Group Theory - Week 2 - Factor Groups and Isomorphism Theorems

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Contents

Based on the notes by Susan J. Sierra, Chapter 2

1 Factor Groups

1.1 Defining Factor Groups

1.1.1 Definition: Factor Group

*Let $N \triangleleft G$. Then, the set of **left cosets**:*

$$G/N = \{gN \mid g \in G\}$$

*(or **right cosets**, since N is normal) defines a group known as the **factor group** (or **quotient group**).*
*G/N is a **group** under the operation:*

$$g_1N \star g_2N = (g_1g_2)N, \qquad \forall g_1, g_2 \in G$$

*Notice, in rings, **ideals** lead to **factor rings**; in **groups**, **normal subgroups** lead to **factor groups**.*

1.1.2 Lemma: Factor Group Operation is Well-Defined

*It is important the the operation on factor groups is **well-defined**: that is, it doesn't depend on the **particular representative** of the coset which we choose. That is, if we apply the operation on g_1N and g_2N, and we have that $g_1N = g_2N$, we better hope that they both give the same answer.*

The operation:

$$g_1N \star g_2N = (g_1g_2)N$$

*is **well-defined**.*

Proof. Consider the following elements of G/H:

$$gN = g'N \qquad hN = h'N$$

where g, h, g', h' are distinct. The operation will be **well-defined** if:

$$gN \star hN = g'N \star h'N$$

This is a routine check:

$$\begin{aligned}
gN \star hN &= (gh)N \\
&= g(hN) \\
&= g(h'N) \\
&= g(Nh'), \qquad \textit{(since } H \textit{ is a normal subgroup)} \\
&= (gN)h' \\
&= (g'N)h' \\
&= g'(Nh') \\
&= g'(h'N) \\
&= (g'h')N \\
&= g'N \star h'N
\end{aligned}$$

so $\star$ is indeed well-defined.

□

1.1.3 Lemma: Factor Group Satisfies Group Axioms

> *The set G/N, where $N \triangleleft G$, is a group under the operation:*
> $$g_1N \star g_2N = (g_1g_2)N, \qquad \forall g_1, g_2 \in G$$

Proof. (1) Existence of Identity

Consider:

$$eN = N \in G/N$$

then, $\forall g \in G$:

$$N \star gN = (eg)N = gN = (ge)N = gN \star N$$

so $eN = N$ is the identity.

(2) Existence of Inverse

Consider the element $gN \in G/N$ for any $g \in G$. Then, $g^{-1}N \in G/N$ and:

$$gN \star g^{-1}N = (gg^{-1})N = N$$
$$g^{-1}N \star gN = (g^{-1}g)N = N$$

so for any $gN \in G/N$, $g^{-1}N$ is an inverse.

(3) Associativity

This will follow from associativity in G:

$$\begin{aligned}
(gN \star hN) \star kN &= (gh)N \star kN \\
&= (gh)kN \\
&= g(hk)N) \\
&= gN \star (hk)N \\
&= gN \star (hN \star kN)
\end{aligned}$$

as required.

□

1.2 Definition: The Canonical Group Homomorphism

The ***canonical map*** *is a function from a* ***group*** *to one of its* ***factor groups****:*

$$can : G \to G/N$$

defined in the most natural way:

$$can(g) = gN$$

By definition, the ***canonical map*** *is a* ***surjective*** *mapping.*

1.2.1 Lemma: The Canonical Map is a Surjective Group Homomorphism

Let $N \triangleleft G$. The ***canonical map*** *can is a* ***group homomorphism****:*

$$can : G \to G/N$$

Proof.

$$\begin{aligned} can(gh) &= (gh)N \\ &= gN \star hN \\ &= can(g) \star can(h) \end{aligned}$$

□

1.3 Theorem: Normal Group iff Kernel of Homomorphism

Let $N \leq G$. Then, $N \triangleleft G$ ***if and only if*** *N is the* ***kernel*** *of a group homomorphism:*

$$\phi_N : G \to H$$

where H is some other group.

Proof.

- ($\Longleftarrow$) We already showed last week that the kernel is a normal subgroup of G.
- ($\Longrightarrow$) Now, suppose that $N \triangleleft G$. We construct a homomorphism ϕ_N, such that:

$$ker(\phi_N) = N$$

where:

$$\phi_N : G \to H$$

and H is another group.
ϕ_N is nothing but the **canonical map**:

$$can : G \to G/N$$

Consider any $g \in G$. Then, by definition:

$$g \in ker(can) \iff can(g) = gN = N$$

But then:

$$\begin{aligned} & gN = N \\ \iff & \exists n_1, n_2 \in N : gn_1 = n_2 \\ \iff & g = n_2 n_1^{-1} \qquad \textit{(by existence of inverse in subgroup)} \\ \iff & g \in N \qquad \textit{(by closure of group operation in subgroup)} \end{aligned}$$

Hence, we have shown that:

$$g \in ker(can) \iff can(g) = gN = N \iff g \in N$$

so it follows that as required:

$$ker(can) = N$$

□

1.4 Factor Group Examples

- all subgroups of $\mathbb{Z}$ are **normal subgroups**. For example consider subgroups of the form:

  $$n\mathbb{Z} = \{nz \mid z \in \mathbb{Z}\}$$

  Then:

  $$g(n\mathbb{Z}) = \{g + m \mid m \in n\mathbb{Z}\} = \{m + g \mid m \in n\mathbb{Z}\} = (n\mathbb{Z})g$$

  where we have used the fact that $\mathbb{Z}$ is abelian. Then, the factor group $\mathbb{Z}/n\mathbb{Z}$ is **isomorphic** to $\mathbb{Z}_n$ - the integers modulo n, where each element $\bar{z}$ is just a coset $z\mathbb{Z}$.

- $\mathbb{Z}_{10}$ is abelian, so its subgroups will be normal. What is the group $\mathbb{Z}_{10}/\{0, 5\}$? We can compute it explicitly:

  $$0 + \{0, 5\} \quad 1 + \{0, 5\} \quad 2 + \{0, 5\} \quad 3 + \{0, 5\} \quad 4 + \{0, 5\}$$

  This is an abelian group of prime order 5; in particular, it must be isomorphic to $\mathbb{Z}_5$. Indeed, we can see that:

  $$(3 + \{0, 5\}) + (4 + \{0, 5\}) = 7 + \{0, 5\} = \{7, 12\} = \{2, 7\} = 2 + \{0, 5\}$$

  In $\mathbb{Z}_5$ we have:

  $$\bar{3} + \bar{4} = \bar{7} = \bar{2}$$

  as expected.

*The above examples show that factor groups tend to have natural isomorphisms for well known groups. This idea is formalised by the **First Isomorphism Theorem**.*

2 The First Isomorphism Theorem

2.1 Theorem: The First Isomorphism Theorem for Groups

Let:

$$\theta : G \to H$$

*be a **group homomorphism**.*
Let:

$$N := ker(\theta)$$

so that $N \triangleleft G$; and, $im(\theta) \leq H$.
*There is an **isomorphism**:*

$$\psi : G/ker(\theta) \to im(\theta)$$

defined by:

$$\psi(gN) = \theta(g)$$

*If θ is **surjective**, then $im(\theta) = H$, and so:*

$$G/ker(\theta) \cong H$$

(Theorem 2.2.1)

Proof. We explicitly show that ψ is an isomorphism. For this we need to:

1. Verify it is **well-defined**
2. Verify it is a **group homomorphism**
3. Verify that it is **injective**
4. Verify that it is **surjective**

(1) Well-Defined

We need to show that for 2 different representatives $g_1, g_2 \in G$ such that:

$$g_1 N = g_2 N$$

we have:

$$\psi(g_1 N) = \psi(g_2 N)$$

Notice, if $g_1 N = g_2 N$, this is equivalent to saying that:

$$g_1^{-1} g_2 \in N$$

Since $g_1^{-1} g_2 \in ker(\theta)$, it follows that:

$$\theta(g_1^{-1} g_2) = \theta(g_1)^{-1}\theta(g_2) = e_H \implies \theta(g_1) = \theta(g_2)$$

so:

$$g_1 N = g_2 N \implies \psi(g_1 N) = \psi(g_2 N)$$

and ψ is well-defined.

(2) Group Homomorphism

Let $g_1, g_2 \in G$. Then:

$$\begin{aligned}\psi(g_1N \star g_2N) &= \psi((g_1g_2)N) \\ &= \theta(g_1g_2) \\ &= \theta(g_1)\theta(g_2) \\ &= \psi(g_1N)\psi(g_2N)\end{aligned}$$

so ψ is a group homomorphism.

(3) Injective

This is essentially the inverse argument of what we did at (1). Assuming that $\psi(g_1N) = \psi(g_2N)$, we claim that:

$$g_1N = g_2N$$

Indeed:

$$\begin{aligned}&\psi(g_1N) = \psi(g_2N) \\ \implies &\theta(g_1) = \theta(g_2) \\ \implies &\theta(g_1)^{-1}\theta(g_2) = \theta(g_1^{-1}g_2) = e_H \\ \implies &g_1^{-1}g_2 \in N \\ \implies &g_1N = g_2N\end{aligned}$$

so ψ is **injective**.

(4) Surjective

Let $h \in im(\theta)$. Then, $\exists g \in G$ such that:

$$\theta(g) = h$$

Hence:

$$\psi(gN) = h$$

so any element in $im(\theta)$ can be mapped to by ψ, so it is **surjective**.

Hence, we have shown that ψ is a **well-defined group isomorphism**, and so:

$$G/ker(\theta) \cong im(\theta)$$

as required.

□

2.2 Theorem: The Universal Property of Factor Groups

Turns out that the ***First Isomorphism Theorem*** *is just a nice consequence of the following* ***universal*** *property.*

Let G be a group and let $N \triangleleft G$.
For ***any*** *homomorphism:*

$$\psi : G \to H$$

with:

$$N \subseteq ker(\psi)$$

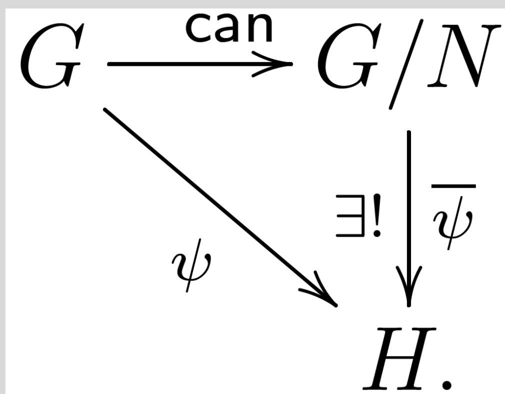
there exists a ***unique homomorphism****:*

$$\bar{\psi} : G/N \to H$$

such that:

$$\psi = \bar{\psi} \circ can$$

where $can : G \to G/N$ is the ***canonical homomorphism****.*
This can be visualised by the following diagram:



(Theorem 2.2.3)

2.3 Corollary: Group Homomorphism Between Images

If:

- $\phi : G \to K$ *is a* ***surjective*** *group homomorphism*
- $\psi : G \to H$ *is a group homomorphism*
- $ker(\phi) \subseteq ker(\psi)$

Then, there is a ***unique*** *group homomorphism:*

$$\bar{\psi} : K \to H$$

such that:

$$\bar{\psi} \circ \phi = \psi$$

(Corollary 2.2.4)

Proof. By the first isomorphism theorem, and since ϕ is surjective, we have that:

$$G/ker(\phi) \cong im(\phi) = K$$

Let $N = ker(\phi)$. Then, the universal property of factor rings applies, and there exists a unique homomorphism:

$$\bar{\psi} : G/ker(\phi) \to H$$

or alternatively:

$$\bar{\psi} : K \to H$$

given by:

$$\bar{\psi}\phi = \psi$$

(we don't need the canonical mapping, since ϕ already maps us to the factor group)

□

3 Theorem: The Second Isomorphism Theorem

Let $N \triangleleft G$ and $H \leq G$. Then:

1. $HN \leq G$
2. $N \triangleleft HN$
3. $H \cap N \triangleleft H$
4. *there exists an isomorphism:*

$$H/(H \cap N) \cong HN/N$$

(Theorem 2.3.7)

Proof. 1. $HN \leq G$

HN is clearly non-empty, since:

$$e \in H, e \in N \implies ee = e \in HN$$

Let $h_1, h_2 \in N$ and $n_1, n_2 \in N$. Then it is sufficient to show that:

$$(h_1 n_1)(h_2 n_2)^{-1} \in HN$$

We have that:

$$(h_1 n_1)(h_2 n_2)^{-1} = h_1 n_1 n_2^{-1} h_2^{-1}$$

Now, notice that:

$$n_1 n_2^{-1} \in N \implies n_1 n_2^{-1} h_2^{-1} \in N h_2^{-1}$$

Since N is a normal subgroup, it thus follows that:

$$n_1 n_2^{-1} h_2^{-1} \in h_2^{-1} N$$

In other words, $\exists n_3 \in N$ such that:

$$n_1 n_2^{-1} h_2^{-1} = h_2^{-1} n_3$$

Hence:

$$(h_1 n_1)(h_2 n_2)^{-1} = h_1 n_1 n_2^{-1} h_2^{-1} = (h_1 h_2^{-1}) n_3 \in HN$$

as required.

2. $N \triangleleft HN$

We first note that $N \leq HN$. This is simple, since $e \in H$ and $eN = N$, so $N \subseteq HN$. Moreover, N is a group, so $N \leq HN$.
Moreover, let $g \in HN$. Then, by group closure we also have $g \in G$. Since $N \triangleleft G$, it is immediate that:

$$gNg^{-1} = N, \qquad \forall g \in HN$$

so $N \triangleleft HN$ as required.

3. $H \cap N \triangleleft H$

Let $a \in H \cap N$ and $h \in H$. Notice:

- $hah^{-1} \in H$, since $a, h \in H$
- $hah^{-1} \in N$, since $h \in H \leq G$, and $a \in N \triangleleft G$

Thus, it follows that $\forall h \in H$:

$$hah^{-1} \in H \cap N$$

so $H \cap N \triangleleft H$ as required.

4. $H/(H \cap N) \cong HN/N$

We need to find a surjective homomorphism of the form:

$$\theta : H \to HN/N$$

such that:

$$ker(\theta) = H \cap N$$

Consider the canonical mapping:

$$can : H \to HN/N$$

given by:

$$can(h) = hN$$

We know that this is a well-defined homomorphism, so we just need to determine its surjectivity and its kernel.
Let $(hn)N \in HN/N$. Since $nN = N$, it follows that:

$$(hn)N = hN = \theta(h)$$

so θ is a surjective mapping.
Moreover:

$$h \in ker(\theta) \iff \theta(h) = hN = N \iff h \in N$$

But $h \in ker(\theta) \iff h \in H$, so it follows that:

$$h \in ker(\theta) \iff h \in H \cap N$$

and so:

$$ker(\theta) = H \cap N$$

Thus, by the First Isomorphism Theorem, it follows that:

$$H/ker(\theta) \cong im(\theta) \implies H/(H \cap N) \cong HN/N$$

as required.

□

4 The Third Isomorphism Theorem

The third isomorphism theorem gives us tools to identify how subgroups of factor groups G/N relate to subgroups of G.

4.1 Proposition: The Canonical Map and Subgroup Preservation

Let G be a group and $N \triangleleft G$. Consider the ***canonical map****:*

$$can : G \to G/N$$

and let:

$$K \leq G/N$$

Then:

1. $can^{-1}(K) \leq G$, *with* $N \subseteq can^{-1}(K)$
2. $can^{-1}(K) \triangleleft G \iff K \triangleleft G/N$

(Proposition 2.3.1)

Proof.

1. $can^{-1}(K) \leq G$, with $N \subseteq can^{-1}(K)$

   Firstly, $can^{-1}(K)$ is non-empty, since $eN \in K \leq G/N$, and $can(e_G) = eN = N$, so $e_G \in can^{-1}(K)$. We now check closure. Let:
   $$h_1, h_2 \in can^{-1}(K)$$
   so that:
   $$can(h_1), can(h_2) \in K$$
   Then:
   $$can(h_1)can(h_2) = can(h_1h_2)$$
   But $can(h_1), can(h_2) \in K$, so $can(h_1h_2) \in K$, by closure of the subgroup, so:
   $$h_1h_2 \in can^{-1}(K)$$
   and so, $can^{-1}(K)$ is closed.
   We now check existence of inverse. Let:
   $$h \in can^{-1}(K)$$
   so that:
   $$can(h) \in K$$
   Since k is a subgroup, $can(h)^{-1}$ exists
   $$can(h)^{-1} = can(h^{-1}) \in K \implies h^{-1} \in can^{-1}(K)$$
   Hence, $can^{-1}(K) \leq G$
   Finally, notice that since $K \leq G/N$, in particular $N \in K$, so $\forall n \in N$, since $can(n) = N$, then $can(n) \in K \implies n \in can^{-1}(K)$, so $N \subseteq can^{-1}(K)$.

2. $can^{-1}(K) \triangleleft G \iff K \triangleleft G/N$

   Suppose that $K \triangleleft G/N$, and let $h \in can^{-1}(K), g \in G$. Then:
   $$can(h) \in K$$
   and:
   $$can(ghg^{-1}) = can(g)can(h)can(g^{-1}) = can(g)can(h)can(g)^{-1}$$
   Since $can(h) \in K \triangleleft G/N$, it follows that:
   $$can(g)can(h)can(g)^{-1} \in K \implies ghg^{-1} \in can^{-1}(K)$$
   so it follows that:
   $$g(can^{-1}(K))g^{-1} \subseteq can^{-1}(K)$$
   and so,
   $$can^{-1}(K) \triangleleft G$$
   On the other hand, assume that $K \not\triangleleft G/N$. Then:
   $$\exists a \in G/N, b \in K \ : \ a^{-1}ba \notin K$$
   Since can is surjective, $\exists g \in G, h \in can^{-1}(K)$ such that:
   $$can(g) = a \qquad can(h) = b$$
   Thus:
   $$can(ghg^{-1}) = can(g)can(h)can(g^{-1}) \notin K$$
   so:
   $$ghg^{-1} \notin can^{-1}(K)$$
   and so:
   $$can^{-1}(K) \not\triangleleft G$$

□

4.2 Proposition: Mapping Factor Subgroups to Subgroups

Let $N \triangleleft G$ and let:

$$can : G \to G/N$$

be the ***canonical map****.*
If:

$$N \leq H \leq G$$

then:

$$H = can^{-1}(can(H))$$

That is, if $H \leq G$, such that H contains a ***normal subgroup*** *of G, then H can be obtained by reverse mapping subgroups of G/N.*
(Proposition 2.3.2)

Proof. Let $g \in can^{-1}(can(H))$. Then:

$$can(g) \in can(H)$$

That is, $\exists h \in H$ such that:

$$can(g) = can(h) \iff can(h^{-1}g) = N \iff h^{-1}g \in N \iff g \in hN$$

But since $N \leq H$, we have that $hN \subseteq H$ so:

$$g \in H$$

Hence:

$$g \in can^{-1}can(H)) \iff g \in H \implies can^{-1}can(H)) = H$$

as required.

□

4.3 Theorem: The Correspondence Theorem

The propositions above allow us to show that the canonical map maps normal subgroups of G containing N to normal subgroups of G/N.

Let G be a group, $N \triangleleft G$ and let:

$$can : G \to G/N$$

be the ***canonical map****.*
The map:

$$H \mapsto can(H)$$

is a ***bijection*** *between* ***subgroups*** *of G containing N, and* ***subgroups*** *of G/N.*
Under this bijection, ***normal subgroups*** *match with* ***normal subgroups****.*
Further, if $N \subseteq A, B$ are subgroups of G, then:

$$can(A) \subseteq can(B) \iff A \subseteq B$$

(Theorem 2.3.3)

Proof. Let $K \leq G/N$. Then, $can^{-1}(K) \leq G$, and $N \subseteq can^{-1}(K)$ by (4.1). But then it follows by (4.2) that we have:

$$H = can^{-1}(can(H)), \qquad K = can(H)$$

In other words, the subgroup H containing N has a direct, bijective mapping to a subgroup $can(H)$ of G/N.
Now, suppose that:

$$N \leq A \leq B \leq G$$

This immediately implies:

$$can(A) \subseteq can(B)$$

Now, suppose that $can(A) \subseteq can(B)$ and let $a \in A$. Then:

$$\exists b \in B \ : \ aN = bN \implies ab^{-1} \in N$$

Thus:

$$\exists n \in N \ : \ ab^{-1} = n \implies a = nb \in B$$

where we have used the fact that that $N \subseteq B$. Thus, $a \in B$, so:

$$A \subseteq B$$

as required. □

4.4 Theorem: The Third Isomorphism Theorem

If $N \leq H \leq G$, with $N, H \triangleleft G$, then:

$$(G/N)/(H/N) \cong G/H$$

(Theorem 2.3.5)

Proof. We want to show that there exists a mapping π such that:

$$\pi : G/N \to G/H$$

is surjective and has kernel:

$$ker(\pi) = H/N$$

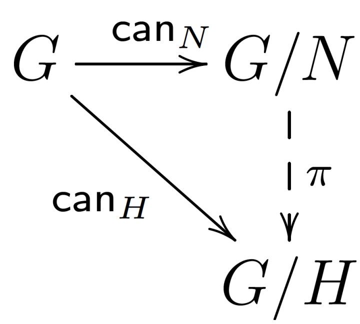
Notice:

$$ker(can_N) = N \subseteq ker(can_H) = H$$

so we can apply the universal property of factor groups, to get that:

$$can_H = \pi \circ can_N$$

Diagrammatically:



can_H is surjective, so π will also be surjective.
Now, assume $gN \in ker(\pi)$. Then:

$$e = \pi(gN) = \pi(can_N(g)) = can_H(g)$$

Hence:

$$gN \in ker(\pi) \iff g \in ker(can_H) = H$$

In other words, gN is in the kernel of π whenever $g \in H$; so the cosets gN are in fact H/N, so:

$$ker(\pi) = H/N$$

Thus, by the First Isomorphism Theorem:

$$(G/N)/ker(\pi) \cong im(\pi) \implies (G/N)/(H/N) \cong (G/H)$$

as required.

□

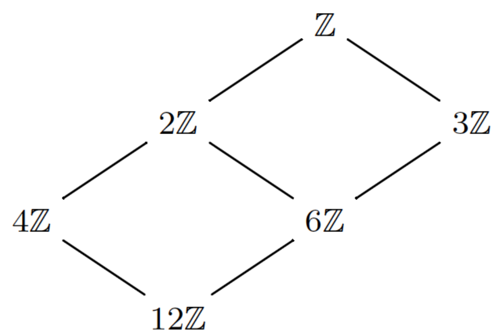
4.5 Worked Examples

1. **Find all subgroups of $\mathbb{Z}_{12}$ together with their inclusions.**

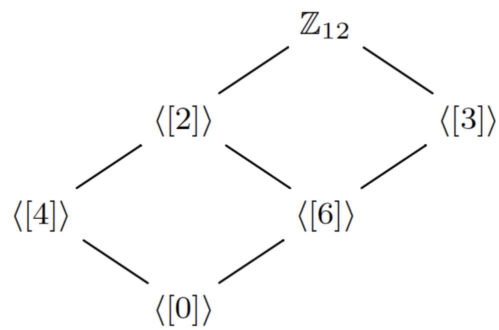
   We can write:

   $$\mathbb{Z}/12\mathbb{Z}$$

   By the correspondence theorem, the subgroups of $\mathbb{Z}_{12}$ will be isomorphic to those subgroups of $\mathbb{Z}$ which contain the normal subgroup $12\mathbb{Z}$.
   The subgroups of $\mathbb{Z}$ containing $12\mathbb{Z}$ are:



So by the Correspondence Theorem, the subgroups of $\mathbb{Z}_{12}$ will be:



where:

$$\langle [n] \rangle = \langle \bar{n} \rangle = can(n)$$

is the cyclic subgroup generated by $\bar{n}$ (i.e $\langle [3] \rangle = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$).

2. Consider the inclusion:

$$10\mathbb{Z} \leq 5\mathbb{Z} \leq \mathbb{Z}$$

By the Third Isomorphism Theorem:

$$(\mathbb{Z}/10\mathbb{Z})/(5\mathbb{Z}/10\mathbb{Z}) \cong \mathbb{Z}/5\mathbb{Z}$$

which we already saw above.

5 Exercises