1,1	20	6	20	3
2,2,1	15	8	15	4
1,1,1,1,1	1	120	1	60

Now, any normal subgroup of A_5 will be a union of these conjugacy classes. As such we require that:

$$|N| = 1 + 12\alpha + 20\beta + 15\nu$$

where the values come from the order of the conjugacy classes, and $\alpha \in [0,1,2], \beta, \nu \in [0,1]$ (the identity element, which is the only elements in the conjugacy class of the identity must be in N for N to be a subgroup). We require that $|N|$ divides $|A_5| = 60 = 60 \times 1 = 30 \times 2 = 20 \times 3 = 15 \times 4 = 12 \times 5$. Notice given this linear combination, the only possibilities are:

$$|N| = 1 \qquad (\alpha = \beta = \nu = 0)$$

$$|N| = 60 \qquad (\alpha = 2, \beta = \nu = 1)$$

Hence, $N = \{\iota\}$ or $N = A_5$, and thus, A_5 is simple.

□

2.5 Simplicity of the Alternating Groups

2.5.1 Lemma: 3-Cycles are Conjugate in Alternating Groups

*If $n \geq 5$ and σ, σ' are **3-cycles** in A_n, then σ, σ' are **conjugate** in A_n:*

$$\exists \tau \in A_n \; : \; \tau\sigma\tau^{-1} = \sigma'$$

(Lemma 6.3.4)

Proof. Assume $\exists \tau \in S_n$, where τ is odd, such that τ commutes with a 3-cycle σ:

$$\tau\sigma = \sigma\tau \implies \tau\sigma\tau^{-1} = \sigma$$

Now, let ν be another odd permutation in S_n. Then $\nu\tau$ will be an even permutation. Then:

$$(\nu\tau)\sigma(\nu\tau)^{-1} = \nu(\tau\sigma\tau^{-1})\nu^{-1} = \nu\sigma\nu^{-1}$$

In other words, for any **odd** permutation ν, we can find a corresponding even permutation $\nu\tau$, such that they conjugate σ to the same value. In other words, any 2 3-cycles, which will be conjugate in S_n, will also be conjugate in A_n, provided that there is an odd permutation with which they commute.

This is because of σ, σ' are conjugate in S_n then $\exists \nu \in S_n$ such that:

$$\nu\sigma\nu^{-1} = \sigma'$$

If ν is even, then $\nu \in A_n$, so σ, σ' are conjugate in A_n. Otherwise, $\tau\nu \in A_n$, so σ, σ' are still conjugate in A_n.

Now, let σ be a 3-cycle in S_n, where $n \geq 5$. Then trivially σ must leave at least 2 elements $\alpha, \beta \in \{1, \ldots, n\}$ **fixed**. Hence, if we define $\tau = (\alpha \; \beta)$, τ and σ will be disjoint, so in particular they must commute. Since τ is a transposition, it is an odd permutation. Hence, by what we have just shown, 3-cycles are **conjugate** in A_n, as required.

□

2.5.2 Lemma: 3-Cycles Generate Alternating Groups

If $n \geq 3$, then A_n is generated by 3-cycles.
(Lemma 6.3.5)

Proof. Let $\sigma \in A_n$. Since σ is even, it can be written as an even number of transpositions. Then, it is sufficient to show that a product of 2 transpositions can be written as a single 3-cycle.

Consider 2 disjoint transpositions:

$$(a\ b)(c\ d) = (a\ c\ b)(c\ d\ a)$$

If the transpositions are not disjoint, without loss of generality we can assume that their first element is common:

$$(a\ b)(a\ c) = (a\ c\ b)$$

Since σ decomposes into an even number of transpositions, by the above we can write σ using 3-cycles (just group transpostions into pairs, and apply the transformations above), and so, any element in A_n can be decomposed into 3-cycles, as required.

□

2.5.3 Lemma: Order of Fixed-Point-Free Subgroups

*We say $\sigma \in S_n$ is **fixed-point-free** if:*

$$\forall i \in [1, n], \quad \sigma(i) \neq i$$

*Then, if $H \leq S_n$, and $\forall \sigma \in H, \sigma \neq \iota$ such that σ is **fixed-point-free**, then $|H| \leq n$.*
(Lemma 6.3.6)

2.5.4 Lemma: Order of Conjugacy Classes of Alternating Groups

If $n \geq 6$ and $\sigma \in A_n$, with $\sigma \neq \iota$, then:

$$|Cl_{A_n}(\sigma)| \geq n$$

(Lemma 6.3.7)

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*Intuitively this lemma says that conjugacy classes in A_n are **big**. For instance, the smallest non-trivial conjugacy class in A_6 has 40 elements (A_6 has 720 elements).*

2.5.5 Theorem: Alternating Groups are Simple When $n \geq 5$

> *A_n is a **simple group** for $n \geq 5$.*
> *(Theorem 6.3.3)*

Proof. We perform induction on n.

① **Base Case:** $n = 5$

We already proved this!

② **Inductive Hypothesis:** $n = k$

Assume this is true for $n = k$. That is, $A_5, A_6, \ldots, A_k$ are all simple groups.

③ **Inductive Step:** $n = k + 1$

Consider the group A_{k+1}. Consider a normal subgroup:

$$H \triangleleft A_{k+1}$$

For $i \in [1, k+1]$ define the set B_i as the set of all even permutations which fix i:

$$B_i = \{\sigma \mid \sigma(i) = i, \quad \sigma \in A_n\}$$

We claim that:

$$B_i \cong A_k$$

This is simple to see: B_i is defined by fixing a unique element out of a set of $k + 1$ total elements; in other words, it permutes all the other k elements, whilst keeping i fixed. This is the definition of a permutation group of k elements. Since B_i only contains even permutations, it must be isomorphic to A_k.

Moreover, recall the Second Isomorphism Theorem:

> *Let $N \triangleleft G$ and $H \leq G$. Then:*
>
> 1. $HN \leq G$
> 2. $N \triangleleft HN$
> 3. $H \cap N \triangleleft H$
> 4. *there exists an isomorphism:*
>
> $$H/(H \cap N) \cong HN/N$$
>
> *(Theorem 2.3.7)*

so since $H \triangleleft A_k$ and $B_i \leq A_k$:

$$H \cap B_i \triangleleft B_i$$

Since $B_i \cong A_k$, and A_k is a simple group by inductive hypothesis, it follows that B_i is simple, so:

$$H \cap B_i = B_i \qquad H \cap B_i = \iota$$

First, assume that:

$$H \cap B_i = B_i$$

This is true **if and only if** $B_i \subseteq H$. Now, since $B_i \cong A_k \subset A_{k+1}$, we know that B_i contains at least one 3-cycle (since 3-cycles are even permutations), and so, H contains a 3-cycle.

But H is a normal subgroup of A_{k+1}, and normal subgroups are unions of conjugacy classes. As such, H must contain the **whole** conjugacy class of 3-cycles in A_{k+1}. But then, since A_{k+1} is an alternating group, it is generated by 3-cycles. Since H contains all the 3-cycles which generate A_{k+1}, it must be the case that $A_{k+1} \subseteq H$. Hence, it follows that $H = A_{k+1}$.

Secondly, assume that:

$$H \cap B_i = \iota$$

In other words, if $\sigma \in H$, then σ fixes no element in $i \in [1, n]$. But then, H is a **fixed-point-free** subgroup of S_{k+1}, so:

$$H \leq k + 1$$

Now take some $\sigma \in H$ such that $\sigma \neq \iota$. We also have that:

$$|Cl_{A_{k+1}}(\sigma)| \geq k + 1$$

But then, recall $Cl_{A_{k+1}}(\sigma)$ won't contain the identity, so:

$$Cl_{A_{k+1}}(\sigma) \cup \{\iota\} \subset H$$

which implies:

$$|H| \geq (k + 1) + 1$$

This is a contradiction, and so, no such non-identity must exist. Hence, $\sigma = \iota$ is the only possible element in H, and so, A_{k+1} must be simple.

□