Honours Analysis - Week 8 - The Interval of Integration and the Fundamental Theorem of Calculus

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Contents

1 Dependence of Integrability on the Interval

As of now we have mainly discussed integrability, assuming that it will be valid on any interval, so long as the function is defined. We now provide Theorems that deal with integrability and the intervals over which we integrate.

1.1 Integrability Over Interval Theorems

For the following theorems, we consider a Lebesgue Integrability. Moreover, we let $J \subset I$, where J, I are intervals. This corresponds to Theorem 4.8 of the notes.

1.1.1 Theorem: Integrability Over Subinterval

*If f is **integrable** over I, then f is **integrable** over J.*

Proof: Integrability Over Subinterval. Assume f is Lebesgue Integrable over I. Then, from the definition of integrability, we can write:

$$f(x) = \sum_{j=1}^{\infty} c_j \mathcal{X}_{J_j}$$

(at least for all $x \in I$ for which $\sum_{j=1}^{\infty} c_j \mathcal{X}_{J_j}$ is absolutely convergent). Moreover, we know that:

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$$

We now need to consider the integrability of f over $J \subset I$. In particular, consider $x \in J$. Notice that if $x \in J$:

$$\mathcal{X}_{J_j} = \mathcal{X}_{J_j \cap J}$$

since:

- the RHS is 1 if $x \in J$ and $x \in J_j$, which automatically means that the LHS will be 1
- if $x \notin J_j$, both sides will be 0

In other words, for $x \in J$, we can write:

$$f(x) = \sum_{j=1}^{\infty} c_j \mathcal{X}_{J_j \cap J}$$

whenever $x \in J$ such that $\sum_{j=1}^{\infty} c_j \mathcal{X}_{J_j \cap J}$ converges absolutely.

Lastly, notice that:

$$\lambda(J_j \cap J) \leq \lambda(J_j)$$

since:

- if $J_j \subseteq J$, then it is easy to see that $J_j \cap J = J_j$

- $J \subseteq J_j$, then clearly $J_j \cap J$ will be a smaller set than J_j
- otherwise, it must be the case that they are disjoint (so the result follows obviously), or they must have some small interval in common, which will still be smaller than J_j (and also smaller than J, but that is not useful for the argument)

So:

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j \cap J) \leq \sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$$

Hence, for $x \in J$, we have shown that f is integrable, as required.

□

1.1.2 Theorem: Integrability Over Bounded Support Subinterval

If f is integrable on J, and $\forall x \in I \backslash J, f(x) = 0$, then f is integrable on I and:

$$\int_I f = \int_J f$$

Proof. I do hate to admit it, but when the notes claim this as "trivial", I must agree.

□

1.1.3 Theorem: Value of Integral Over Subinterval

If f is integrable on I, and $\forall x \in I, f(x) \geq 0$ then:

$$\int_J f \leq \int_I f$$

Proof: Value of Integral Over Subinterval. By the Theorem on Integrability Over Subintervals (1.1.1), we know that f is integrable on J. We can express this in terms of integrals by introducing a function $g(x) = f(x)\mathcal{X}_J$. In other words, $g(x) = f(x)$ when $x \in J$, and is 0 otherwise. From this we have that:

$$\int_J f = \int_I g$$

Now, notice that for any $x \in I$, $g(x) \leq f(x)$. This is simply because $f(x) \geq 0$ if $x \in I \backslash J$, but $g(x) = 0$ if $x \in I \backslash J$.

Recall the following property of the Lebesgue Integral from Week 6:

If $f \geq 0$ on I, then $\int_I f \geq 0$.
If $f \geq g$ on I, then $\int_I f \geq \int_I g$.

Hence, it follows that:

$$\int_J f = \int_I g \leq \int_I f$$

□

1.1.4 Theorem: Integrability Given Union of Disjoint Intervals

This is probably the most useful out of all the Theorems here.

Suppose that I can be written as a ***disjoint union*** *of intervals I_n.*
Assume f is ***integrable*** *on each I_n.*
Then, f is ***integrable*** *on I* ***if and only if****:*

$$\sum_{n=1}^{\infty} \int_{I_n} |f| < \infty$$

in which case:

$$\int_I f = \sum_{n=1}^{\infty} \int_{I_n} f$$

Proof: Integrability Given Union of Disjoint Intervals. Define:

$$f_n(x) = f(x)\mathcal{X}_{I_n}$$

where I_n is a set of intervals whose union is I. Notice that for any $x \in I$:

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

Moreover,

$$|f(x)| = \sum_{n=1}^{\infty} |f_n(x)|$$

This is easy to see, since if $x \in I_n$, then $\sum_{n=1}^{\infty} |f_n(x)| = |f_n(x)| = |f(x)|$.

We begin by proving the forward statement: if $\sum_{n=1}^{\infty} \int_{I_n} |f| < \infty$, then f is integrable.

Notice, $f_n(x) = f(x)$ on I_n, and is 0 otherwise, so by (1.1.2), and by noticing that $\int_{I_n} f_n = \int_{I_n} f$

$$\int_I f_n = \int_{I_n} f_n \implies \int_I f_n = \int_{I_n} f$$

and similarly:

$$\int_I |f_n| = \int_{I_n} |f|$$

Notice, from our assumption:

$$\sum_{n=1}^{\infty} \int_{I_n} |f| < \infty \implies \sum_{n=1}^{\infty} \int_I |f_n| < \infty$$

Recall Week 7's Theorem on integrability of series:

Suppose f_n is a sequence of functions, each of which is ***integrable*** *on some I.*

(a) ***If****:*

$$\sum_{n=1}^{\infty} \int_I |f_n| < \infty$$

(the sum of integrals of each function in the sequence is convergent) ***and*** *f is a function on I, such that,*

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

for any x, such that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$[a] , ***then*** *$f$ is* ***integrable*** *on I, and its integral is:*

$$\int_I f = \sum_{n=1}^{\infty} \int_I f_n < \infty$$

(b) If we further have that for any $n \in \mathbb{N}$, $f_n \geq 0$, and

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

for all $x \in I$ (except for possibly finitely many points, in which we can allow that $\sum_{n=1}^{\infty} f_n(x) = \infty$).
Then f is integrable on I ***if and only if****:*

$$\sum_{n=1}^{\infty} \int_I f_n < \infty$$

[Theorem 4.3]

[a] this is just saying that we require the x to be such that the sum converges (to f)

So it follows that if $\sum_{n=1}^{\infty} \int_{I_n} |f| < \infty$, then f is integrable.

We now prove the converse: if f is integrable, then $\sum_{n=1}^{\infty} \int_{I_n} |f| < \infty$.

Recall the following property of Lebesgue Integrable functions:

*If f is **integrable**, then $|f|$ is **integrable** on I, and:*

$$\left|\int_I f\right| \leq \int_I |f|$$

Moreover, notice that for any $x \in I$:

$$|f(x)| = \sum_{n=1}^{\infty} |f_n(x)| \implies |f(x)| \geq \sum_{j=1}^{n} |f_j(x)|$$

since the equality only holds after "adding infinitely many terms".

Using all of the above, consider:

$$\sum_{j=1}^{n} \int_{I_j} |f|$$

which is just:

$$\sum_{j=1}^{n} \int_{I} |f_j|$$

Since we are considering a finite sum, we can apply linearity of the Lebesgue Integral, which leads to:

$$\int_I \left(\sum_{j=1}^{n} |f_j| \right)$$

But we know that $|f(x)| \geq \sum_{j=1}^{n} |f_j(x)|$, so it follows that:

$$\sum_{j=1}^{n} \int_{I_j} |f| = \int_I \left(\sum_{j=1}^{n} |f_j| \right) \leq \int_I |f|$$

But since $|f|$ is integrable, $\int_I |f|$ is finite, so it follows that:

$$\sum_{j=1}^{n} \int_{I_j} |f| < \infty$$

as required.

□

1.2 Cauchy Integral Criterion of Summability of Series

This Theorem is particularly useful later on, when we consider the Fundamental Theorem of Calculus. In particular, we can use it to justify the p-series test.

*Let $p \in \mathbb{Z}$ and suppose that f is a **non-negative, non-increasing** function on the interval $[p, \infty)$.*
*f is integrable on $[p, \infty)$ **if and only if**:*

$$\sum_{n=p}^{\infty} f(n) < \infty$$

Proof: Cauchy Integral Criterion of Summability of Series. Recall, if a function is defined on a closed interval, and is monotone, then the function is Riemann (and so Lebesgue) integrable on said interval. Moreover, since f is non-negative:

$$\int_I f = \int_I |f|$$

Consider the set of intervals $I_n = [n, n+1)$ for $n \geq p$. Clearly,

$$\bigcup_{n=p}^{\infty} I_n = [p, \infty)$$

Since f is monotone non-increasing, and non-negative, it follows that for any $x \in I_n$:

$$f(n+1) \leq f(x) \leq f(n)$$

Consider integrating over the interval I_n. Notice, $f(n+1)$ and $f(n)$ are constants, so:

$$\int_{I_n} f(n) = f(n) \times (n + 1 - n) = f(n)$$

$$\int_{I_n} f(n+1) = f(n+1) \times (n + 1 - n) = f(n+1)$$

So we get the following inequality:

$$f(n+1) \leq \int_{I_n} f(x) \leq f(n)$$

Lastly, consider an infinite summation, from $n = p$ to ∞:

$$\sum_{n=p}^{\infty} f(n+1) \leq \sum_{n=p}^{\infty} \int_{I_n} f(x) \leq \sum_{n=p}^{\infty} f(n)$$

Now, if $\sum_{n=p}^{\infty} f(n) < \infty$, then it is easy to see that $\sum_{n=p}^{\infty} \int_{I_n} f(x)$ is bounded, which by the Theorem above (1.1.4) means that f is integrable over I, since the union of I_n forms I, and $|f(x)| = f(x)$.

Alternatively, assume that f is integrable. Again by the theorem above (1.1.4), this is true if and only if:

$$\sum_{n=1}^{\infty} \int_{I_n} f(x) < \infty$$

which is the same as saying that $\sum_{n=p}^{\infty} \int_{I_n} f(x)$ is bounded. But then, if we notice that:

$$\sum_{n=p+1}^{\infty} f(n) = \sum_{n=p}^{\infty} f(n+1)$$

It follows that $\sum_{n=p+1}^{\infty} f(n)$ is bounded, which means that:

$$\sum_{n=p}^{\infty} f(n) < \infty$$

as required.

If we look again at the inequality we obtain:

$$\sum_{n=p+1}^{\infty} f(n) \leq \sum_{n=p}^{\infty} \int_{I_n} f(x) \leq \sum_{n=p}^{\infty} f(n)$$

And notice that:

$$\sum_{n=p}^{\infty} \int_{I_n} f(x) = \int_{p}^{p+1} f + \int_{p+1}^{p+2} f + \ldots = \int_{p}^{\infty} f$$

we can see that:

$$\int_{p}^{\infty} f \leq \sum_{n=p}^{\infty} f(n)$$

and since $\sum_{n=p+1}^{\infty} f(n) + f(p) = \sum_{n=p}^{\infty} f(n)$:

$$\int_{p}^{\infty} f \geq \sum_{n=p}^{\infty} f(n) - f(p) \implies \sum_{n=p}^{\infty} f(n) \leq f(p) + \int_{p}^{\infty} f$$

so:

$$\int_{p}^{\infty} f \leq \sum_{n=p}^{\infty} f(n) \leq f(p) + \int_{p}^{\infty} f$$

□

1.3 Integration Notation

If we have intervals:

- $[a, b]$
- $[a, b)$
- $(a, b]$
- (a, b)

We can represent the integral of f over all of the intervals via:

$$\int_{a}^{b} f$$

We also have that:

- $\int_a^b f + \int_b^c f = \int_a^c f$
- $\int_a^b f = -\int_b^a f$

These 2 properties will be particularly useful when proving the FTC, and this notation is easier and more general.

2 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is **fundamental** because it's the Theorem that relates derivatvies and integrals as inverses.

2.1 Theorem: FTC Version 1 - Derivative of Integral

> *Let I be an interval, and let $g : I \to \mathbb{R}$ be an* ***integrable*** *function on I. Define a function $G(x)$, for any $x \in I$, and some fixed $x_0 \in I$, such that:*
>
> $$G(x) = \int_{x_0}^{x} g$$
>
> *Further, suppose g is* ***continuous*** *at some $x \in I$ (if x is an endpoint of I, we just need to consider right/left continuity).*
> *Then, it must be the case that G is* ***differentiable*** *at x, and:*
>
> $$G'(x) = g'(x)$$
>
> *(if x is an endpoint, we consider right/left differentiability) [Theorem 4.10]*

Proof: Fundamental Theorem of Calculus: The Derivative of an Integral. We need to show that $G(x)$ is differentiable, and moreover, $G'(x) = g(x)$.

Recall, G is differentiable at a point x if and only if:

$$\lim_{h \to 0} \frac{G(x+h) - G(x)}{h}$$

is defined.

Secondly, since g is defined to be continuous at $x \in I$ (here we assume that x is an interior point of I), it follows that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that:

$$|t - x| < \delta \implies |f(t) - f(x)| < \delta$$

where we are taking g as a function of t (since x is an arbitrary point of the interval).

Lets pick h small enought, such that if $x \in I$, then $x + h \in I$ too. Since we want G to be differentiable, and we want it to evaluate to g, lets consider the definition of the limit:

$$\left| \frac{G(x+h) - G(x)}{h} - g(x) \right|$$

If we can show that the expression above is less than ε, then if we take the limit as $h \to 0$, we will have shown that $G'(x) = g(x)$. Here we are considering $h > 0$, but the argument is essentially the same if $h < 0$.

Thus:

$$
\begin{aligned}
&\left|\frac{G(x+h)-G(x)}{h}-g(x)\right| \\
=&\frac{1}{h}\left|G(x+h)-G(x)-hg(x)\right| \\
=&\frac{1}{h}\left|\int_{x_0}^{x+h} g(t)dt-\int_{x_0}^{x} g(t)dt-hg(x)\right| \\
=&\frac{1}{h}\left|\int_{x_0}^{x} g(t)dt+\int_{x}^{x+h} g(t)dt-\int_{x_0}^{x} g(t)dt-hg(x)\right| \\
=&\frac{1}{h}\left|\int_{x}^{x+h} g(t)dt-hg(x)\right|
\end{aligned}
$$

Notice, $g(x)$ is just taken as a constant. We can conveniently write it as an integral:

$$\int_{x}^{x+h} g(x)dt = g(x)(x+h-x) = hg(x) \implies g(x) = \frac{1}{h}\int_{x}^{x+h} g(x)dt$$

Thus, we get:

$$\left|\frac{G(x+h)-G(x)}{h}-g(x)\right| = \frac{1}{h}\left|\int_{x}^{x+h}(g(t)-g(x))dt\right|$$

But by properties of integrals:

$$
\begin{aligned}
&\left|\frac{G(x+h)-G(x)}{h}-g(x)\right| \\
=&\frac{1}{h}\left|\int_{x}^{x+h}(g(t)-g(x))dt\right| \\
\leq&\frac{1}{h}\int_{x}^{x+h}|g(t)-g(x)|dt
\end{aligned}
$$

Since we are considering t in an interval with endpoints x and $x+h$ (these are the limits of the integral), it then follows that if $h<\delta$, for any such t, $|t-x|<h<\delta$. Hence, by continuity, it must be the case that $|g(t)-g(x)|<\varepsilon$, so:

$$\frac{1}{h}\int_{x}^{x+h}|g(t)-g(x)|dt < \frac{1}{h}\varepsilon(x+h-x) = \varepsilon$$

In other words:

$$\frac{G(x+h)-G(x)}{h} \to g(x)$$

so $G'(x) = g(x)$, as required.

□

2.2 Theorem: FTC Version 2 - Integral of Derivative

> *Let I be an interval, and let $f : I \to \mathbb{R}$ be an* ***differentiable*** *function on I, with f' being* ***continuous*** *on I. Then, for any $a, b \in I$:*
>
> $$\int_a^b f' = f(b) - f(a)$$

Fundamental Theorem of Calculus: The Integral of a Derivative. We know that f' is integrable on an interval defined by a and b, so in particular it is integrable on any subinterval of $[a, b]$, by (1.1.1).

Define:

$$G(x) = \int_a^x f'$$

By the Fundamental Theorem of Calculus above, we know that $G'(x)$ exists on (a, b) (since the above will be valid on any subinterval $[a, b]$). Moreover, $G'(x) = f'(x)$.

Hence, it follows that $\exists k \in \mathbb{R}$, such that $G - f = k$, since:

$$(G - f)' = G' - f' = 0$$

(can also use Rolle's Theorem/Mean Value Theorem)

Now, consider:

$$\int_a^b f'$$

By definition of G, we must have:

$$\int_a^b f' = G(b)$$

Moreover, notice that $G(a) = 0$, since $\int_a^a f' = 0$. Thus:

$$\int_a^b f' = G(b) - G(a) = (f(b) + k) - (f(a) + k) = f(b) - f(a)$$

as required.

□

> *Now that the Fundamental Theorem of Calculus is defined, we understand integrals and derivatives to be "inverses" of each other, which allows us to use our standard integration toolkit, such as integration by substitution, integration by parts, or just finding an antiderivative.*

3 Exercises

1. **Prove that if $p > 1$, the series $\sum_{n=1}^{p} \frac{1}{n^p}$ converges. Show that if $p \leq 1$, the series diverges.**

   Recall, the Cauchy Integral Criterion of Summability of Series. Given a non-negative, non-increasing function on some interval $[p, \infty)$, we have that f is integrable on the interval if and only:

   $$\sum_{n=p}^{\infty} f(n)$$

   converges.

   For this case, we need to consider:

   $$f(x) = \frac{1}{x^p}$$

   Indeed, if f is integrable, $\sum_{n=p}^{\infty} \frac{1}{n^p}$ converges, then so does $\sum_{n=1}^{\infty} \frac{1}{n^p}$, since f is clearly always positive and decreasing.

   Now that we are armed with the FTC, computing integrals is a lot easier. Consider:

   $$\int_p^{\infty} \frac{1}{x^p} dx$$

   We can't directly integrate this, since we have ∞ as an upper limit. However, we can make use of the Theorem above (1.1.4), which tells us that if we can split the integration interval into disjoint subintervals I_n, the final integral is just the sum of the integrals of f over each I_n.

   Indeed, if we use $I_n = [n, n+1)$ for $n \geq p$, we can see that $\int_{I_n} f(x)dx$ is defined, so the Theorem applies, and:

   $$\int_p^{\infty} \frac{1}{x^p} dx = \sum_{n=p}^{\infty} \int_n^{n+1} \frac{1}{x^p} dx$$

   We need to consider 3 different cases:

   (1) $\boldsymbol{p > 1}$

   If $p > 1$, we can use the fact that $\frac{1}{(-p+1)x^{p-1}}$ is an antiderivative of f, so by the FTC:

   $$\begin{aligned}\int_n^{n+1} \frac{1}{x^p} dx &= \left[\frac{1}{(-p+1)x^{p-1}}\right]_n^{n+1} \\ &= \frac{1}{(-p+1)(n+1)^{p-1}} - \frac{1}{(-p+1)n^{p-1}} \\ &= \frac{1}{-p+1}\left(\frac{1}{(n+1)^{p-1}} - \frac{1}{n^{p-1}}\right)\end{aligned}$$

But then, we obtain a telescoping series:

$$\begin{aligned}
&\int_p^\infty \frac{1}{x^p}dx \\
&= \sum_{n=p}^{\infty} \int_n^{n+1} \frac{1}{x^p}dx \\
&= \frac{1}{-p+1} \sum_{n=p}^{\infty} \left(\frac{1}{(n+1)^{p-1}} - \frac{1}{n^{p-1}} \right) \\
&= \frac{1}{-p+1} \lim_{k\to\infty} \left[\sum_{n=p}^{k} \left(\frac{1}{(n+1)^{p-1}} - \frac{1}{n^{p-1}} \right) \right] \\
&= \frac{1}{-p+1} \lim_{k\to\infty} \left[\frac{1}{(p+1)^{p-1}} - \frac{1}{p^{p-1}} + \frac{1}{(p+2)^{p-1}} - \frac{1}{(p+1)^{p-1}} + \ldots + \frac{1}{k^{p-1}} - \frac{1}{(k-1)^{p-1}} + \frac{1}{(k+1)^{p-1}} - \frac{1}{k^{p-1}} \right] \\
&= \frac{1}{-p+1} \lim_{k\to\infty} \left[-\frac{1}{p^{p-1}} + \frac{1}{(k+1)^{p-1}} \right] \\
&= \frac{1}{(1-p)p^{p-1}}
\end{aligned}$$

where we have used the fact that if $p > 1$, we have that $\lim_{k\to\infty} \left[\frac{1}{(k+1)^{p-1}}\right] = 0$. Hence, it follows that $\sum_{n=p}^{\infty} \int_n^{n+1} \frac{1}{x^p}dx$ converges, so the integral $\int_p^\infty \frac{1}{x^p}$ exists, and so, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

② $p < 1$

If $p < 1$, we have the same derivation as above, but notice that:

$$\lim_{k\to\infty} \frac{1}{(k+1)^{p-1}} = \infty$$

so the integral will not exists, and so, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ will diverge.

③ $p = 1$

If $p = 1$, then the antiderivative defined above won't work, since we are considering:

$$\int_n^{n+1} \frac{1}{x}$$

which has antiderivative $\ln x$. We follow similar logic:

$$\int_n^{n+1} \frac{1}{x} = \ln(n+1) - \ln(n) =$$

Which also leads to a telescoping series:

$$\begin{aligned}\int_1^\infty \frac{1}{x}dx &= \sum_{n=1}^\infty \int_n^{n+1} \frac{1}{x}dx \\ &= \sum_{n=1}^\infty \ln(n+1) - \ln(n) \\ &= \lim_{k\to\infty} [\ln(2) - \ln(1) + \ln(3) - \ln(2) + \ldots + \ln(k+1) - ln(k)] \\ &= \lim_{k\to\infty} [-\ln(1) + \ln(k+1)] \\ &= \lim_{k\to\infty} [\ln(k+1)] \\ &= \infty\end{aligned}$$

So the integral diverges, and so, $\sum_{n=1}^\infty \frac{1}{n}$ diverges.

2. **Determine for which $p \in \mathbb{R}$ we get that f is integrable on I:**

(a) $\frac{1}{x^p}, \quad I = (1, \infty)$

In the work above, we have shown that $\int_p^\infty \frac{1}{x}^p$:

- converges if $p > 1$
- diverges if $p \leq 1$

We also know that:

$$\int_1^p \frac{1}{x^p} + \int_p^\infty \frac{1}{x^p} = \int_1^\infty \frac{1}{x^p}$$

so $\int_1^\infty \frac{1}{x}^p$ converges if and only if $\int_p^\infty \frac{1}{x}^p$ converges. Thus, $\int_1^\infty \frac{1}{x}^p$ converges if and only if $p > 1$

(b) $\frac{1}{x^p}, \quad I = (0, 1)$

We again apply (1.1.4), partitioning $I = (0, 1)$ via $I_n = [\frac{1}{n+1}, \frac{1}{n})$. Then:

$$\int_0^1 \frac{1}{x^p}dx$$

is integrable if and only if:

$$\sum_{n=1}^\infty \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{1}{x^p}dx$$

converges.

We work in a similar way as above. If $p \neq 1$, we can use the fact that $\frac{1}{(-p+1)x^{p-1}}$ is an antiderivative of f, so by the FTC:

$$\begin{aligned}\int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{1}{x^p}dx &= \left[\frac{1}{(-p+1)x^{p-1}}\right]_{\frac{1}{n+1}}^{\frac{1}{n}} \\ &= \frac{n^{p-1}}{1-p} - \frac{(n+1)^{p-1}}{1-p} \\ &= \frac{1}{1-p}\left(n^{p-1} - (n+1)^{p-1}\right)\end{aligned}$$

But then, we obtain a telescoping series:

$$\begin{aligned}
&\int_0^1 \frac{1}{x^p}dx \\
&= \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{1}{x^p}dx \\
&= \frac{1}{1-p} \sum_{n=1}^{\infty} \left(n^{p-1} - (n+1)^{p-1}\right) \\
&= \frac{1}{1-p} \lim_{k\to\infty} \left[\sum_{n=1}^{k} \left(n^{p-1} - (n+1)^{p-1}\right)\right] \\
&= \frac{1}{1-p} \lim_{k\to\infty} \left[1 - 2^{p-1} + 2^{p-1} - 3^{p-1} + \ldots + k^{p-1} - (k+1)^{p-1}\right] \\
&= \frac{1}{1-p} \lim_{k\to\infty} \left[1 - (k+1)^{p-1}\right]
\end{aligned}$$

If $p > 1$, this limit will diverge, but if $p < 1$, the limit will exist, and so the integral is defined.

Alternatively, if we consider $p = 1$:

$$\begin{aligned}
&\int_0^1 \frac{1}{x^p}dx \\
&= \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{1}{x}dx \\
&= \sum_{n=1}^{\infty} (\ln(n+1) - \ln(n)) \\
&= \lim_{k\to\infty} \left[\sum_{n=1}^{k} (\ln(n+1) - \ln(n)\right] \\
&= \lim_{k\to\infty} [\ln(2) - \ln(1) + \ln(3) - \ln(2) + \ldots + \ln(k+1) - \ln(k)] \\
&= \lim_{k\to\infty} [-\ln(1) + \ln(k+1)] \\
&= \infty
\end{aligned}$$

so the integral diverges. Thus, the integral converges on $(0, 1)$ if and only if $p < 1$

3. **Determine whether $f(x)$ is integrable over I:**

   (a) $\sin(x), \quad I = (0, \infty)$

   Once again we shall use (1.1.4). Noticing that $\sin(x)$ is 2π periodic, define $I_n = (n\pi, (n+1)\pi$. $\sin(x)$ is integrable on the interval if and only if:

$$\sum_{n=1}^{\infty} \int_{I_n} |sin(x)|dx < \infty$$

   We thus consider the integral over I_n, and applying the substitution $t = x - n\pi$:

$$\int_{n\pi}^{(n+1)\pi} |sin(x)|dx = \int_0^{\pi} |sin(t + n\pi)|dt$$

The sum of angles formula tells us that:

$$sin(t+n\pi) = \sin(t)\cos(n\pi) + \sin(n\pi)\cos(t) = (-1)^n \sin(t)$$

so $|sin(t+n\pi)| = |sin(t)|$, and for $t \in [0,\pi]$, $sin(t) \geq 0$ so:

$$\int_{n\pi}^{(n+1)\pi} |sin(x)|dx = \int_0^{\pi} \sin(t) = [-\cos(pi) + \cos(0)] = 2$$

But this then means that $\sum_{n=1}^{\infty} \int_{I_n} |sin(x)|dx$ diverges, and so f is not integrable on $(0,\infty)$.

(b) $\frac{1-\cos(x)}{x^2}$, $\quad I = (0,\infty)$

We split I via $I_n = (n, n+1]$ for $n \geq 0$. The function is integrable if and only if:

$$\sum_{n=0}^{\infty} \int_{I_n} \left| \frac{1-\cos(x)}{x^2} \right| dx$$

exists.

This is true if and only if f is integrable on each I_n (noticing that $|f| = f$, since the function is always non-negative). Computing the integral is not nice, so instead we show that it can be bounded. We first consider the case $n \geq 1$:

$$\begin{aligned}
&\int_{I_n} \frac{1-\cos(x)}{x^2} dx \\
=&\int_n^{n+1} \frac{1-\cos(x)}{x^2} dx \\
\leq&\int_n^{n+1} \frac{2}{x^2} dx \\
=&\left[-\frac{2}{x}\right]_n^{n+1} \\
=&\frac{2}{n} - \frac{2}{n+1}
\end{aligned}$$

Using this bounded we can then compute:

$$\begin{aligned}
sum_{n=1}\infty \int_{I_n} \left| \frac{1-\cos(x)}{x^2} \right| dx \\
&\leq \sum_{n=1} \infty \frac{2}{n} - \frac{2}{n+1} \\
&= \lim_{k\to\infty} \sum_{n=1} k \frac{2}{n} - \frac{2}{n+1} \\
&= \lim_{k\to\infty} \left(\frac{2}{1} - \frac{2}{2} + \frac{2}{2} - \frac{2}{3} + \ldots + \frac{2}{k} - \frac{2}{k+1} \right) \\
&= \lim_{k\to\infty} \left(\frac{2}{1} - \frac{2}{k+1} \right) \\
&=2
\end{aligned}$$

We now just need to consider the interval $I_0 = (0,1]$. Indeed, f is integrable on this interval. The only point of potential discontinuity will be at $x = 0$, since we are dividing by x^2. If we show that the function doesn't diverge at $x = 0$, then we can define $f(0) = \lim_{x\to 0} \frac{1-\cos(x)}{x^2}$, which will make the function continuous, and so integrable (since we consider a closed, bounded interval). We can use L'Hopital's Rule:

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{\sin(x)}{2x} = \lim_{x\to 0} \frac{\cos(x)}{2} = \frac{1}{2}$$

So if $f(0) = \frac{1}{2}$, the function will be continuous, and so integrable on I_0. Thus,

$$\int_0^\infty \frac{1-\cos(x)}{x^2} = \sum_{n=0} \infty \int_{I_n} < \infty$$

so the function is integrable.

4. **Let** $F(x) : \mathbb{R} \to \mathbb{R}$ **be continuous. Find** $F'(x)$ **for:**

(a) $F(x) = \int_{x^2}^1 f(t)dt$

Define:

$$G(x) = \int_1^x f(t)dt$$

By the FTC:

$$G'(x) = f(x)$$

Furthermore, notice that:

$$F(x) = -G(x^2)$$

So differentiating both sides:

$$F'(x) = -2xG'(x^2) = -2xf(x^2)$$

(b) $F(x) = \int_0^x f(t-x)dx$

Lets redefine the integrable into a new variable $y = t - x$:

$$F(x) = \int_{-x}^0 f(y)dy = -\int_0^{-x} f(y)dy$$

Moreover, define:

$$G(x) = \int_0^x f(y)dy$$

Notice,

$$F(x) = -\int_0^{-x} f(y)dy = -G(-x)$$

Moreover, by FTC:

$$G'(x) = f(x)$$

So it follows that:

$$F'(x) = G'(-x) = f(-x)$$

4 Workshop

We consider a function $f : [a, b] \to \mathbb{R}$ which is continuous on a closed, bounded interval $[a, b]$.

1. **Let:**

$$g(x) = \begin{cases} f(x), & x \in [a, b] \\ 0, & otherwise \end{cases}$$

For each $n \in \mathbb{N}$, consider the function $\phi_n : \mathbb{R} \to \mathbb{R}$ defined by:

$$\phi_n(x) = \inf\left\{ g(y) \mid y \in \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right]\right\}, \qquad \forall x \in \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)$$

Prove that each ϕ_n is a step function for any such g.

> *This is a rather straightforward exercise, but we have to be careful that we tick all the boxes which define a step function.*

Notice, on each interval $\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)$, $\phi_n(x)$ is constant (since the infimum is taken over the same set). Moreover, this constant value will be bounded, since $g(x) \le f(x)$, and by the Extreme Value Theorem, since f is continuous over a closed bounded interval, it is bounded on said interval. Lastly, ϕ_n has bounded support: that is, it can only be non-zero on a finite interval (namely $[a, b]$). Thus, $\phi_n(x)$ must be a step function.

2. **Prove that the sequence (ϕ_n) is monotone increasing:**

$$\phi_1 \le \phi_2 \le \ldots \le \ldots \qquad \forall x \in \mathbb{R}$$

> *The key is that the definition is rather lax, and we can just pick any integer i which we want when defining the values over which the ϕ_n will operate.*

Consider $x \in \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)$. We must have that $\exists c \in \mathbb{R}$ such that:

$$\phi_n(x) = c$$

But now notice, we can write this interval as a union of disjoint intervals:

$$\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right) = \left[\frac{2i}{2^{n+1}}, \frac{2(i+1)}{2^{n+1}}\right) = \left[\frac{2i}{2^{n+1}}, \frac{2i+1}{2^n}\right) \cup \left[\frac{2i+1}{2^{n+1}}, \frac{2i+2}{2^{n+1}}\right)$$

But then, if $x \in \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)$ we must have that:

$$\phi_{n+1}(x) \ge \phi_n(x)$$

as the infimum used to compute the value of $\phi_{n+1}(x)$ depends on the subintervals which are used to compute the infimum of $\phi_n(x)$

3. **Prove that:**

$$g(x) = \lim_{n\to\infty} \phi_n(x), \qquad \forall x \in (a, b)$$

4. **Finally, argue that:**

$$\int_{(a,b)} \phi_n \leq M(b-a)$$

where:

$$M = \sup_{x\in[a,b]} f(x)$$

and hence apply Theorem 4.4 of the lecture notes to conclude that f is integrable on any interval with endpoints a, b and that:

$$\int_{(a,b)} f = \lim_{n\to\infty} \int_{(a,b)} \phi_n$$

Since $M = \sup_{x\in[a,b]} f(x)$ then:

$$M = \sup_{x\in[a,b]} f(x) \geq \inf_{x\in[a,b]} f(x) = \inf_{x\in[a,b]} g(x) = \phi_n(x)$$

Thus, by part b) of Theorem 4.2, if we integrate over (a, b), the inequality is preserved, so:

$$\int_{(a,b)} \phi_n \leq \int_{(a,b)} M = M(b-a) < \infty$$

Which in particular means that:

$$\sup_{n\in\mathbb{N}} \int_{(a,b)} \phi_n \leq M(b-a) < \infty$$

Recall the Monotone Convergence Theorem:

Suppose that f_n is a sequence of:

- ***monotone***
- ***non-decreasing***
- ***integrable***

functions on an interval I:

$$f_1(x) \leq f_2(x) \leq \dots$$

For any $x \in I$, define:

$$f(x) = \lim_{n\to\infty} f_n(x)$$

We allow that for some x, this limit diverges to infinity: we are not concerned with particular points. Notice, that if f_n is a bounded sequence, it will necessarily converge, since its monotone.
*Then, f must be **integrable** on I **if and only if**:*

$$\sup_{n\in\mathbb{N}} \int_I f_n = \lim_{n\to\infty} \int_I f_n < \infty$$

(this equivalence might not be immediately obvious, but it is due to the fact that f_n is non-decreasing, so $f_n \leq f_{n+1} \implies \int_I f_n \leq \int_I f_{n+1}$, so the supremum must coincide with the limit)
Moreover, we have that:

$$\int_I f = \lim_{n\to\infty} \int_I f_n$$

---

*There is an equivalent result if the sequence of functions is **monotone**, **non-increasing**, in which we just need to check that:*

$$\inf_{n\in\mathbb{N}} \int_I f_n$$

exists for

$$\int_I f = \lim_{n\to\infty} \int_I f_n$$

[Theorem 4.4]

Now, the ϕ_n are monotone, non-decreasing, and we just showed that the supremum of the integral of each ϕ_n exists. Moreover, we have shown that on (a, b):

$$\phi_n(x) \to f(x)$$

Thus, the Monotone Convergence Theorem applies, and f is integrable. In particular:

$$\int_{(a,b)} f = \lim_{n\to\infty} \int_{(a,b)} \phi_n$$

5. **Modify the above argument for the case $f : (a,b) \to \mathbb{R}$ where f is bounded and continuous on (a,b). One difference is lack of uniform continuity. Does this make a difference?**

6. **Let $f(x) = n$ for any:**

$$x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], \qquad n \in \mathbb{N}$$

Prove that f is not integrable on $(0,1]$.

As a hint we are given to consider:

$$f_n = \mathcal{X}_{\left(0,\frac{1}{n}\right]} \qquad \sum_{n=1}^{\infty} f_n$$

We can see that:

$$f(x) = \sum_{n=1}^{\infty} f_n$$

> *At this point, the solutions and my answer differ: I use Theorem 4.3, whilst the solutions use Monotone Convergence Theorem.*

Self: I consider Theorem 4.3, part b)

Suppose f_n is a sequence of functions, each of which is ***integrable*** *on some I.*

(a) ***If****:*

$$\sum_{n=1}^{\infty} \int_I |f_n| < \infty$$

(the sum of integrals of each function in the sequence is convergent) ***and*** *f is a function on I, such that,*

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

for any x, such that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$[a] , ***then*** *$f$ is* ***integrable*** *on I, and its integral is:*

$$\int_I f = \sum_{n=1}^{\infty} \int_I f_n < \infty$$

(b) If we further have that for any $n \in \mathbb{N}$, $f_n \geq 0$, and

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

for all $x \in I$ (except for possibly finitely many points, in which we can allow that $\sum_{n=1}^{\infty} f_n(x) = \infty$).
Then f is integrable on I ***if and only if****:*

$$\sum_{n=1}^{\infty} \int_I f_n < \infty$$

[Theorem 4.3]

[a] this is just saying that we require the x to be such that the sum converges (to f)

Clearly, each $f_n \geq 0$, so the Theorem applies, and f is integrable if and only if:

$$\sum_{n=1}^{\infty} \int_{(0,1]} \mathcal{X}_{(0,\frac{1}{n}]} < \infty$$

However,

$$\sum_{n=1}^{\infty} \int_{(0,1]} \mathcal{X}_{(0,\frac{1}{n}]}(x) = \sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges, and so f can't be integrable.

In the solutions, they apply monotone convergence (Theorem 4.4). Notice, we can define $\forall x \in (0,1]$:

$$\phi_n(x) = \sum_{j=1}^{n} f_j$$

So:

$$\lim_{n\to\infty} \phi_n(x) = f(x)$$

The sequence of ϕ_n is monotone, non-decreasing and integrable (since it is a step function - as it is a finite sum of characteristic functions).

Hence, the monotone convergence theorem applies, and f is integrable on $(0,1]$ if and only if:

$$\sup_{n\in\mathbb{N}} \int_{(0,1]} \phi_n < \infty$$

However:

$$\sup_{n\in\mathbb{N}} \int_{(0,1]} \phi_n = \sup_{n\in\mathbb{N}} \left(\sum_{j=1}^{n} \frac{1}{n} \right)$$

and this supremum is unbounded as n increases (Harmonic Series), so f can't be integrable.

When I first did this, I didn't consider the hint, and just used $f_n(x) = n\mathcal{X}_{I_n}(x)$, where $I_n = \left(\frac{1}{n+1}, \frac{1}{n}\right]$. It also works, but its a bit longer to show that it leads to divergence.

7. **For $x > 0$ define:**

$$L(x) = \int_1^x \frac{dt}{t}$$

Show that:

- $L(xy) = L(x) + L(y)$
- **Show that** $L'(x) = \frac{1}{x}$
- **Show that L is the inverse to the exponential function $E(x)$, defined by the power series** $E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

Notice, since $\frac{1}{t}$ is continuous $\forall t > 0$, in particular it is Riemann (and so Lebesgue) integrable (Theorem 4.7). Hence, $L(x)$ is well defined.

We compute:

$$L(xy) = \int_1^{xy} \frac{dt}{t} = \int_1^x \frac{dt}{t} + \int_x^{xy} \frac{dt}{t} = L(x) + \int_x^{xy} \frac{dt}{t}$$

But notice, if we let $u = \frac{t}{x}$, then $dt = x\ du$ so:

$$\int_x^{xy} \frac{dt}{t} = \int_1^u \frac{du}{u} == \int_1^y \frac{dy}{y} = L(y)$$

where changing the name of the variable won't change the integral. Thus:

$$L(xy) = L(x) + L(y)$$

Again, since $\frac{1}{t}$ is continuous on the interval of itnegration, by the Fundamental Theorem of Calculus (Theorem 4.10), it must be the case that L is differentiable at x, and:

$$L'(x) = \frac{1}{x}$$

Lastly, consider:

$$\phi(x) = L \circ E(x)$$

Notice, ϕ is defined $\forall x \in \mathbb{R}$, since we showed that $\forall x \in \mathbb{R}, E(x) > 0$, and $L(x)$ is defined precisely when $x > 0$. We can differentiate ϕ for any $x \in \mathbb{R}$ by the chain rule:

$$\phi'(x) = L'(E(x)) \times E'(x) = \frac{1}{E(x)} \times E(x) = 1$$

Now, consider the interval $[0, x]$ with $x \in \mathbb{R}$. ϕ is continuous and differentiable on this interval, so by the mean value theorem:

$$1 = \frac{\phi(x) - \phi(0)}{x - 0} = \frac{\phi(x)}{x} \implies \phi(x) = x$$

where we have used the fact that:

$$\phi(0) = L(E(0)) = \int_1^1 \frac{dt}{t} = 0$$

Hence:

$$L \circ E(x) = x$$

so L is the left inverse of E.

The solutions now use the fact that a function is surjective ***if and only*** *if it has a right inverse; they then show that E is surjective on the set of positive numbers (to do this, they use the fact that $E(x) \to \infty$ as $x \to \infty$, whilst $E(x) \to 0$ as $x \to -\infty$, and then apply the Intermediate Value Theorem). This then means that E has a right inverse, and since it has a left inverse, then these must be equal, from which we get that:*

$$L^{-1} = E$$

I don't like this method though, so I'll just give mine.

We now need to consider $E(L(x))$. Since we know that $L(E(x)) = x$ for any $x > 0$, it follows that $L(E(x)) > 0$, so the following is true for any $x > 0$:

$$L(E(L(x))) = L(x)$$

where we have just made the argument of E be $L(x)$ (this is always valid, as E is defined for any $x \in \mathbb{R}$). Moreover, it is easy to see from the definition of L that L is monotone increasing on any interval $(0, x]$. Since:

$$L(x) = \int_1^x \frac{1}{t} dt$$

We know $\frac{1}{t} > 0$ whenever $t \in (0, x]$. Then, for any $u, v \in (0, x]$, with $u < v$ we have $L(u) < L(v)$ since:

$$L(v) = \int_1^v \frac{1}{t} dt = \int_1^u \frac{1}{t} dt + \int_u^v \frac{1}{t} dt = L(u) + \int_u^v \frac{1}{t} dt$$

Since $u < v$, and $\frac{1}{t} > 0, t \in (0, x]$, it follows that:

$$\int_u^v \frac{1}{t} dt > 0$$

from which it follows that $L(u) < L(v)$. But then, since L is monotonically increasing, $L(E(L(x))) = L(x)$ is true if and only if:

$$E(L(x)) = x$$

In other words, we have shown that $\forall x > 0$, we have $E(L(x)) = L(E(x)) = x$, so it follow that L is the inverse of E whenever $x > 0$.

8. **Let $g : [a, b] \to \mathbb{R}$ (with $a < b$) be continuous and nonnegative. If $\int_a^b g = 0$, show that $g = 0$ on $[a, b]$.**

> *Again, I don't like the solution from the notes, so I provide my own.*

g is continuous, and so integrable, so define a function $G(x)$ via:

$$G(x) = \int_a^x g(t) dt$$

where $x \in [a, b]$. By the Fundamental Theorem of Calculus (Theorem 4.10), since g is continuous on $[a, b]$, it follows that G is differentiable, and:

$$G'(x) = g(x)$$

Now, notice since $x \in [a, b]$, by properties of integrals :

$$\int_a^b g(t) dt = \int_a^x g(t) dt + \int_x^b g(t) dt$$

The fact that g is non-negative on all of $[a, b]$ implies that in particular, using Theorem 4.4, b):

$$\int_a^x g(t) dt \geq 0$$

$$\int_x^b g(t) dt \geq 0$$

But we know that $\int_a^b g(t) dt = 0$. In other words, $\int_a^b g(t) dt = \int_a^x g(t) dt + \int_x^b g(t) dt$ will hold if and only if $\int_a^x g(t) dt = \int_x^b g(t) dt = 0$. But then, we have defined $G(x) = \int_a^x g(t) dt$, so it follows that:

$$G(x) = 0$$

If we differentiate $G(x)$, we see $G'(x) = 0$. We have shown by the Fundamental Theorem of Calculus that $G'(x) = g(x)$, so it follows that for any $x \in [a, b]$:

$$g(x) = 0$$

as required.