Fundamentals of Pure Mathematics: Theorems and Definitions

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Contents

1 Week 1: Symmetries and Groups

1.1 Symmetries and Graphs

- **Graph**: a finite set of vertices joined by edges
- **Valency**: the valency of a vertex is the number of edges emerging from it
- **Isomorphism**: a *bijection* between 2 graphs that *preserves edges*
  - If Γ_1, Γ_2 are graphs, with vertices given by V_1, V_2, then an **isomorphism** f from Γ_1 to Γ_2 is:

    $$f : V_1 \to V_2$$

    and $v_1, v_2 \in V_1$ are connected by an edge in Γ_1 **if and only if** $f(v_1), f(v_2) \in V_2$ are connected by an edge in Γ_2.
- **Isomorphic Graphs**: graphs for which there exists an isomorphism
- **Symmetry**: an isomorphism from a graph to itself

1.2 Groups

- **Binary Operation**: $*$ is a rule on a set S, such that:

  $$(a, b) \in S \times S \implies a * b \in S$$

  and $a * b$ is unique.

  ✓ $x, y \in \mathbb{Q}, (x, y) \to x^y$

  × $x, y \in \mathbb{Q}$, (x $= \frac{a}{b}$, y $= \frac{c}{d}$) $\to \frac{a+c}{b+d}$
- **Group**: a set G is a group under operation $*$ if it satisfies the *group axioms*
  - **Closure Under $*$**: $\forall g, h \in G,\ g * h \in G$
  - **Associativity**: $\forall g, h, k \in G,\ g * (h * k) = (g * h) * k$
  - **Identity**: $\exists e \in G : \forall g \in G,\ e * g = g * e = g$
  - **Existence of Inverse**: $\forall g \in G,\ \exists g^{-1} \in G : g * g^{-1} = g^{-1} * g = e$
- **Order**: the order of group G is the number of elements it contains, and is denoted by $|G|$
- **Abelian Group**: a group in which all of its elements commute:

  $$\forall g, h \in G,\ g * h = h * g$$

- **Product Group**: if $(G, *_G)$ and $(H, *_H)$ are groups, then we define the product group as:

$$G \times H = \{(g, h) \mid g \in G, h \in H\}$$

  - $(g_1, h_1) * (g_2, h_2) = (g_1 *_G g_2, h_1 *_H h_2)$
  - $e_{G \times H} = (e_G, e_H)$
  - $(g, h)^{-1} = (g^{-1}, h^{-1})$

- **Order of Product Group**: the order of a product group $G \times H$ is $|G||H|$

- **Unique Element Product**: if G is a group, and $g, h \in G$, then $\exists k_1, k_2 \in G$, which are **unique**, such that:

$$k_1 * g = h$$

$$g * k_2 = h$$

- **Cancellation Law**: if $s * g = t * g$, then $s = t$ (if $g * s = g * t$, then $s = t$ also)

- **Uniqueness of Inverses**: inverses in groups are unique. If $g \in G$, there is a unique $h \in G$ such that $gh = e$, namely $h = g^{-1}$

  - $e * e = e \implies e = e^{-1}$
  - $(g^{-1})^{-1} = g$

- **Uniqueness of Identity**: a group has only 1 identity. If $g * h = h$, then $g = e$

- **Symmetries as Groups**: the symmetries of a graph form a group under composition

- **The Symmetric Group**: S_n is a symmetric group representing the *set of all permutations of n elements*

  - an edge-less graph with n nodes has symmetries given precisely by a permutation with n elements
  - $|S_n| = n!$
  - S_n is abelian only when $n = 2$

- **The Dihedral Group**: D_n is a dihedral group representing the *set of all symmetries of a regular n-gon*

  - n symmetries correspond to rotations, and n symmetries correspond reflections on an axis, so $|D_n| = 2n$
  - D_n is **not** abelian

- typically described with $g = \frac{2\pi}{n}$ anti-clockwise rotation, and h being a reflection across a line
- represent as:

$$D_n = \{e, g, g^2, \cdots, g^{n-1}, h, gh, \cdots, g^{n-1}h\}$$

- $g^n = h^2 = e$
- $hg^k = g^{n-k}h$

- $\boldsymbol{GL(n, \mathbb{R})}$: group representing the set of all $n \times n$ matrices of real numbers under matrix multiplication

2 Week 3: Subgroups and Lagrange's Theorem

2.1 Subgroups

- **Subgroup**: a subgroup H of G is a non-empty subset of G, which is a group itself. $H \subseteq G$ is a subgroup if and only if:
  - $H \neq \emptyset$
  - $h, h' \in H \implies h * h' \in H$
  - $h \in H \implies h^{-1} \in H$

  If G is finite, H is a subgroup if and only if:
  - $H \neq \emptyset$
  - $h, k \in H \implies hk^{-1} \in H$
- **Trivial Subgroup**: $\{e_G\} \leq G$
  - trivially, G is a subgroup of itself, so $G \leq G$
- **Normal Subgroup**: if $H \leq G$, and:

  $$ghg^{-1} \in H,\ \forall g \in G,\ \forall h \in H$$

  then H is a normal subgroup, and we say $H \triangleleft G$
- **Identity in Subgroup**: if $H \leq G$, then $e_H = e_G$
- **Inverse in Subgroup**: if $H \leq G$, then for any element $h \in H$, we must also have $h^{-1} \in H$

2.2 Cyclic Groups

- **Order of a Group Element**: if G is a group, and $g \in G$, then the order of g, $o(g)$, is the smallest natural number n such that:

  $$g^n = e_G$$

  - if such an n doesn't exist, $o(g) = \infty$
- **Order in Finite Group**: the elements in a finite group have *finite order*
- $\langle \boldsymbol{g} \rangle$: if $g \in G$, then $\langle g \rangle$ is a subgroup of G, given by:

  $$\langle g \rangle = \{e, g, g^2, \cdots, g^{o(g)-1}\}$$

- **Cyclic Group**: G is cyclic if:

  $$\exists g \in G : \langle g \rangle = G$$

  - g is a *generator of* G
  - always abelian
  - a group G is cyclic **if and only if** it contains an element of order $|G|$. For example, $\mathbb{Z}_6$ is cyclic, as 5 has order 6 ($5 \to 5, 10 \to 4, 15 \to 3, 20 \to 2, 25 \to 1, 30 \to 0 = e$)
  - $\mathbb{Z}_n^+$ is cyclic, as 1 is always a generator

- **Cyclic Subgroups**: $H \leq G$ is cyclic if $\exists h \in H : \langle h \rangle = H$
  - the rotations of D_n form a cyclic subgroup, although D_n itself is not cyclic

- **Subgroups of Cyclic Groups**: any subgroup of a cyclic group is cyclic.
  - For example $\langle 2 \rangle = \{0, 2, 4, 6\}$ is a subgroup of $\mathbb{Z}_8$

- **Product of Cyclic Groups**: if G is a cyclic group of order m, and H is a cyclic group of order n, then $G \times H$ is cyclic **if and only if** m is coprime to n

2.3 Cosets

- **Left Coset**: if $H \leq G$, the left coset of H in G, is the subset of G given by gH:

$$gH = \{gh \mid \forall h \in H\}$$

  - a **Right Coset** is given by Hg
  - a group G is abelian **if and only if** $gH = Hg$
  - $\forall g \in G,\ g \in gH, Hg$

- **Sets of Cosets**: G/H denotes the set of all left cosets; $G \backslash H$ denotes the set of all right cosets

- **Cosets From Elements of the Same Group**: applying an element of a group to the whole group leads to the left/right cosets being equal to the group:

$$\forall h \in H,\ hH = Hh = H$$

- **Equivalences Across Coset Properties**: if $g_1, g_2 \in G$, the following are equivalent:

$$g_1 H = g_2 H$$

$$\exists h \in H : g_2 = g_1 h$$

$$g_2 \in g_1 H \ \therefore\ g_1^{-1} g_2 \in H$$

- **Cosets as Equivalence Relations**: if $g_1, g_2 \in G$, and we define the relation $g_1 \sim g_2$ as $g_1 H = g_2 H$. Then $\sim$ defines an equivalence relation on G.

- this means that cosets can be used to partition a group (cosets correspond to equivalence classes)

- **Order of Cosets**: if $H \leq G$ and H is finite, then $|gH| = |H|$

2.4 Lagrange's Theorem

- **Lagrange's Theorem**: if G is a finite group, and $H \leq G$, then H divides G

- **Order of Group Elements Divide Order of Group**: if $g \in G$, by Lagrange's Theorem, $o(g)$ divides $|G|$

- **Order of Group Leads to Identity**: if $g \in G$, then by Lagrange's Theorem:
$$g^{|G|} = e$$

- **Index of a Subgroup**: the number of distinct left cosets of H in G, given that $H \leq G$. By Lagrange's Theorem:
$$|G/H| = \frac{|G|}{|H|}$$
  - this also applies for right cosets, so the number of distinct left cosets is the same as the number of distinct right cosets

- **Groups of Prime Order**: if the order of a group G is prime, then G is cyclic (and so, abelian)

- **Abelian Groups Given Order**: if a group G is such that $|G| < 6$, then G is *abelian*

- **Fermat's Little Theorem**: if p is prime, then:
$$a^{p-1} \equiv a \ mod \ p$$

- **Abelian Groups From Inverses**: a group G is abelian if all of its elements are their own inverse:
  - then $g^2 = e$, so we must have $gh = (gh)^{-1} = h^{-1}g^{-1} = hg$. Hence, abelian.

3 Week 5: Going Between Groups

3.1 Morphisms

- **Group Homomorphisms**: if G and H are groups, then a map $\phi : G \to H$ is a group homomorphism if:

$$\phi(x *_G y) = \phi(x) *_H \phi(y), \ \forall x, y \in G$$

- **Group Isomorphism**: a group homomorphism which is a bijection
- **Isomorphic Groups**: 2 groups are isomorphic, if there exists an isomorphism between the groups
  - we denote isomorphic groups using $G \cong H$
  - isomorphic groups are *algebraically indistinguishable*, as the ismorphism allows us to match up group elements perfectly
  - all cyclic groups of order n are isomorphic (thus can refer to *the* cyclic group). Thus, all groups of prime order p are isomorphic.
  - $D_3 \cong S_3$
  - isomorphisms are equivalence relations
- **Group and Graph Isomorphisms**: if 2 graphs are isomorphic, then their symmetric groups are also isomorphic
- **Inverse Isomorphism**: the inverse of an isomorphism is also an isomorphism
- **Group Automorphism**: a group isomorphism from a group to itself ($\phi : G \to G$)
- **Identity Under Homomorphism**: homomorphisms preserve identities:

$$\phi(e_G) = e_H$$

- **Powers of Homomorphisms**: $\phi(g^n) = \phi(g)^n$
- **Inverses of Homomorphisms**: $\phi(g^{-1}) = (\phi(g))^{-1}$
- **Order of Group Elements Under Injective Homomorphisms**: if a homomorphism is *injective*, then the order of g is the same as the order of $\phi(g)$
- **Homomorphisms on Subgroups**: if $G' \leq G$, then $\phi(G') \leq H$

3.2 Image and Kernel

- **Image of a Group Homomorphism**: the image of a group homomorphism ϕ is all the elements that ϕ can map to:

$$im(\phi) = \{h = \phi(g) \mid g \in G\} \subseteq H$$

  - the image of a homormorphism is a subgroup of H ($im(\phi) \leq H$)

- **Kernel of a Group Homomorphism**: the kernel of a group homomorphism ϕ is the set of all elements that map to e_H under ϕ:

$$ker(\phi) = \{g \mid \phi(g) = e_H,\ g \in G\}$$

  - the kernel of a homomorphism is a *normal subgroup* of G ($ker(\phi) \lhd G$)

- **Injective Homomorphisms and their Kernel**: a group homomorphism ϕ is injective **if and only if** $ker(\phi) = \{e_G\}$

- **Injective Homomorphisms and Isomorphic Groups**: if a group homomorphism ϕ is injective, then ϕ defines an isomorphism $G \cong im(\phi)$

3.3 Products of Groups and Isomorphisms

- **Decomposing Group into Product of Subgroups**: if $H, K \leq G$, then $\phi : H \times K \to G$ defines a group isomorphism **if and only if**:
  - $H \cap K = \{e\}$
    * then, we have that ϕ is bijective and $|HK| = |H||K|$, where $HK = \{hk \mid h \in H,\ k \in K\}$
  - $hk = kh,\ \forall h \in H,\ \forall k \in K$
    * alongisde the condition $H \cap K = \{e\}$, this shows that $HK \leq G$, and furthermore ϕ defines an isomorphism such that $H \times K \cong HK$
  - $G = HK$
    * then, from the above point, since $H \times K \cong HK$ and $HK = G$, then we have shown that ϕ is an isomorphism such that $H \times K \cong G$

- **Properties of Isomorphic Groups**: if $G \cong H$, then:
  - $|G| = |H|$
  - G is abelian **if and only if** H is abelian
  - G is cyclic **if and only if** H is cyclic
  - G and H have the same number of elements of each order

- **Order of Product of Groups**: the order of an element $(g, h) \in G \times H$ is given by:

$$o((g, h)) = lcm(o(g), o(h))$$

4 Week 4: Groups Acting

4.1 Group Actions

- **Left Action**: if G is a group and X is a non-empty set, then the map:

  $$G \times X \to X$$

  defined by:

  $$(g, x) \to g \cdot x$$

  is a left action of G on X. We mean that applying any element of G to any element in X produces an element in X (so G can be thought as a symmetry group of G, although this isn't true in practice). It has the following properties:

  - $e \cdot x = x$
  - $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$

  We can think of D_n acting on a set $\{1, 2, \cdots, n\}$ which represents the vertices of a regular n-gon. Alternatively, D_n also acts on $\{T, B\}$, the top and bottom faces of the n-gon.

- **Right Action**: defined similarly by the map:

  $$(g, x) \to xg^{-1}$$

- **Trivial Group Action**: for any group G and non-empty set X, we can define a trivial (left or right) action:

  $$(g, x) \to x$$

- **Group Acting on Itself**: there are 3 ways in which a group can act on itself. Let G be a group, and $X = G$, which $g \in G$ and $h \in X$. Then:

  - *Left Action*: $(g, h) \to gh$
  - *Right Action*: $(g, h) \to hg^{-1}$
  - *Conjugate Action*: $(g, h) \to ghg^{-1}$

- **Kernel of an Action**: the kernel N can be thought of as the set of all trivial actions given by a group G acting on a set X:

  $$N = \{g \mid g \cdot x = x, \ \forall x \in X, \ g \in G\}$$

  - the kernel of an action is a subgroup of the acting group ($N \leq G$). In fact, it is a normal subgroup, so $N \triangleleft G$

- **Faithful Action**: an action such that the only element of G which fixes everything in X is the identity e. Thus, an action is faithful if:

  $$N = \{e\}$$

- a group acting on itself is always faithful, as e is the only element of the group for which $g \cdot x = gx = x$
- D_3 is faithful when acting on the vertices of a triangle, but not faithful if it acts on the faces
- the group of rotational symmetries acting on the faces of a platonic solid is faithful

4.2 Orbits and Stabilisers

- **Stabilizer of Set Element x**: let G be a group acting on X, with $x \in X$. Define the Stabilizer of x by:

$$Stab_G(x) = \{g \mid g \cdot x = x,\ g \in G\} \subseteq G$$

  - think of stabilizer as the elements of G that fix x (i.e all the g that lead to a trivial action for a particular x)
  - $Stab_G(x) \leq G$
  - the kernel of the action is composed by all trivial actions, which is precisely given by:

$$N = \bigcap_{x \in X} Stab_G(x)$$

- **Orbit of Set Element x**: let G be a group acting on X, with $x \in X$. Define the Orbit of x by:

$$Orb_G(x) = \{g \cdot x \mid g \in G\} \subseteq X$$

  - the set of all elements that can be reached from x by applying the actions in G

- **Orbits as Equivalence Classes**: we can define an equivalence relation $\sim$ via:

$$x \sim y \iff y = g \cdot x,\ g \in G$$

  Then, the orbits of G define the equivalence classes of this relation. Thus, when G acts on X, the orbits partition X

  - if $H \leq G$, then if H acts on G via a left action $(h, g) \to hg$, the orbits of the action are precisely the right cosets of H in G. Notice that, cosets partition G, and these cosets are precisely the orbits of H, which indeed partition G.

- **Transitive Action**: an action of G on X such that $\forall x, y \in X$, $\exists g \in G$ such that $y = g \cdot x$.

  - this means that we can define all of X as a single orbit under G
  - D_n acts transitively on the vertices of an n-gon, as we can always find a rotation g^t which maps a vertex x_1 to a vertex x_2
  - S_n acts transitively

4.3 The Orbit-Stabiliser Theorem

- **Orbit-Stabilizer Theorem**: let G be a finite group acting on a set X, and let $x \in X$. Then:

$$|Orb_G(x)| \times |Stab_G(x)| = |G|$$

4.4 Cauchy's Theorem

- **Cauchy's Theorem**: let G be a group and p be prime. If p divides $|G|$, then G contains an element of order p

5 Week 9: Pólya Counting and Conjugacy

5.1 Pólya Counting

- **Fixed Point Set**: if G is a group acting on X, and $g \in G$, the fixed point set is the set fo all elements in X that are fixed under an action g:

$$Fix(g) = \{x \mid x \in X,\ g \cdot x = x\}$$

  - we can redefine the kernel of an action in terms of the fixed point set:

$$N = \{g \mid g \cdot x = x,\ \forall x \in X,\ g \in G\} = \{g \mid Fix(g) = X,\ g \in G\}$$

- **Pólya Counting**: if G is a finite group acting on a finite set X, then:

$$\#\ of\ orbits\ in\ X = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|$$

  - we can use PC to find the number of distinct colourings that can be applied to a graph. If G is the group of symmetries of the graph, consider the set X of all possible colourings of the graph (if there are n nodes and k colours, there are k^n total colourings). Then, if G acts on X, the orbits of X correspond to all the unique colourings (each orbit represents all colourings that can be reached via symmetry from one of the colourings within the orbit). Thus, the total number of colourings is the toal number of orbits in X, which can be found via Pólya.

5.2 Conjugacy

- **Conjugacy Classes of a Group**: if G is a group which acts on itself via a *conjugacy action*, then the orbits of G (under the action G) are the conjugacy classes

  - since orbits partition the set, G is partitioned by its conjugacy classes

– the conjugacy class of some element $h \in G$ is:

$$Conj(h) = Orb_G(h) = \{k \mid ghg^{-1} = k,\ g \in G\}$$

- **Abelian Groups From 1-Element Conjugacy Classes**: if G is abelian, then each $h \in G$ defines its own conjugacy class, as $\forall g \in G$:

$$g \cdot h = ghg^{-1} = hgg^{-1} = h$$

- **Conjugate Elements**: $h, k \in G$ are conjugate if they are in the same conjugacy class:

$$\exists g, g' \in G : ghg^{-1} = k \iff g'kg'^{-1} = h$$

in this case with $g = g^{-1}$. Notice that $ghg^{-1} = k \implies k \in Conj(h)$ and $g'kg'^{-1} = h \implies h \in Conj(k)$

- **Conjugates Look The Same**: we can define an automorphism:

$$\phi_G : G \to G,\ \phi_G(h) = ghg^{-1}$$

  – for example, in D_4, $\{g, g^3\}$ form a conjugacy class and they each correspond to 90^{o} rotations
  – conjugate elements have the same order
  – $|H| = gHg^{-1}$
  – if conjugates form a subgroup of G, then they are a normal subgroup

- **The Class Equation**: we can define the order of a group in terms of its conjugacy classes:

$$|G| = \sum_i |Conj(i)|$$

  – $Conj(i)$ is an orbit of G, so in particular $|Conj(i)|$ divides $|G|$ by OST

- **Identity as a Conjugacy Class**: the identity always forms its own conjugacy class, $\{e\}$

5.3 Centres and Centralizers of a Group

- **Centre of a Group**: a set containing all elements of a group G which commute with any other element:

$$C(G) = \{g \mid g \in G,\ gh = hg,\ \forall h \in G\}$$

  – the centre of a group is actually the kernel of the conjugation action:

$$\begin{aligned} N &= \{g \mid g \cdot h = h,\ \forall h \in G,\ g \in G\} \\ &= \{g \mid ghg^{-1} = h,\ \forall h \in G,\ g \in G\} \\ &= \{g \mid gh = hg,\ \forall h \in G,\ g \in G\} \end{aligned}$$

- since the kernel of a group action is a normal subgroup of the acting group, $C(G) \triangleleft G$
- $C(G) = G$ **if and only if** G is abelian
- $C(S_n) = \{e\}$

- **Central Element**: any element $g \in G$ such that $g \in C(G)$

- **Centralizer of a Group**: the stabiliser of an element $g \in G$ when G acts on itself via a conjugate action

$$C(g) = \{h \mid h \cdot g = hgh^{-1} = g,\ h \in G\} = \{h \mid hg = gh,\ h \in G\}$$

  - in other words, the centralizer is the set of all elements $h \in G$ which commute with a particular element $g \in G$
  - $C(G)$ is the set of all elements which commute with every other element, so it follows that:

$$C(G) = \bigcap_{g \in G} C(g)$$

  - since $C(g) = Stab_G(g)$, $C(g) \leq G$
  - morever the center is a subgroup of any centralizer ($C(G) \leq C(g)$)

- **Order of Group In Terms of Centralizer and Conjugacy Class**: from OST

$$|Conj(g)||C(g)| = |G|$$

  - under conjugate action, $C(g)$ is just the stabilizer of g
  - under conjugate action, $Conj(g)$ represents the conjugacy class of g, which are the orbits of g in G

- **Centre as a Union of 1-Element Conjugacy Classes**: the centre of a group $C(G)$ can be viewed as the union of all 1-element conjugacy classes:

$$\{g\} = Conj(g) \iff g \in C(G)$$

  - if g is conjugate to itself only, then it must commute with all other elements of G ($hgh^{-1} = g \iff gh = hg, \forall h \in G$), so by definition it must be part of $C(G)$

- **Prime Powers as Order of a Group**: if $|G| = p^k$ for some prime p, then it must be the case that $|C(G)| \geq p$

- **Abelian Groups from Order of Group**: if a group has order p^2, for prime p, then the group is *abelian*

5.4 Permutations as Cycles

- **Length of a Cycle**: a permutation given by $\begin{pmatrix} a_1 & a_2 & \cdots & a_r \end{pmatrix}$ has cycle length r

- **Disjoint Cycles**: cycles which share no element
  - disjoint cycles commute
  - any element of S_n can be written as a product of disjoint cycles

- **Cycle Type**: specifies the structure of a product of cycles, using the notation n^k to represent that an n-cycle occurs k times.
  - $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}$ has cycle shape $2, 3^2$

- **Number of Elements of a Cycle Type**: the number of elements of cycle type $1^{m_1}, 2^{m_2}, \cdots, n^{m_n}$ is given by:

$$\frac{n!}{m_1!m_2!\cdots m_n!1^{m_1}2^{m_2}\cdots n^{m_n}}$$

- **Applying Conjugate Action to Disjoint Cycles**: consider a permutation given as a product of disjoint cycles:

$$\sigma = \begin{pmatrix} a_1 & \cdots & a_r \end{pmatrix} \begin{pmatrix} b_1 & \cdots & b_s \end{pmatrix} \cdots$$

  Then, $\forall \tau \in S_n$,

$$\tau\sigma\tau^{-1} = \begin{pmatrix} \tau(a_1) & \cdots & \tau(a_r) \end{pmatrix} \begin{pmatrix} \tau(b_1) & \cdots & \tau(b_s) \end{pmatrix} \cdots$$

- **Conjugate Permutations**: 2 permutations are conjugate **if and only if** they have the same cycle type

6 Analysis

6.1 Real Numbers and Sequences

- **Maximum Element of a Set**: for a set S, if $\exists s' \in S : s \leq s',\ \forall s \in S$, s' is the maximum of S
- **Minimum Element of a Set**: for a set S, if $\exists s' \in S : s \geq s',\ \forall s \in S$, s' is the minimum of S
- **Closed Interval**: an interval containing all elements between the endpoints, and including the endpoints (except possibly infinities):

$$[a, b] = \{x \mid a \leq x \leq b,\ x \in \mathbb{R}\}$$

- **Open Interval**: an interval containing all elements between the endpoints, and excluding the endpoints:

$$(a, b) = \{x \mid a < x < b,\ x \in \mathbb{R}\}$$

- **Upper Bound of a Set**: if E is a set, and $\exists M \in \mathbb{R} : \forall a \in E, a \leq M$, M is an upper bound of E, and E is bounded above.
- **Supremum of a Set**: the smallest of the upper bounds to a set, denote $sup\ E$ for a set E
- **Upper Bound of a Set**: if E is a set, and $\exists m \in \mathbb{R} : \forall a \in E, a \geq m$, m is a lower bound of E, and E is bounded below
- **Infimum of a Set**: the largest of the lower bounds to a set, denote $inf\ E$ for a set E
- **Bounded Set**: set which is bounded above and below
- **Completeness Axiom**: if $E \subset \mathbb{R}$ is *non-empty* and *bounded above*, then $sup\ E$ exists and it is a real number
  - *Corollary*: every non-empty, bounded below set has a real $inf\ E$
- **Approximation Property of Suprema**: if a set E has a supremum, $\forall \epsilon > 0,\ \exists a \in E : sup\ E - \epsilon < a \leq sup\ E$
- **Supremum in Subset of Natural Numbers**: the supremum of a subset of natural numbers is contained within the set
- **Archimidean Principle**: if $a, b \in \mathbb{R}^+,\ \exists n \in \mathbb{N} : b < na$
- **Sequence of Real Numbers**: a function $f : \mathbb{N} \to \mathbb{R}$. The n^{th} term of the sequence is defined by $x_n = f(n)$

- **Convergent Sequence**: a sequence $(x_n)_{n\in\mathbb{N}}$ converges to $L \in \mathbb{R}$ if:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} : \forall n \geq N$$

$$|x_n - L| < \epsilon$$

and we say:

$$\lim_{n\to\infty} x_n = L$$

- **Absolute Value of Limits**: if $\lim_{n\to\infty} x_n$ exists, then so does $\lim_{n\to\infty} |x_n|$ and $\lim_{n\to\infty} |x_n| = |\lim_{n\to\infty} x_n|$
- **Limit to 0**: $\lim_{n\to\infty} x_n \to 0 \iff \lim_{n\to\infty} |x_n| \to 0$
- **Product of Convergent and Bounded Sequence**: if $x_n \to 0$ and y_n is bounded, then $x_n y_n \to 0$
- **Limit to Supremum**: if a set E has a finite supremum $sup\ E$, then there exists a sequence of terms in E, namely x_n, such that $x_n \to sup\ E$
- **Bounded Sequence and Convergence**: if a sequence converges, then it is bounded
- **Divergent Sequence to Infinity**: x_n diverges to $+\infty$ if:

$$\forall M \in \mathbb{R},\ \exists N \in \mathbb{N} : \forall n \geq N,\ x_n > M$$

- **Divergent Sequence to Negative Infinity**: x_n diverges to $+\infty$ if:

$$\forall m \in \mathbb{R},\ \exists N \in \mathbb{N} : \forall n \geq N,\ x_n < m$$

- **Monotonic Sequence**: a sequence which is either increasing or decreasing
- **Monotonic Convergence Theorem**: if a sequence is monotone and bounded, it is convergent, and it converges to the supremum/infimum of the set of elements
  - a monotone, unbounded sequence diverges to $+/-\infty$
- **Subsequence**: if $(s_n)_{n\in\mathbb{N}}$ is a sequence, $(a_{n_k})_{k\in\mathbb{N}}$ defines a subsequence, where $n_1 < n_2 < \cdots$ is an increasing sequence.
- **Convergence of Subsequence**: a subsequence of a convergent sequence is convergent, and converges to the same limit
- **Bolzano-Weierstrass Theorem**: any bounded sequence has a convergent subsequence
  - there exists a subsequence of (x_n) which converges to $t \in \mathbb{R}$ **if and only if** there are infinitely many $n \in \mathbb{N}$ such that $|x_n - t| < \epsilon,\ \forall \epsilon > 0$
- **Relationship Between Unbounded Sequence and Diverging Subsequence**: a sequence is unbounded **if and only if** it contains a diverging subsequence

6.2 Infinite Series of Real Numbers

- **Infinite Series**: if a_n is a sequence, an infinite series is:

$$\sum_{k=1}^{\infty} a_k$$

- **Partial Sum**: $S_n = \sum_{k=1}^{n} a_k$
- **Convergence of Infinite Series**: an infinite series converges **if and only if** its sequence of partial sums converges. Then, the limit of the sequence is defined as the value of the inifinite series.
  - a series diverges if its sequence of partial sums diverges
- **Geometric Series**: a series of the form:

$$\sum_{k=0}^{\infty} ar^k$$

  The partial sum is:

$$S_n = a\frac{1-r^n}{1-r}$$

  The series converges if $|r| < 1$, and ti converges to:

$$\lim_{n\to\infty} S_n = \frac{a}{1-r}$$

- **Harmonic Series**: $\sum_{k=1}^{\infty} \frac{1}{k}$
  - the Harmonic Series diverges
- **Convergence of Series and Sequence**: if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$
- **Divergence Test**: if $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ diverges
- **Sum of Convergent Sum**: if $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then:

$$\sum_{n=1}^{\infty} a_n + b_n = A + B$$

- **Telescoping Series**: $\sum_{n=1}^{\infty}(a_n - a_{n+1} = a_1 - \lim_{n\to\infty} a_n$
- **Convergence of Series with Positive Terms**: if $a_n \geq 0$, $\sum_{n=1}^{\infty} a_n$ converges **if and only if** the sequence of partial sums is bounded:

$$\sum_{n=1}^{\infty} a_n = S \iff \exists M > 0 : |\sum_{k=1}^{n} a_k| \leq M$$

- **Comparison Test**: if $0 \leq a_n \leq b_n$ for some $n \geq N \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} a_n \textit{ diverges} \implies \sum_{n=1}^{\infty} b_n \textit{ diverges}$$

$$\sum_{n=1}^{\infty} b_n \textit{ converges} \implies \sum_{n=1}^{\infty} a_n \textit{ converges}$$

- **p-series Test**: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges **if and only if** $p > 1$. If $p < 1$, the series diverges.

- **Limit Comparison Test**: if $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, $\sum a_n$ converges **if and only if** $\sum b_n$ converges.

  - $L = 0$ **and** $\sum b_n$ converges $\implies \sum a_n$ converges
  - $L = 0$ **and** $\sum b_n$ diverges $\implies \sum a_n$ diverges

- **Cauchy's Condensation Test**: if a_n is a decreasing sequence of non-negative terms, $\sum a_n$ converges **if and only if** $\sum 2^n a_{2n}$ converges

- **Root Test**: let $\sqrt[n]{a_n} \to L$.

  - $0 \leq L < 1 \implies \sum a_n$ converges
  - diverges if $L > 1$

- **Ratio Test**: let $\frac{a_{n+1}}{a_n} \to L$.

  - $0 \leq L < 1 \implies \sum a_n$ converges
  - diverges if $L > 1$

- **Cauchy's Criterion**: $\sum a_n$ converges **if and only if** $\forall \epsilon > 0$, $\exists N$: $\forall m, n, m \geq n \geq N$ and

$$\left| \sum_{k=n}^{m} a_k \right| < \epsilon$$

- **Alternating Series Test**: if a_n is a decreasing sequence of non-negative real numbers, and $a_n \to 0$, then:

$$\sum (-1)^n a_n$$

  converges

- **Absolute Convergence**: $\sum a_n$ converges absolutely if and only if $\sum |a_n|$ converges. If a series converges absolutely, it converges

- **Conditional Convergence**: a series $\sum a_n$ converges conditionally if it converges, but $\sum |a_n|$ diverges

6.3 Continuity

- **Continuity at a Point**: let f be a function defined by $f : dom(f) \to \mathbb{R}$, and let $a \in dom(f)$. f is continuous at a, if, for any sequence x_n whose terms all lie in $dom(f)$, and $x_n \to a$, then $f(x_n) \to f(a)$.

- **Continuity on an Interval**: let $S \subseteq dom(f)$. If f is continuous $\forall a \in S$, then f is continuous n the interval S

- **Continuous Function**: a function that is continuous on $dom(f)$

- **Polynomials are Continuous**

- **Continuity Transformations**: if f and g are continuous functions on a common domain, then $f + g$ and fg are continuous on that domain

- **Continuity of Compositions**: if f is continuous at a, and g is continuous at $f(a)$, then $g \circ f$ is continuous at a

- **$\epsilon - \delta$ Definition of Continuity**: let $f : dom(f) \to \mathbb{R}$. Then, f is continuous at $a \in dom(f)$ **if and only if**:

  $$\forall \epsilon > 0,\ \exists \delta > 0$$

  such that whenever $x \in dom(f)$ **and** $|x - a| < \delta$, then:

  $$|f(x) - f(a)| < \epsilon$$

- **Bounded Functions**: let $f : E \to \mathbb{R}$. Then, f is bounded on E if $\forall x \in E,\ |f(x)| \leq M \in \mathbb{R}$

  - continuous functions on closed, bounded intervals are always bounded

- **Extreme Value Theorem**: let $I \subseteq \mathbb{R}$ be a *closed* and *bounded* interval. Let $f : I \to \mathbb{R}$ be a continuous function on I. Then, f is *bounded on I*. Moreover, let:

  $$m = inf\{f(x) \mid x \in I\}$$

  $$M = sup\{f(x) \mid x \in I\}$$

  Then, $\exists x_m, x_M \in I$ such that:

  $$f(x_m) = m$$

  $$f(x_M) = M$$

  - a continuous function on a closed, bounded interval is not only bounded, but it also achieves its maximum and minimum within the interval

- **Intermediate Value Theorem**: let I be a non-degenerate interval, and let $f : I \to \mathbb{R}$ be a continuous function on I. If $a, b \in I,\ a < b$, then on the interval (a, b) f attains all values between $f(a)$ and $f(b)$. That is, if y_0 is between $f(a)$ and $f(b)$ (non-inclusive), then there exists $x_0 \in (a, b)$, such that:
$$f(x_0) = y_0$$
- **Bolzano's Theorem**: if f is continuous on the closed, bounded interval $[a, b]$, and $f(a)f(b) < 0$, then $\exists c \in (a, b)$ such that $f(c) = 0$
  - this is a corollary of the IVT. It says that if a function every goes from being positive to negative (or viceversa), then it must be 0 at some point in between
- **Image of a Function as an Interval**: let $f : [a, b] \to \mathbb{R}$ be continuous. Then, the image of f is a closed, bounded interval (possibly degenerate i.e a point)
- **Image of a Function as an Interval Leads to Continuity**: let $f : [a, b] \to \mathbb{R}$ be a *strictly increasing function*. If the image of f is an interval, then f is continuous on $[a, b]$
- **Continuous Inverse of a Function**: let $f : [a, b] \to \mathbb{R}$ be a continuous, strictly increasing function. Then, $f^{-1} : [f(a), f(b)] \to \mathbb{R}$ is a continuous, strictly increasing function

6.4 Limits of Functions

- **Limits at a Point Not In An Interval**: let $f : dom(f) \to \mathbb{R}$, and $a \in \overline{\mathbb{R}}$. Then:
$$\lim_{x \to a^{dom(f)}} f(x) = L \in \overline{\mathbb{R}}$$
if for every sequence x_n (with all terms being in $dom(f) \backslash \{a\}$) with $x_n \to a$, we have $\lim_{n \to \infty} f(x_n) = L$
  - this definition says that a function need not be defined at a for a limit to exist.
- **Two-Sided Limit**: let $a \in \mathbb{R}$, and let f be defined on an interval I, with $a \in I$. Then, define:
$$\lim_{x \to a} f(x)$$
as $\lim_{x \to a^{I \backslash \{a\}}} f(x)$
- **Right-Handed Limit**: let $a \in \mathbb{R}$, and let f be defined on an interval I, with $a \in I$. Then, define:
$$\lim_{x \to a^+} f(x)$$
as $\lim_{x \to a^{I \cap (a, \infty)}} f(x)$

- **Left-Handed Limit**: let $a \in \mathbb{R}$, and let f be defined on an interval I, with $a \in I$. Then, define:
$$\lim_{x \to a^-} f(x)$$
as $\lim_{x \to a^{I \cap (-\infty, a)}} f(x)$

- **Limit to Positive Infinity**: let f be defined on an open interval (b, ∞). Then, define:
$$\lim_{x \to \infty} f(x)$$
as $\lim_{x \to \infty^I}$f(x)

- **Limit to Negative Infinity**: let f be defined on an open interval $(-\infty, b)$. Then, define:
$$\lim_{x \to -\infty} f(x)$$
as $\lim_{x \to \infty^I}$f(x)

- **Unique Limits**: since limits are unique, our choice, the interval that we choose to select the sequence x_n doesn't matter

- **$\epsilon - \delta$ Definition of the Right-Hand Limit**: let $a \in \mathbb{R}$, and define an open interval I, with a as a left end point. Then, $\lim_{x \to a^+} f(x) = L$ if:
$$\forall \epsilon > 0, \ \exists \delta > 0$$
such that, if $x \in I$:
$$a < x < a + \delta \implies |f(x) - L| < \epsilon$$

- **$\epsilon - \delta$ Definition of the Limit**: let f be defined on D, and let $c \in D$. Then, $\lim_{x \to a} f(x) = L \in \mathbb{R}$ if:
$$\forall \epsilon > 0, \ \exists \delta > 0$$
such that $\forall x \in D$ if $0 < |x - c| < \delta$, then:
$$|f(x) - L| < \delta$$

- **Two-Sided Limit in Terms of One-Sided Limits**: $\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

- **Continuity and Limits**: f is continuous at a **if and only if**:
$$\lim_{x \to a} f(x) = f(a)$$

6.5 Differentiability

- **Differentiability at a Point**: let $f : I \to \mathbb{R}$ be a function, and let I be an open interval. If $x_0 \in I$, then f is differentiable at x_0 if the following limit is defined:
$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$
Alternatively, if the following limit is defined:
$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$
  - we denote the value of the limit with $f'(x_0)$, the *derivative of f at* x_0
- **Right-Hand Derivative**: $f'(x_0^+) = \lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}$, $f : [x_0, b) \to \mathbb{R}$
- **Left-Hand Derivative**: $f'(x_0^-) = \lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}$, $f : (a, x_0] \to \mathbb{R}$
- **Derivative as One-Sided Limits**: f is differentiable at x_0 if $f'(x_0^+) = f'(x_0^-)$
- **Continuity and Differentiability**: if f is differentiable at x_0, for $x_0 \in I$, where I is an open interval, then f is continuous at x_0
- **Differentiability on an Interval**: a function $f : I \to \mathbb{R}$ is differentiable on an interval I if it is differentiable $\forall x \in I$
  - $\forall a \in I$, $f'(a)$ exists
  - alternatively, differentiable on an interval if $f_I'(x_0) = \lim_{x \to x_0^I} \frac{f(x)-f(x_0)}{x-x_0}$
  - if x_0 is not an endpoint, $f_I'(x_0) = f'(a)$
- **Differentiability at an Endpoint**: at an endpoint, the differnetiability of a function is defined in terms of one sided limits. That is, if $I = [a, b]$, then, f is differentiable at a if:
$$\lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h}$$
and differentiable at b if:
$$\lim_{h \to 0^-} \frac{f(b + h) - f(b)}{h}$$
- **Continuous Differentiability**: f is continuously differentiable on an interval if it is differentiable, and the derivative is continuous on the interval
- **Rolle's Theorem**: let $a, b \in \mathbb{R}$, and $a < b$. If f is:
  1. *continuous* on $[a, b]$

2. *differentiable* on (a, b)
3. $f(a) = f(b)$

then $\exists c \in (a, b) : f'(c) = 0$.

- **Mean Value Theorem**: let $a, b \in \mathbb{R}$ and $a < b$.
  1. if f is continuous on $[a, b]$, and differentiable on (a, b), then $\exists c \in (a, b)$, such that:
  $$f'(c) = \frac{f(b) - f(a)}{b - a}$$
  2. if f, g are continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ such that:
  $$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$$
- **Fermat's Theorem**: let I be an open interval, with $c \in I$ and $f : I \to \mathbb{R}$. Let f bedifferentiable at c. If f has a local minimum/maximum at c, then $f'(c) = 0$
  - c is a critical point of f
  - c must not be an end point (hence the requirement of open interval
- **Increasing Function**: if f is a function, and it is differentiable on an interval (a, b), and $\forall x \in I,\ f'(x) > 0$, then f is an increasing function on $[a, b]$
  - the converse is not necessarily true. x^3 is strictly increasing on $[-1, 1]$, but $f'(0) = 0$
- **Decreasing Function**: if f is a function, and it is differentiable on an interval (a, b), and $\forall x \in I,\ f'(x) < 0$, then f is an increasing function on $[a, b]$
- **Monotone Function**: a function that is increasing or decreasing on an interval
- **Continuity and Monotonicity**: let f be injective and continuous on an interval I. Then, f is strictly monotone on I, and the inverse function f^{-1} is continuous and strictly monotone on $f(I)$
- **Inverse Function Theorem**: let f be injective and continuous on an open interval I. If $a \in f(I)$, and f' exists at $f^{-1}(a)$, and is non-zero, then f^{-1} is differentiable at a, and:
$$(f^{-1})'(a) = \frac{1}{f`(f^{-1}(a)}$$

- **L'Hôpital's Rule**: let $a \in \overline{\mathbb{R}}$, and let I be an interval that either contains a, or has it as an endpoint. Let f, g be differentiable on $I \backslash \{a\}$, and $g(x), g'(x) \neq 0,\ \forall x \in I \backslash \{a\}$. If:

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

leads to an indeterminate form, then, if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists (and is in $\overline{\mathbb{R}}$), then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

- **Intermediate Value Theorem For Derivatives (Darboux**: let f be a differentiable function on an open interval I. Then, for $a, b \in I$, $f'(a) \neq f'(b)$,

$$\forall \gamma : f'(a) < \gamma < f'(b),\ \exists c \in I : f'(c) = \gamma$$

- **Taylor Polynomial**: let $n \in \mathbb{R}$, and $a, b \in \overline{\mathbb{R}}$ with $a < b$. If $f : (a, b) \to \mathbb{R}$ is a function n times differentiable at $x_0 \in (a, b)$, then the Taylor Polynomial of degree n of f at x_0 is:

$$P_n^{f,x_0} = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

- **Taylor's Formula**: let $n \in \mathbb{R}$, and $a, b \in \overline{\mathbb{R}}$ with $a < b$. If $f : (a, b) \to \mathbb{R}$ is a function $n + 1$ times differentiable on (a, b) (f^{n+1} exists), then $\forall x, x_0 \in (a, b)$, there is a c between x and x_0 (dependent on n, x, x_0), such that:

$$f(x) = P_n^{f,x_0} + \frac{f^{(n+1)}(c)}{n + 1!}(x - x_0)^{n+1}$$

- **Second Derivative Test**: if $f : I \to \mathbb{R}$ is twice continuously differentiable in an open interval I, and x_0 is a critical point of f:
  - $f''(x_0) > 0$ means that there is a local minimum at x_0
  - $f''(x_0 < 0$ means that there is a local maximum at x_0