Group Theory - Week 3 - Representations of Groups

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October 2022

Contents

1 Cayley Tables

A ***Cayley*** *or* ***multiplication table*** *is an* ***array*** *recording the* ***group structure*** *of a* ***finite group****, by listing all possible products of group elements:*

	g_1	g_2	$\cdots$	g_n
g_1	g_1^2	g_1g_2	$\cdots$	g_1g_n
g_2	g_2g_1	g_2^2	$\cdots$	g_2g_n
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

	()	(12)	(13)	(23)	(123)	(132)
()	()	(12)	(13)	(23)	(123)	(132)
(12)	(12)	()	(132)	(123)	(23)	(13)
(13)	(13)	(123)	()	(132)	(12)	(23)
(23)	(23)	(132)	(123)	()	(13)	(12)
(123)	(123)	(13)	(23)	(12)	(132)	()
(132)	(132)	(23)	(12)	(13)	()	(123)

Figure 1: Multiplication table for S_3.

2 Group Presentations

2.1 Definition: Presentation of a Group

*A **group presentation** is defined by a set of **generators** $x_1, \ldots, x_m$ and a set of **relations** on the generators $r_1, \ldots, r_n$:*

$$\langle x_1, \ldots, x_m \mid r_1, \ldots, r_n \rangle$$

this defines the following group:

- *a group generated by all possible combinations (**words**) of $x_1, \ldots, x_m$ and their inverses $x_1^{-1}, \ldots, x_m^{-1}$ (for example, if we use symbols x, y, z, possible group elements will be x^2yz^{-1} and $x^{-3}z^5$)*
- *constrained by the relations $r_1 = e, \ldots, r_n = e$, where $\forall i \in [1, n], r_n \in \{x_1, \ldots, x_m\}$ (for example, we might require that $z^2 = e$, in which case x^2yz^{-1} would become x^2yz and $x^{-3}z^5$ would become $x^{-3}z$)*
- *satisfying the **group axioms** (this essentially imposes associativity, since the remaining axioms are trivially satisfied from definition)*

(Definition 3.2.3)

2.2 Definition: Free Groups

*A **free group** on **generators** $x_1, \ldots, x_m$ is a group which can be defined via a **group presentation** without **relations**.*
*In other words, it is the group produced by all combinations (**words**) of the symbols $x_1, \ldots, x_m$ and their inverses, subject to group axioms and under the operation of **concatenation**.*
A free group can be written as:

$$\langle x_1, \ldots, x_m \mid - \rangle = \langle x_1, \ldots, x_m \rangle$$

2.3 Examples of Group Presentations

2.3.1 Group Presentation of Cyclic Groups

Consider the group presentation:

$$A = \langle x \mid x^n = e \rangle$$

Then this defines a set:

$$\{x, x^2, \ldots, x^{n-1}, e\}$$

which is the form of any cylic group C_n of order n, so $A \cong C_n$.

2.3.2 Group Presentation of the Integers

Consider the group presentation:

$$A = \{x \mid -\}$$

A is a free group of the form:

$$\{\ldots, x^{-2}, x^{-1}, e, x, x^2, \ldots\}$$

This is just a group which is generated by a single element. In particular:

$$\phi : x^a \to a$$

defines an isomorphism from A to $\mathbb{Z}$, so $A \cong \mathbb{Z}$

2.3.3 Group Presentation of $\mathbb{Z} \times \mathbb{Z}$

Consider the group presentation:

$$A = \langle x, y \mid xyx^{-1}y^{-1} \rangle$$

That is, we have a relation:

$$xyx^{-1}y^{-1} = e \implies xy = yx$$

Hence, A will be a **commutative** group. However, we know more: given any $g \in A$, the fact that we can permute the symbols x, y implies that $\exists i, j \in \mathbb{Z}$ such that:

$$g = x^i y^j$$

In particular:

$$\phi : x^i y^j \to (i, j)$$

defines a group isomorphism between A and $\mathbb{Z} \times \mathbb{Z}$.

2.3.4 Group Presentation of the Trivial Group

Consider the group presentation:

$$A = \langle x \mid x^3 = x^2 \rangle$$

A is a group, so the cancellation property, alongside the relation imply that:

$$x = e$$

Hence, A must be the trivial group:

$$A = \{e\}$$

Novikov's Theorem *states that, in general, there is no algorithm which can decided whether a group presentation defines the trivial group.*
This doesn't mean that there aren't algorithms for determining this, just that there is no single algorithm which can decide for ***all*** *group presentations.*
In fact, in general it is not possible to determine whether a ***word*** *(like x^3 or x^2yz^{-1}) is itself the identity, given just the group presentation.*

2.4 Extended Example: The Dihedral Group D_5 and the Universal Property of Free Groups

2.4.1 Defining the Group Presentation E

We now analyse the group presentation:

$$E = \left\langle a, b \mid a^2, b^5, (ab)^2 \right\rangle$$

We begin by looking at how the relations affect the group structure:

$$a^2 = e \implies a^{-1} = a$$

$$b^5 = e \implies b^{-1} = b^4$$

$$(ab)^2 = abab = e \implies aba = b^{-1} \implies ba = a^{-1}b^{-1} = ab^4$$

This is a crucially important piece of information: as with $\mathbb{Z} \times \mathbb{Z}$, the fact that $ba = ab^4$ implies that **any** element of E can be written in the form:

$$a^i b^j, \qquad i \in [0, 1], j \in [0, 4]$$

In particular, this means that we can list all the elements of E:

$$E = \{e, b, b^2, b^3, b^4, a, ab, ab^2, ab^3, ab^4\}$$

However, the group presentation doesn't tell us whether all these elements are unique (there might be some way of combining the relations which allows us to equate 2 elements); all we can say is that $|E| \leq 10$.

2.4.2 Proposition: The Universal Property of Free Groups

*Now, we take a step back, and define the **Universal Property of Free Groups**:*

*Let G be a **group** generated by a set:*

$$\{s_1, \dots, s_n\}$$

*Consider the **free group**:*

$$F = \langle S_1, \dots, S_n \rangle$$

defined by the letters $S_1, \dots, S_n$.
*Then, there exists a **unique surjective homomorphism**:*

$$\pi : F \to G$$

given by:

$$\pi(S_i) = s_i, \qquad \forall i \in [1, n]$$

*Here we note that G is a group, so it may have some restrictions on its elements; on the other hand, F is a free group, so it is composed by **all** possible words derived from S_i.*

2.4.3 Discovering E

Now, we define 2 symbols A, B which generate the free group $\langle A, B\rangle$. Then, by the universal property of free groups, we have a unique surjective homomorphism:

$$\pi : \langle A, B\rangle \to E$$

$$A \mapsto a \in E$$

$$B \mapsto b \in E$$

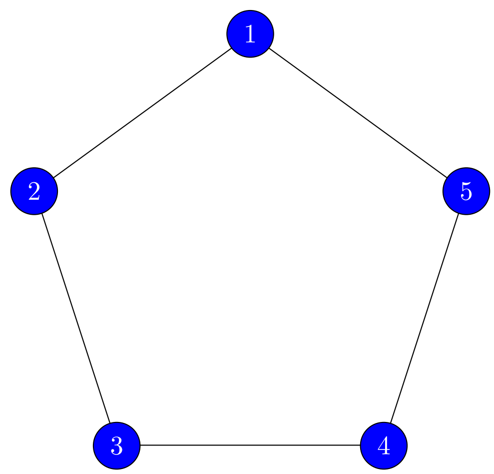
For example,

$$\pi(A^3B^2A^5B^6) = a^3b^2a^5b^6 = ab^2ab = a^2b^8b = b^4$$

This is surjective, since by the definition of E, we can write $x \in E$ via a^ib^j, so:

$$\pi(A^iB^j) = x$$

Now, recall the dihedral group D_5, which gives the symmetries of a regular pentagon:

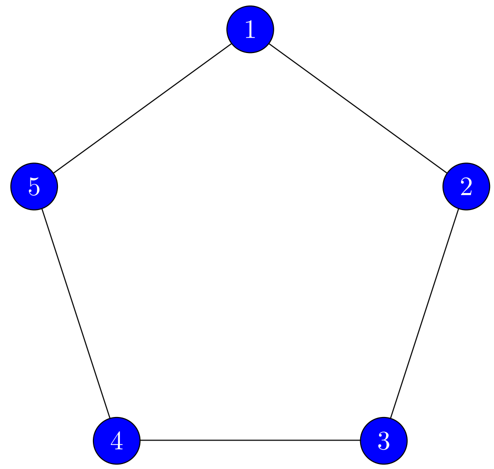


This is composed of 2 elements: g (reflection about vertex 1) and h (rotation by $\frac{2\pi}{5}$ anticlockwise).

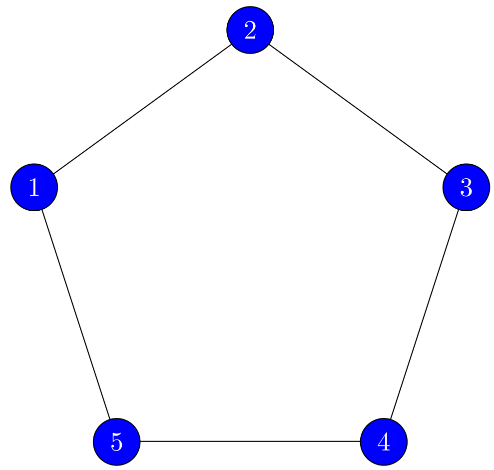
Now, notice D_5 has many similarities with E:

- $g^2 = e$ (similarly, $a^2 = e$)
- $h^5 = e$ (similarly, $h^5 = e$)
- h generates a normal subgroup, and $g^{-1}hg = h^4$. This means that $hg = gh^4 \implies (gh)^2 = e$.

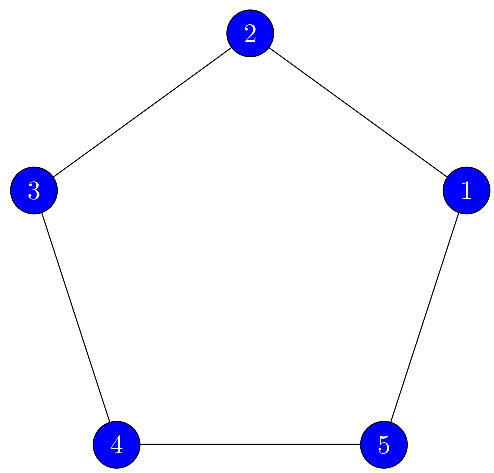
We can see that $g^{-1}hg = h^4$ geometrically. If we apply g, we reflect the vertices:



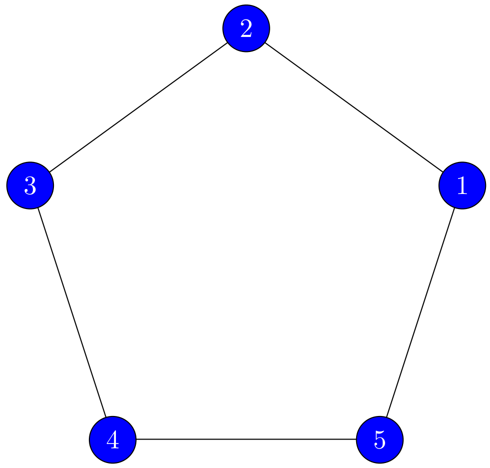
We then rotate by $\frac{2\pi}{5}$ anticlockwise:



Finally, we reflect again by the top vertex (2):



Alternatively, if we had done the rotation $h^4 = h^{-1}$ (so a $\frac{2\pi}{5}$ clockwise rotation):



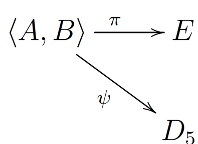
Now, in a similar vein to the work above, we can define a free group homomorphism:

$$\psi : \langle A, B \rangle \to D_5$$

$$\psi(A) = g$$

$$\psi(B) = h$$

We now have the following mappings:



Now, lets consider $ker(\pi)$. This contains $A^2, B^5, (AB)^2$, alongside all those elements in $\langle A, B \rangle$ which, due to the logical consequences defined by E, are mapped to $e \in E$. Notice, all these elements must also be contained in $ker(\psi)$, since D_5 contains all the relations defining E, so in particular all the logical consequences imposed on π apply to ψ, so $ker(\pi) \subseteq ker(\psi)$ (since we don't know all the relations which are applicable to D_5 only).

But now, recall the Corollary to the Universal Property of Factor Groups:

If:

- $\phi : G \to K$ *is a* ***surjective*** *group homomorphism*
- $\psi : G \to H$ *is a group homomorphism*
- $ker(\phi) \subseteq ker(\psi)$

Then, there is a ***unique*** *group homomorphism:*

$$\bar{\psi} : K \to H$$

such that:

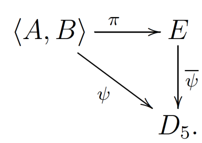
$$\bar{\psi} \circ \phi = \psi$$

(Corollary 2.2.4)

Hence, there exists a unique homomorphism:

$$\bar{\psi} : E \to D_5$$

$$\bar{\psi}\pi = \psi$$



where:

$$\bar{\psi}(a) = g \qquad \bar{\psi}(b) = h$$

Notice, this means that $\bar{\psi}$ is a surjective mapping, and so:

$$|E| \geq |D_5| = 10$$

But since $|E| \leq 10$, $\bar{\psi}$ must be an isomorphism, and so, $E \cong D_5$.

In general:

$$\forall n \geq 3, \qquad \left\langle a, b \mid a^2, b^n, (ab)^2 \right\rangle \cong D_n$$