Honours Analysis - Week 9 - Fatoux Lemma and Dominated Convergence Theorem

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November 2021

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3 Workshop **8**

In Week 7, we discussed the integrability of sequences and series of function. Whilst for series we gave quite robust statements, in terms of sequences, we only have the Monotone Convergence Theorem, which is quite restrictive:

Suppose that f_n is a sequence of **monotone, non-decreasing, integrable** functions on an interval I:

$$f_1(x) \leq f_2(x) \leq \dots$$

For any $x \in I$, define:

$$f(x) = \lim_{n\to\infty} f_n(x)$$

We allow that for some x, this limit diverges to infinity: we are not concerned with particular points. Notice, that if f_n is a bounded sequence, it will necessarily converge, since its monotone.

Then, f must be integrable on I **if and only if**:

$$\sup_{n\in\mathbb{N}} \int_I f_n = \lim_{n\to\infty} \int_I f_n < \infty$$

(this equivalence might not be immediately obvious, but it is due to the fact that f_n is non-decreasing, so $f_n \leq f_{n+1} \implies \int_I f_n \leq \int_I f_{n+1}$, so the supremum must coincide with the limit)

Moreover, we have that:

$$\int_I f = \lim_{n\to\infty} \int_I f_n$$

There is an equivalent result if the sequence of functions is monotone, non-increasing, in which we just need to check that:

$$\inf_{n\in\mathbb{N}} \int_I f_n$$

exists for

$$\int_I f = \lim_{n\to\infty} \int_I f_n$$

1 Fatoux Lemma

The Fatoux Lemma is a useful building block for the next theorem, the ***Dominated Convergence Theorem****.*

Let f_n be a ***sequence*** *of* ***non-negative, integrable*** *functions on an interval I. Let:*

$$f(x) = \liminf_{n\to\infty} f_n(x), \qquad \forall x \in I$$

(recall, intuitively, the limit inferior is the smallest value to which any subsequence can tend to).
If we have that:

$$\liminf_{n\to\infty} \int_I f_n(x) < \infty$$

then f is ***integrable*** *on I, and we can bound the value of the integral:*

$$\int_I f \le \liminf_{n\to\infty} \int_I f_n(x)$$

[Lemma 4.2]

Proof: Fatoux Lemma. We shall make use of properties of the limit inferior, which you can find here.

By Completeness of the Real Numbers, and since the sequence of f_n is non-negative, we know that $f_n(x) \ge 0$ means that its infimum exists, and hence, so does the limit inferior.

Define the following sequence of functions:

$$g_n(x) = \inf_{k\ge n} f_n(x), \qquad \forall x \in I$$

For example:

$$g_1 = \inf\{f_1, f_2, \dots,\}$$
$$g_2 = \inf\{f_2, f_3, \dots,\}$$
$$g_3 = \inf\{f_3, f_4, \dots,\}$$

From the definition of g_n, it follows that for every $n \ge 1$:

- the sequence g_n is (pointwise) non-decreasing (for example, we can notice that $g_1 = \min\{f_1, g_2\}$, so clearly $g_n \le g_{n+1}$)
- by the same argument as above ($g_n = \min\{f_b, g_{n+1}\}$), we must have that $g_n \ge f_n$
- lastly, g_n must be integrable, since:

$$g_n(x) = \lim_{k\to\infty} \min\{f_n, f_{n+1}, \dots, f_k\}$$

Recall, one of the properties of Lebesgue Integrals is that $\min\{f, g\}$ is integrable if f and g are integrable, so in particular $\min\{f_n, f_{n+1}, \ldots, f_k\}$ is integrable:

$$\lim_{k\to\infty} \int_I \min\{f_n, f_{n+1}, \ldots, f_k\} < \infty$$

Moreover, $\min\{f_n, f_{n+1}, \ldots, f_k\}$ is a **monotone, non-increasing** sequence of functions which converge to g_n. Hence, by the Monotone Convergence Theorem, it must be the case that g_n is integrable.

Furthermore, it is easy to see that:

$$f(x) = \lim_{n\to\infty} g_n$$

by the definition of limit inferior

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\inf_{k\geq n} x_k \right)$$

We have also shown that the sequence of g_n is **monotone, non-decreasing**. Consider:

$$\lim_{n\to\infty} \int_I g_n$$

(I recommend visiting this shorter proof)

Consider:

$$\lim_{n\to\infty} \int_I g_n$$

Since each g_n is integrable on I, the limit exists and coincides with the limit inferior:

$$\lim_{n\to\infty} \int_I g_n = \liminf_{n\to\infty} \int_I g_n(x)$$

But as we have shown above, $f_n \geq g_n$, so:

$$\lim_{n\to\infty} \int_I g_n \leq \liminf_{n\to\infty} \int_I f_n(x)$$

and by the Theorem's Assumption, $\liminf_{n\to\infty} \int_I f_n(x) < \varepsilon$.

Overall, we have then shown that $\lim_{n\to\infty} \int_I g_n < \varepsilon$, and that g_n is monotone, non-decreasing. Furthermore, $f(x) = \lim_{n\to\infty} g_n(x)$. Hence, f must be integrable by the Monotone Convergence Theorem, and:

$$\int_I f = \int_I \left(\lim_{n\to\infty} g_n \right) = \lim_{n\to\infty} \int_I g_n \leq \liminf_{n\to\infty} \int_I f_n(x)$$

as required.

$\square$

2 Dominated Convergence Theorem

The Dominated Convergence Theorem gives us conditions under which function limits and integrals can be exchanged. This requires finding a **dominating function**.

2.1 The Dominated Convergence Theorem

*Let f_n be a sequence of **integrable** functions on an interval I, and assume that:*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

*Further assume that the sequence f_n is **dominated** by a integrable function g:*

$$|f_n(x)| \leq g(x), \qquad \forall x \in I, \forall n \geq 1$$

with:

$$\int_I g < \infty$$

*Then, the function f is **integrable** on I and:*

$$\int_I f = \int_I \left(\lim_{n\to\infty} f_n\right) = \lim_{n\to\infty} \int_I f_n$$

[Theorem 4.12]

Proof: Dominated Convergence Theorem. This is easily provable by considering the Fatoux Lemma. For that, we need to construct a non-negative sequence of integrable functions on I, whose limit inferior converges.

Notice:

$$|f_n(x)| \leq g(x) \implies -g(x) \leq f_n(x) \leq g(x)$$

If we add g throughout, in particular we see that:

$$0 \leq f_n(x) + g(x)$$

Lets use this as the sequence of functions to consider.

Notice that:

$$\lim_{n\to\infty} (f_n + g)(x) = (f + g)(x)$$

since g is independent of n. Moreover, since the limit exists, it must be the case that it is equal to the limit inferior, so:

$$\liminf_{n\to\infty} (f_n + g)(x) = \lim_{n\to\infty} (f_n + g)(x) = (f + g)(x)$$

Moreover, since $f_n \leq g$, it follows that:

$$\liminf_{n\to\infty} \int_I (f_n + g)(x) \leq \liminf_{n\to\infty} \int_I 2g(x) < \infty$$

By Fatoux Lemma, it follows that:

$$\int_I f(x) + g(x) \leq \liminf_{n\to\infty} \int_I f_n(x) + g(x)$$

which means that:

$$\int_I f(x) \leq \int_I f_n(x)$$

where we have used the linearity of the Lebesgue Integral.

Working similarly as above, we can also see that, if we substract $f_n(x)$ from the inequality $-g(x) \leq f_n(x) \leq g(x)$ we see that:

$$g(x) - f_n(x) \geq 0$$

We use this again with Fatoux Lemma, noticing that:

$$\limsup_{n\to\infty}(g - f_n)(x) = \lim_{n\to\infty}(g - f_n)(x) = (g - f_n)(x)$$

Again, since $f_n \geq -g$:

$$\limsup_{n\to\infty} \int_I (g - f_n)(x) \leq \limsup_{n\to\infty} \int_I 2g(x) < \infty$$

Hence, Fatoux Lemma applies again, and:

$$\int_I g(x) - f(x) \leq \limsup_{n\to\infty} \int_I g(x) - f_n(x)$$

cancelling out the $\int_I g$ term:

$$-\int_I f(x) \leq -\limsup_{n\to\infty} \int_I f_n(x) \implies \int_I f(x) \geq \limsup_{n\to\infty} \int_I f_n(x)$$

Overall, we have shown that:

$$\limsup_{n\to\infty} \int_I f_n(x) \leq \int_I f(x) \leq \liminf_{n\to\infty} \int_I f_n(x)$$

But notice that it is always true by properties of the limit inferior and superior that:

$$\limsup_{n\to\infty} \int_I f_n(x) \geq \liminf_{n\to\infty} \int_I f_n(x)$$

Hence, under the assumptions of the Dominated Convergence Theorem:

$$\limsup_{n\to\infty} \int_I f_n(x) = \liminf_{n\to\infty} \int_I f_n(x) = \lim_{n\to\infty} \int_I f_n(x)$$

and so:

$$\int_I f(x) = \lim_{n\to\infty} \int_I f_n(x)$$

as required.

□

2.2 Integrability of Uniformly Continuous Functions

The following is a Theorem which we claimed all the way back in Week 4, but gave no proof. Now that we have the Dominated Convergence Theorem, the proof is trivial.

Let (a, b) be a bounded interval, and suppose that

$$f_n : (a, b) \to \mathbb{R}$$

are ***integrable*** *functions which* ***converge uniformly*** *to a function f. Then, f is* ***integrable*** *on (a, b), and:*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

[Theorem 4.13]

Notice, there are 2 key differences when compared to the Dominated Convergence Theorem:

- DCT requires only pointwise convergence, whilst this theorem requires uniform convergence
- DCT poses no restriction on the interval I, whilst this theorem required a bounded interval

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Proof: Interchanging Integral and Limit for Uniformly Convergent Sequence of Functions. Since f_n converges uniformly to f, for any $\varepsilon > 0$, we can find $n \geq N$ such that $\forall x \in (a, b)$:

$$|f_n(x) - f(x)| < \varepsilon$$

In particular, we can pick ε such that for $n \geq N$:

$$|f_n(x) - f_N(x)| \leq 1$$

for every $x \in (a, b)$. Define:

$$g(x) = |f_N(x)| + 1$$

The absolute value of a function is Lebesgue integrable if the function is Lebesgue integrable. Moreover, we are integrating 1 over a bounded interval, so it is also integrable. Hence, $g(x)$ is the sum of 2 integrable functions, and so, g is integrable. Moreover, it is easy to see that:

$$|f_n(x)| \leq g(x)$$

But then, $f_n \to f$ and f_n is dominated by g so:

$$\int_I f = \lim_{n\to\infty} \int_I f_n$$

by the Dominated Convergence Theorem. □

3 Workshop

> *Recall the concept of Riemann integral. A function $f : [a,b] \to \mathbb{R}$ is Riemann integrable* ***if and only if*** *$\forall \varepsilon > 0$ there exists a partition $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ such that:*
>
> $$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \lambda(I_j) < \varepsilon, \qquad I_j = (x_{j-1}, x_j)$$
>
> *This is* ***Lemma 4.1*** *of the Lecture Notes.*

1. **Discuss Riemann integrability of the Dirichlet function $\mathcal{X}_{\mathbb{Q}\cap[0,1]}$.**

   Notice, the rationals and the irrationals are dense, so on any open subinterval of $[0,1]$, we can always find at least one rational and at least on irrational. This means that no matter how we segment $[0,1]$ into subintervals I_j, we can always find at least one rational and one irrational, such that:

   $$\sup_{x,y \in I_j} |f(x) - f(y)| \lambda(I_j) = 1, \qquad \forall j \in [1,n]$$

   This then means that for example with $\varepsilon = 0.5$:

   $$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \lambda(I_j) = \sum_{j=1}^{n} \lambda(I_j) = 1 > \epsilon$$

   so the Dirichlet Function won't be Riemann integrable.

2. **Prove or disprove these 2 statements:**

   (a) **If f is Riemann integrable on $[a,b]$ then so is $|f|$.**
   This is true. If we apply the reverse triangle inequality, $\forall x, y \in [a,b]$:

   $$||f(x)| - |f(y)|| \leq |f(x) - f(y)|$$

   Since f is integrable, then:

   $$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \lambda(I_j) < \varepsilon, \qquad I_j = (x_{j-1}, x_j)$$

   But then:

   $$\sum_{j=1}^{n} \sup_{x,y \in I_j} ||f(x)| - |f(y)|| \lambda(I_j) \leq \sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \lambda(I_j) < \varepsilon, \qquad I_j = (x_{j-1}, x_j)$$

   so $|f|$ must be Riemann integrable.

   (b) **If $|f|$ is Riemann integrable on $[a,b]$, then so is f.**
   This is false. Let:

   $$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ -1, & x \in \mathbb{Q}^C \cap [0,1] \end{cases}$$

   Then, $\forall x, y \in [0,1]$:

   $$|f(x)| = 1 \implies |f(x)| - |f(y)| = 0$$

So $\forall \varepsilon > 0$:

$$\sum_{j=1}^{n} \sup_{x,y \in I_j} ||f(x)| - |f(y)||\lambda(I_j) = 0 < \varepsilon$$

and so, $|f|$ is Riemann integrable.
However, f isn't Riemann integrable. Again, since the rationals and irrationals are dense, on any subinterval I_j we have:

$$\sup_{x,y \in I_j} |f(x) - f(y)| = 2$$

so:

$$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)|\lambda(I_j) = \sum_{j=1}^{n} 2\lambda(I_j) = 2$$

so if $|f|$ is Riemann integrable, f need not be Riemann integrable.

We now consider Lebesgue integrability again.

Let $-\infty \leq a < b \leq \infty$. Suppose that a function f is integrable on the interval (a, v), $\forall v \in (a, b)$. Then, f is integrable on the interval (a, b) ***if and only if*** *there exists $M < \infty$ such that:*

$$\int_a^v |f| \leq M, \qquad \forall v \in (a, b)$$

Moreover, if this holds:

$$\int_a^b f = \lim_{v \to b^-} \int_a^v f$$

where if $b = \infty$ we understand $\lim_{v \to b^-}$ to be $\lim_{v \to \infty}$.

3. **Formulate an analogous statement on integrability on (a, b) under the assumption of integrability of f on intervals (u, b) where $u \in (a, b)$.**

Let $-\infty \leq a < b \leq \infty$. Suppose that function f is integrable on the interval (u, b), where $u \in (a, b)$. Then f is integrable on the interval (a, b) if and only if there exists $M < \infty$ such that:

$$\int_u^b |f| \leq M, \quad \forall u \in (a, b)$$

Additionally, if this condition holds, then:

$$\int_a^b f = \lim_{u \to a^+} \int_u^b f$$

where if $a = \infty$ we understand $\lim\limits_{u \to a^+}$ to be $\lim\limits_{u \to \infty}$).

4. **Consider a sequence (v_n) of numbers:**

$$a < v_1 < v_2 < \ldots < b$$

such that:

$$\lim_{n\to\infty} v_n = b$$

Let:

$$I_1 = (a, v_1] \qquad I_j = (v_{j-1}, v_j] \qquad j = 2, 3, \ldots$$

Deduce that:

$$\int_a^v |f| \leq M, \qquad \forall v \in (a,b) \iff \sum_{j=1}^{n} \int_{I_j} |f| \leq M, \qquad n \in \mathbb{N}$$

Deduce from this integrability of f on:

$$(a, b) = \bigcup_{j\in\mathbb{N} I_j}$$

by quoting a correct theorem from the lecture notes, and finally deduce that:

$$\int_a^b f = \lim_{v\to b^-} \int_a^v f$$

We begin by noticing that:

$$\sum_{j=1}^{n} \int_{I_j} |f| = \int_a^{v_n} |f|$$

Now, if $\sum_{j=1}^{n} \int_{I_j} |f| \leq M$, recall Theorem 4.8, part c) of the notes:

If f is integrable on I, and $\forall x \in I, f(x) \geq 0$ then:

$$\int_J f \leq \int_I f$$

where J is a subinterval of I.

We can always pick $n \in \mathbb{N}$ such that $v_n \geq v$, so that:

$$\int_a^v |f| \leq \int_a^{v_n} |f| \leq M$$

Similarly, if $\int_a^v |f| \leq M$, we can always pick $n \in \mathbb{N}$ such that $v_n \leq v$ so:

$$\int_a^{v_n} |f| \leq \int_a^v |f| \leq M$$

Hence, we get that:

$$\int_a^v |f| \leq M, \qquad \forall v \in (a,b) \iff \sum_{j=1}^{n} \int_{I_j} |f| \leq M, \qquad n \in \mathbb{N}$$

Now, recall Theorem 4.8, part d) of the notes:

> *Suppose that I can be written as a* ***disjoint union*** *of intervals I_n.*
> *Assume f is* ***integrable*** *on each I_n.*
> *Then, f is* ***integrable*** *on I* ***if and only if****:*
>
> $$\sum_{n=1}^{\infty} \int_{I_n} |f| < \infty$$
>
> *in which case:*
>
> $$\int_I f = \sum_{n=1}^{\infty} \int_{I_n} f$$

Now, $\forall n \in \mathbb{N}$, we can always find $u \in (a, b)$ such that:

$$\sum_{j=1}^{n} \int_{I_j} |f| \leq \int_a^v |f| \leq M$$

which means that also:

$$\sum_{j=1}^{\infty} \int_{I_j} |f| \leq M$$

Hence, Theorem 4.8, part d) applies, and so, f is integrable over $(a, b) = \bigcup_{j \in \mathbb{N} I_j}$ and:

$$\int_a^b f = \sum_{j=1}^{\infty} \int_{I_j} f$$

But notice:

$$\begin{aligned} \int_a^b f &= \sum_{j=1}^{\infty} \int_{I_j} f \\ &= \lim_{n \to \infty} \sum_{j=1}^{n} \int_{I_j} f \\ &= \lim_{n \to \infty} \int_a^{v_n} f \\ &= \lim_{v \to b^-} \int_a^{v} f \end{aligned}$$

as required.

5. **Prove the converse, assume that f is integrable on (a, b) and show that $\exists M < \infty$ such that:**

$$\sum_{j=1}^{n} \int_{I_j} |f| \leq M, \qquad n \in \mathbb{N}$$

By properties of the Lebesgue integral, if f is integrable, then $|f|$ is integrable, and so $\exists M \in \mathbb{R}$:

$$\int_a^b |f| \leq M$$

But then, since $v \in (a, b)$:

$$\int_a^v |f| \leq \int_a^b |f| \leq M$$

so the result follow by the equivalence

$$\int_a^v |f| \leq M, \qquad \forall v \in (a,b) \iff \sum_{j=1}^{n} \int_{I_j} |f| \leq M, \qquad n \in \mathbb{N}$$

as required.

6. **Analogously, prove the statement formulated in Q3.**

> *This is taken directly from my homework answers, so it is unnecessarily long and over detailed. The above is more similar to the solutions provided.*

We first prove that if

$$\int_u^b |f| \leq M, \quad \forall u \in (a,b)$$

then f is integrable on (a, b) and

$$\int_a^b f = \lim_{u \to a^+} \int_u^b f$$

Lets consider a sequence $(u_n)_{n \in \mathbb{N}}$ of numbers, such that:

$$a < \ldots < u_2 < u_1 < b$$

with:

$$\lim_{n \to \infty} u_n = a$$

We can then partition the interval (a, b) via:

$$I_1 = [u_1, b)$$

$$I_2 = [u_2, u_1)$$

$$\vdots$$

$$I_j = [u_j, u_{j-1}]$$

$$\vdots$$

such that:

$$\bigcup_{j=1}^{\infty} I_j = (a,b)$$

Lets define $|f_j| = \mathcal{X}_{I_j}|f|$. It follows that $\forall x \in (u_n, b)$:

$$\sum_{j=1}^{n} |f_j(x)| = |f(x)|$$

Since f is integrable on any subinterval of (u, b), in particular it is integrable on the interval from u_n to b:

$$\int_{u_n}^{b} |f| = \int_{u_n}^{b} \sum_{j=1}^{n} |f_j| = \sum_{j=1}^{n} \int_{u_n}^{b} |f_j| = \sum_{j=1}^{n} \int_{I_j} |f|$$

where we have made us of the linearity of the Lebesgue Integral, alongside Theorem 4.8, part b). But then, if $\forall u \in (a, b)$ we have:

$$\int_{u}^{b} |f| \leq M$$

since $u_n \in (a, b)$, and $(u_n, b) \subseteq (u, b)$ for any $u \in (a, b)$, it follows by Theorem 4.8, part c) that:

$$\int_{u_n}^{b} |f| = \sum_{j=1}^{n} \int_{I_j} |f| \leq \int_{u}^{b} |f| \leq M$$

Hence, it follows that if $\int_u^b |f| \leq M$ then:

$$\sum_{j=1}^{n} \int_{I_j} |f| \leq M$$

Notice, the inequality $\sum_{j=1}^{n} \int_{I_j} |f| \leq \int_u^b |f|$ holds $\forall n \in \mathbb{N}$, so $\sum_{j=1}^{n} \int_{I_j} |f| \leq M$ is also true $\forall n \in \mathbb{N}$. It must then be the case that:

$$\lim_{n\to\infty} \sum_{j=1}^{n} \int_{I_j} |f| = \sum_{j=1}^{\infty} \int_{I_j} |f| \leq M$$

Notice, we have written $\bigcup_{j=1}^{\infty} I_j = (a, b)$, where each I_j is disjoint. Moreover, by the statement of the question, we know that f is intergrable on any interval (u, b) where $u \in (a, b)$, so by properties of Lebesgue Integrability, so is $|f|$. Thus, since we have shown that:

$$\sum_{j=1}^{\infty} \int_{I_j} |f| < \infty$$

It follows by Theorem 4.8, part d) of the notes that f is integrable over (a, b), and:

$$\int_{a}^{b} f = \sum_{j=1}^{\infty} \int_{I_j} f$$

Finally, notice that, from the work above:

$$\sum_{j=1}^{\infty} \int_{I_j} f = \lim_{n\to\infty} \sum_{j=1}^{n} \int_{I_j} f$$

but:

$$\lim_{n\to\infty} \sum_{j=1}^{n} \int_{I_j} f = \lim_{n\to\infty} \int_{u_n}^{b} f = \lim_{u_n\to a^+} \int_{u_n}^{b} f = \lim_{u\to a^+} \int_{u}^{b} f$$

(since $u_n \to a^+$ as $n \to \infty$, and we can always find $u \in (a,b)$ such that $\forall n \in \mathbb{N}$, $u_n \geq u$). Then, it must be the case that:

$$\int_a^b f = \lim_{u \to a^+} \int_u^b f$$

We now show that if f is integrable on (a,b), then

$$\int_u^b |f| \leq M, \quad \forall u \in (a,b)$$

Since f is integrable on (a,b), by properties of Lebesgue Integrability, $|f|$ is also integrable on the interval, so $\exists M < \infty$ such that:

$$\int_a^b |f| \leq M$$

Hence, by Theorem 4.8, part c) of the notes, for any $u \in (a,b)$, since $(u,b) \subset (a,b)$ it must be the case that:

$$\int_u^b |f| \leq \int_a^b |f| \leq M$$

7. **Show that $f(x) = e^{-x}\cos(x)$ is integrable on $[0,\infty)$, and calculate the value of the integral.**

Notice, f is a product of continuous functions, so in particular, it is continuous on any interval $[0,v], v \in \mathbb{R}$, and so, integrable on such intervals.

We apply the Theorem above. Notice:

$$\int_0^v |f(x)| \; dx \int_0^v |e^{-x}||\cos(x)| \; dx \leq \int_0^v e^{-x} = -\left[e^{-v} - e^0\right] = 1 - e^{-v} \leq 1$$

Hence, the theorem applies, and the function is integrable on $[0,\infty)$.

*At this point, remember we have just shown that the integral is **bounded**, but the bound we have computed requires approximations, which don't necessarily reflect the value of the integral (for example, using an absolute value, or removing terms from the integral).*

We compute:

$$\int_0^\infty e^{-x}\cos(x) \; dx = \lim_{v \to \infty} \int_0^v e^{-x}\cos(x) \; dx$$

We can apply integration by parts with:

$$u = e^{-x} \qquad du = -e^{-x}$$

$$dv = \cos(x) \qquad v = \sin(x)$$

So:

$$\int e^{-x}\cos(x) = e^{-x}\sin(x) + \int e^{-x}\sin(x) \; dx$$

Applying integration by parts again:

$$u = e^{-x} \qquad du = -e^{-x}$$

$$dv = \sin(x) \qquad v = -\cos(x)$$

So:

$$\int e^{-x}\sin(x)\ dx = -e^{-x}\cos(x) - \int e^{-x}\cos(x)$$

Hence, we have that:

$$I = e^{-x}\sin(x) - e^{-x}\cos(x) - I \implies I = \frac{e^{-x}}{2}\left(sin(x) - \cos(x)\right)$$

Thus:

$$\begin{aligned}\int_0^\infty e^{-x}\cos(x)\ dx &= \lim_{v\to\infty}\int_0^v e^{-x}\cos(x)\ dx \\ &= \lim_{v\to\infty}\left[\frac{e^{-v}}{2}\left(sin(v)-\cos(v)\right) - \frac{e^0}{2}\left(sin(0)-\cos(0)\right)\right] \\ &= \lim_{v\to\infty}\left[\frac{e^{-v}}{2}\left(sin(v)-\cos(v)\right) + \frac{1}{2}\right]\end{aligned}$$

Now, since $\sin(v)$ and $\cos(v)$ are bounded, it follows that:

$$\lim_{v\to\infty}\left[\frac{e^{-v}}{2}\left(sin(v)-\cos(v)\right)+\frac{1}{2}\right] = 0 + \frac{1}{2} = \frac{1}{2}$$

8. **Discuss integrability of:**

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

and find the value of the integral if it exists.

Notice, $\frac{1}{\sqrt{1-x^2}}$ is continuous on all $(0,1)$, and positive, so:

$$\int_0^v |f| = \int_0^v \frac{dx}{\sqrt{1-x^2}} = \arcsin(v) - \arcsin(0) = \arcsin(v)$$

Now, $\arcsin(x)$ is bounded on $[0,1]$, such that:

$$\arcsin(v) \leq \frac{\pi}{2}$$

Hence, $\frac{1}{\sqrt{1-x^2}}$ is integrable on $(0,1)$, and:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{v\to 1^-}[\arcsin(v)] = \frac{\pi}{2}$$

by continuity.

9. **Discuss integrability of:**

$$\int_0^1 \ln(x)\ dx$$

and find the value of the integral if it exists.

We compute:

$$\int_u^1 |\ln(x)|\ dx = \int_u^1 -\ln(x)\ dx = -[x\ln(x) - x]_u^1 = -(0 - 1 - u\ln(u) + u) = 1 - u(\ln(u) - 1)$$

Notice, since $u(\ln(u) - 1) \to 0$ as $u \to 0$ (i.e by L'Hôpital), it follows that:

$$\int_u^1 |\ln(x)| \ dx \leq 1$$

so the function is integrable over $(0, 1)$, and:

$$\int_0^1 \ln(x) \ dx = \lim_{u \to 0+} (-1 + u(\ln(u) - 1)) = -1$$

10. **Explain why the function:**

$$f(x) = \frac{(-1)^{[x]}}{[x]}, \qquad x \geq 1$$

is not integrable on $[1, \infty)$ even though the limit:

$$\lim_{v \to \infty} \int_1^v f(x) \ dx$$

exists.

It doesn't satisfy the condition necessary to apply the theorem. If we let $I_j = [j, j+1)$, then:

$$\int_{I_j} |f| = \int_j^{j+1} \frac{1}{j} = \frac{1}{j}$$

Hence:

$$\sum_{j=1}^n \int_{I_j} |f| = \sum_{j=1}^n \frac{1}{j}$$

However, this sum diverges as $n \to \infty$, so in particular it can be made arbitrarily large, and so, we can choose n such that $\forall M \in \mathbb{R}$:

$$\sum_{j=1}^n \frac{1}{j} \geq M \implies \int_1^v |f(x)| \ dx \geq M$$

Hence, the theorem won't apply, and so, the function isn't integrable on $[1, \infty)$.

11. **Are the functions in Q7-Q9 Riemann integrable? Explain your answers?**

They aren't.

- $e^{-x} \cos(x)$ isn't Riemann integrable on $(0, \infty)$, since it has unbounded support, and Riemann integrability is defined by the fact that f can be approximated arbitrarily way by step functions, so if f has unbounded support, it means it can't be represented by step functions, and so, it can't be Riemann integrable
- again, the requirement of being approximated by step functions means that we require that f is bounded, but neither $\ln(x)$ or $\frac{1}{\sqrt{1-x^2}}$ are bounded on $(0, 1)$, and so, can't be Riemann integrable