Honours Analysis - Week 6 - The Lebesgue Integral

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Contents

1 Motivation **2**

2 The Indicator/Characteristic Function **2**
2.1 Defining the Characteristic Function . . . 2

3 The Step Function **3**
3.1 Defining Step Function . . . 3
3.2 Theorem: Sum of Step Functions is a Step Function . . . 4
3.3 Theorem: Constructing Step Functions from Other Step Functions . . . 5
3.3.1 Theorem: Step Functions are a Vector Space . . . 5
3.3.2 Theorem: Absolute Value of Step Function is a Step Function . . . 5
3.3.3 Theorem: Maximum and Minimum of Step Functions is a Step Function . . . 6
3.3.4 Theorem: Product of Step Functions is a Step Function . . . 6
3.4 Theorem: Step Functions as Sums of Characteristic Functions . . . 6
3.5 The Integral of the Step Function . . . 8

4 Lebesgue Integrable Functions **8**
4.1 Integrals: Intuition Using Step Functions . . . 8
4.2 Defining Lebesgue Integrable Functions . . . 10
4.3 Theorem: Integral of Step Function Independent of Interval . . . 11
4.4 Corollary: Step Functions are Lebesgue Integrable . . . 11
4.5 Properties of Lebesgue Integrals . . . 12
4.5.1 Theorem: Linearity of Lebesgue Integral . . . 12
4.5.2 Theorem: Positivity of Lebesgue Integral . . . 14
4.5.3 Theorem: Lebesgue Integral of Absolute Value . . . 14
4.5.4 Theorem: Lebesgue Integral of Max/Min . . . 15
4.5.5 Theorem: Lebesgue Integrability of Function Products . . . 15
4.5.6 Theorem: Bounded Functions and Lebesgue Integrability . . . 15

5 Exercises **16**

6 Workshop **20**

1 Motivation

- *seek to define integrals of real functions as to represent the notion of "area under the curve"*
- *mainly focus on* ***definite*** *integrals (so that $\int f$ is a real number rather than a function)*
- *want it to have desirable features:* ***linearity*** *and* ***positivity***
- *want usual rules (recognition of antiderivatives, product rule, substitution etc...) to hold rigorously, validating the usual techniques for calculating integrals*
- *also want to consider situations in which order of integration and summation can be swapped*

2 The Indicator/Characteristic Function

2.1 Defining the Characteristic Function

- **How can bounded intervals be described?**
  - consider $a, b \in \mathbb{R}, a < b$
  - if E is a **bounded interval**, it has one of the following forms:
    * $[a, a] = a$ (interval containing only the element a)
    * $[a, b]$
    * $[a, b)$
    * $(a, b]$
    * (a, b)
- **What is the length of an interval?**
  - consider a **bounded interval** E
  - we denote its length via:

$$\lambda(E)$$

  - if E is defined by the bounds $a, b \in \mathbb{R},\ a \leq b$, then, independent of whether E is open, closed or half-open:

$$\lambda(E) = b - a$$

- **What is a characteristic function?**
  - a function over a **bounded interval** $E \subseteq \mathbb{R}$:

$$\mathcal{X}_E : \mathbb{R} \to \mathbb{R}$$

  - defined as:

$$\mathcal{X}_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$$

- **What is the integral of a characteristic function?**
  - using the principles outlined in the motivation pushes us to define:

$$\int \mathcal{X}_E := \lambda(E)$$

3 The Step Function

3.1 Defining Step Function

- **What is a step function?**
  - a function $\phi : \mathbb{R} \to \mathbb{R}$ which is **constant** on **discrete intervals** of the real line
  - its value on different intervals can vary.
- **How can we formally describe a step function?**
  - more formally consider the real numbers:

$$x_0 < x_1 < \ldots < x_n, \qquad n \in \mathbb{N}$$

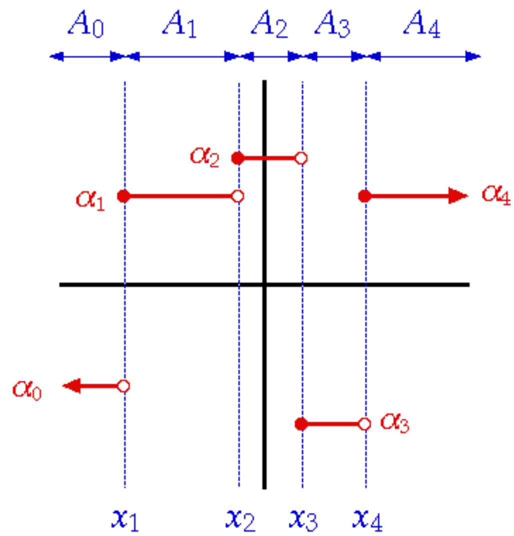
  - can define a **step function with respect to** $\{x_0, x_1, \ldots, x_n\}$ via:

$$\phi(x) = \begin{cases} 0, & x < x_0 \ or \ x > x_n \\ c_j, & x \in (x_j, x_{j+1}) \end{cases}$$

    where $0 \leq j \leq n-1$ and $c_j \in \mathbb{R}$.
  - alternatively, ϕ is a step function if and only if it can be defined as:

$$\phi(x) = \sum_{j=1}^{n} c_j \mathcal{X}_{I_j}(x), \qquad I_j = (x_{j-1}, x_j)$$

- **Are step functions defined at the endpoints of the intervals (x_j, x_{j+1})?**
  - these can be defined, but we don't formally define the value of $\phi(x_j)$
  - in particular ϕ is continuous on all of $\mathbb{R}$, except possibly at $\{x_0, x_1, \ldots, x_n\}$
  - each of $\{x_0, x_1, \ldots, x_n\}$ is called a **potential jump point of ϕ**
- **Are step functions bounded?**
  - from their definition, step functions must be bounded
  - in particular, there always exists a bounded interval $I \subset \mathbb{R}$, such that if $x \notin I$, we have $\phi(x) = 0$ (this I can be defined by x_0 or x_n)



3.2 Theorem: Sum of Step Functions is a Step Function

Let ϕ and ψ be ***step functions****. Then, $\phi + \psi$ is also a step function. [Example 4.1]*

Proof: Sum of Step Functions. Since ϕ and ψ are step functions, in particular:

- ϕ is a step function with respect to a set $\{x_0, x_1, \ldots, x_n\}$
- ψ is a step function with respect to a set $\{y_0, y_1, \ldots, y_m\}$

Then, consider the bounded, finite set:

$$\{z_0, z_1, \ldots, z_k\} = \{x_0, x_1, \ldots, x_n\} \cup \{y_0, y_1, \ldots, y_m\}$$

where $k \leq m + n$ (it can be the case that there are elements in common in both sets).

Notice, it must be the case that:

- ϕ is constant on any interval (z_j, z_{j+1}), since in particular, each $(z_j, z_{j+1}$ must be a subinterval (either the same size or smaller) than any interval defined by $\{x_0, x_1, \ldots, x_n\}$
- ψ is constant on any interval (z_j, z_{j+1}), since in particular, each $(z_j, z_{j+1}$ must be a subinterval (either the same size or smaller) than any interval defined by $\{y_0, y_1, \ldots, y_m\}$

In particular, since ϕ and ψ are both constant on any interval (z_j, z_{j+1}) defined by $\{z_0, z_1, \ldots, z_k\}$, it must be the case that $\phi + \psi$ must also be constant on any interval (z_j, z_{j+1}).

In particular if $x \in (z_j, z_{j+1})$, and $\phi(x) = c_j,\ \psi(x) = d_j$, then:

$$(\phi + \psi)(x) = c_j + d_j$$

Lastly, assuming that $\{z_0, z_1, \ldots, z_k\}, \{x_0, x_1, \ldots, x_n\}, \{y_0, y_1, \ldots, y_m\}$ are all ordered, it is easy to see that that $\phi(x) = 0, \forall x < z_0 | x > z_k$ and $\psi(x) = 0, \forall x < z_0 | x > z_k$. In other words, $\phi + \psi$ is also zero outside of $[z_0, z_k]$.

Thus, we have shown that $\phi + \psi$ is a step function with respect to $\{z_0, z_1, \ldots, z_k\}$, as required.

□

3.3 Theorem: Constructing Step Functions from Other Step Functions

These are all part of ***Exercise 4.1****.*

3.3.1 Theorem: Step Functions are a Vector Space

The class of ***step functions*** *defines a* ***vector space****.*
If ϕ, ψ are step functions, and $\alpha, \beta \in \mathbb{R}$, then:

$$\alpha\phi + \beta\phi$$

is also a ***step function****.*

Proof. If

$$\phi(x) = \begin{cases} 0, & x < x_0 \ \textit{or} \ x > x_n \\ c_j, & x \in (x_j, x_{j+1}) \end{cases}$$

then:

$$\alpha\phi(x) = \begin{cases} 0, & x < x_0 \ \textit{or} \ x > x_n \\ \alpha c_j, & x \in (x_j, x_{j+1}) \end{cases}$$

so $\alpha\phi$ is also a step function Since the sum of step functions is a step function, it follows that $\alpha\phi + \beta\psi$ is a step function. □

3.3.2 Theorem: Absolute Value of Step Function is a Step Function

If ϕ is a ***step function****, then $|\phi|$ is a* ***step function****.*

Proof. If

$$\phi(x) = \begin{cases} 0, & x < x_0 \ \textit{or} \ x > x_n \\ c_j, & x \in (x_j, x_{j+1}) \end{cases}$$

then:

$$|\phi(x)| = \begin{cases} 0, & x < x_0 \ \textit{or} \ x > x_n \\ |c_j|, & x \in (x_j, x_{j+1}) \end{cases}$$

so $|\phi|$ is also a step function. □

3.3.3 Theorem: Maximum and Minimum of Step Functions is a Step Function

Let ϕ, ψ be step functions. Then, $\max\{\phi, \psi\}$ and $\min\{\phi, \psi\}$ are step functions.

Proof. We know that[1]:

$$\max\{\phi, \psi\} = \frac{\phi + \psi + |\phi - \psi|}{2}$$

which is a linear combination of step functions, and so is a step function.

Similarly, we know that:

$$\min\{\phi, \psi\} = \frac{\phi + \psi - |\phi - \psi|}{2}$$

which is a linear combination of step functions, and so is a step function.

□

3.3.4 Theorem: Product of Step Functions is a Step Function

If ϕ, ψ are step functions, then $\phi\psi$ is a step function.

3.4 Theorem: Step Functions as Sums of Characteristic Functions

We formally prove the intuitive result which we presented intuitively above.

*ϕ is a step function **if and only if** it can be written in the form:*

$$\phi = \sum_{j=1}^{n} c_j \mathcal{X}_{J_j}$$

for some n, c_j, and bounded intervals J_j.

Proof: Step Function as Sum of Characteristic Functions. Intuitively, this makes a lot of sense. If we look at the definition of a step function:

$$\phi(x) = \begin{cases} 0, & x < x_0 \text{ or } x > x_n \\ c_j, & x \in (x_j, x_{j+1}) \end{cases}$$

and of a characteristic function:

$$\mathcal{X}_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$$

[1] http://caseychu.io/posts/minimum-and-maximum-of-two-functions/

Then if we let $J_j = (x_j, x_{j+1})$ (or (x_{j-1}, x_j) as in the formulation of the theorem), we can see that:

$$\phi(x) = \begin{cases} 0, & x < x_0 \ or \ x > x_n \\ c_j, & x \in (x_j, x_{j+1}) \end{cases} \iff \phi(x) = \begin{cases} 0 (= \mathcal{X}_{J_j}(x)), & \forall j, x \notin J_j \\ c_j \mathcal{X}_{J_j}(x), & x \in J_j \end{cases}$$

We argue more formally, however.

Firstly, we show that if

$$\phi = \sum_{j=1}^{n} c_j \mathcal{X}_{J_j}$$

then ϕ is a step function.

(This can be proven by the fact that the sum of 2 step functions is a step function, and then arguing that each $c_j \mathcal{X}_{J_j}$ is a step function with respect to the end points of J_j. This is what is said in the notes (basically). In the videos they go from first principles, which is the proof below.)

If ϕ is indeed a step function, then we should be able to define the set of points with respect to which ϕ is a step function.

Since each J_j is a bounded intervals, and we are considering n such intervals, then the set of all endpoints of each J_j must be finite. Define this set as:

$$A = \{a_0, a_1, \dots, a_k\}$$

with $a_0 < a_1 < \dots < a_k$. We claim ϕ is a step function with respect to A, as:

- if $x < a_0$ or $x > a_k$, we know by construction that for any j, $\mathcal{X}_{J_j}(x) = 0$, since any such x is beyond any of the endpoints of any J_j
- if $x \in [a_0, a_k]$, there must exist at least one interval (a_{j-1}, a_j), such that $c_j \mathcal{X}_{J_j}(x) = c_j$. Consider any interval (a_{j-1}, a_j). Then either $(a_{j-1}, a_j) \subset J_j$, in which case $\mathcal{X}_{J_j}(x) = 1$ so $c_j \mathcal{X}_{J_j}(x) = c_j$; or $(a_{j-1}, a_j) \cap J_j = \emptyset$, in which case $\mathcal{X}_{J_j}(x) = 0$ so $c_j \mathcal{X}_{J_j}(x) = 0$

Thus, ϕ satisfies all the properties of a step function, with respect to A.

Now, we show that if ϕ is a step function, it must have the form:

$$\phi = \sum_{j=1}^{n} c_j \mathcal{X}_{J_j}$$

Since ϕ is a step function, it must be so with respect to some set:

$$X = \{x_0, x_1, \dots, x_n\}$$

Then, it is easy to see that, $\forall x \notin X$:

$$\phi(x) = \sum_{j=1}^{n} c_j \mathcal{X}_{(J_j}(x)$$

where c_j is a constant, and $J_j = (x_{j-1}, x_j)$. In order to fix the fact that $\sum_{j=1}^{n} c_j \mathcal{X}_{(J_j}$ doesn't equal ϕ on X, we introduce an additional term:

$$\phi(x) = \sum_{j=1}^{n} c_j \mathcal{X}_{(J_j}(x) + \sum_{i=0}^{n} \phi(x_i) \mathcal{X}_{\{x_i\}}(x)$$

Notice, with this new formulation, we are able to account for whichever value ϕ takes at each value in X, since $\mathcal{X}_{\{x_i\}}(x)$ is 1 only if $x = x_i$.

We have shown that any step function ϕ can thus be expressed as in the form above, as required.

□

3.5 The Integral of the Step Function

- **How can we define the integral of a step function?**
  - let $\phi(x)$ be a step function with respect to $\{x_0, x_1, \dots, x_n\}$
  - we can express $\phi(x)$ as:

    $$\phi(x) = \sum_{j=1}^{n} c_j \mathcal{X}_{J_j}(x)$$

    where c_j is a constant, and $J_j = (x_{j-1}, x_j)$
  - under our desired property of **linearity**, and given the finite sum, we can define:

    $$\begin{aligned} \int \phi(x) &= \int \sum_{j=1}^{n} c_j \mathcal{X}_{J_j}(x) \\ &= \sum_{j=1}^{n} c_j \int \mathcal{X}_{J_j}(x) \\ &= \sum_{j=1}^{n} c_j \lambda(J_j) \\ &= \sum_{j=1}^{n} c_j (x_j - x_{j-1}) \end{aligned}$$

- **Since a step function can be represented in many ways, is their integral always the same?**
  - yes, **independent** on the intervals with which we describe a step function, the integral always evaluates to the same value
  - this shows that the integral is **well-defined**: it only depend on the inherent function, and not necessarily its representation
  - this will be presented more formally when discussing **Lebesgue integrals**

4 Lebesgue Integrable Functions

4.1 Integrals: Intuition Using Step Functions

- **Can you approximate non-negative, continuous functions by using step functions?**
  - yes. In fact as we add more and more infinitesimally small intervals, we can perfectly describe a continuous function. Formally, for any continuous function $f(x)$ on some interval I, there exists some step function, such that::

    $$f(x) = \sum_{j=1}^{\infty} c_j \mathcal{X}_{J_j}$$

    where each $J_j \subset I$, and $c_j \geq 0$.

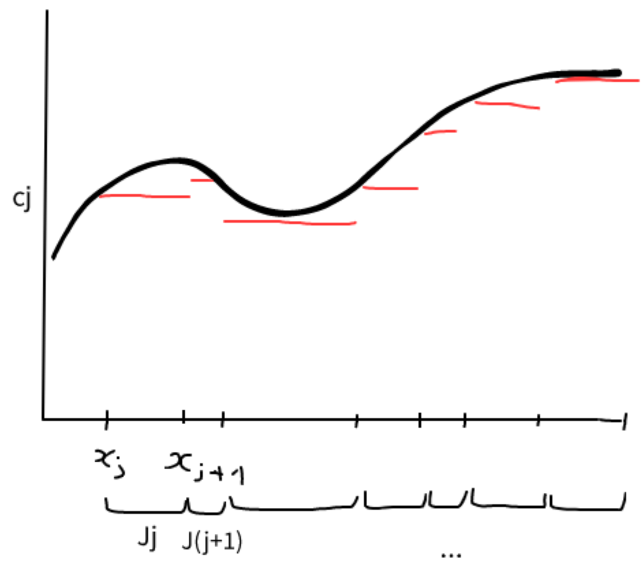
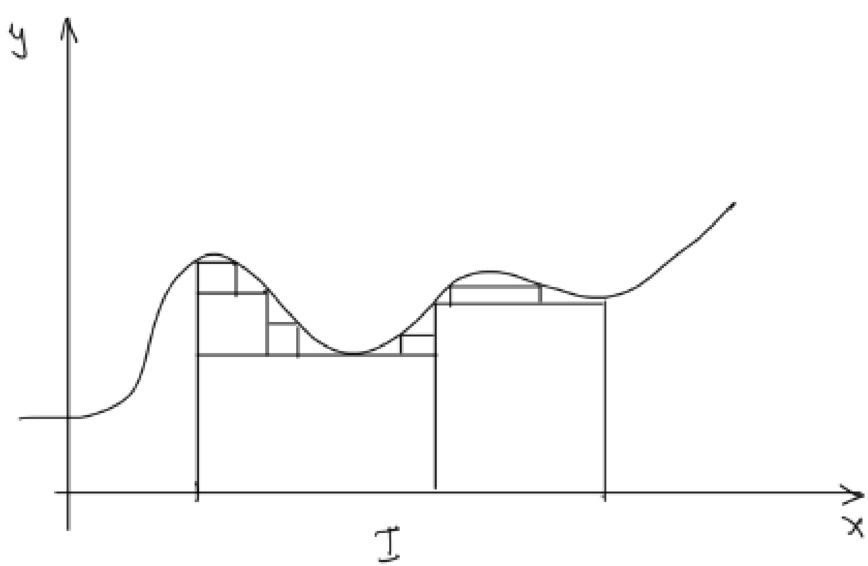


Figure 1: As the intervals J_j become smaller, we can better approximate the function.

- **How can we use step functions to find the area under a non-negative continuous function?**
  - since we can express $f(x)$ via a step function, and we can integrate step functions, it follows that:

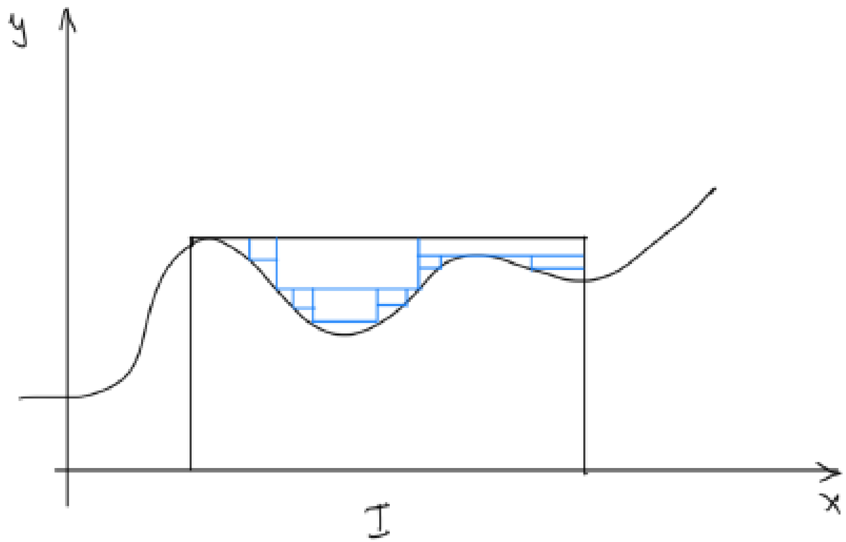
$$\int f(x) = \sum_{j=1}^{n} c_j \lambda(J_j)$$

  - diagramatically, this can be thought of as filling the curve with non-overlapping rectangles of height c_j and width $\lambda(J_j)$.



  - this idea is due to Archimedes, but he used triangles

- **What if the continuous function is sometimes negative?**
  - then we modify the argument in 2 ways:
    - we allow any c_j (positive or negative)
    - instead of considering area in terms of rectangles **under** the curve, we consider rectangles **above** the curve, and find the area under the cruve by substracting rectangle areas



  - to avoid the possibility that $\sum_{j=1}^{n} c_j \lambda(J_j)$ is conditionally convergent (and so, that adding areas of rectangles in different orders affects the value of the series), we enforce that:

$$\sum_{j=1}^{n} |c_j| \lambda(J_j) < \infty$$

4.2 Defining Lebesgue Integrable Functions

- **What is a Lebesgue Integrable function?**
  - consider a function

$$f : I \to \mathbb{R}$$

  - f is **Lebesgue Integrable** on an interval I if we can represent it as a **convergent step function**, and said step function series has a defined integral
  - more rigorously, f is **Lebesgue Integrable** if there exist:
    - $c_j \in \mathbb{R}$
    - bounded intervals $J_j \subset I, j \in \mathbb{N}$

    such that the series:

$$\sum_{j=1}^{\infty} c_j \lambda(J_j)$$

    is **absolutely convergent** (so $\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$), and for any $x \in I$ for which:

$$\sum_{j=1}^{\infty} |c_j| \mathcal{X}_{J_j}(x) < \infty$$

we have that:

$$f(x) = \sum_{j=1}^{\infty} c_j \mathcal{X}_{J_j}(x)$$

– we call the number $\int_I f$ the **integral of** f **over** I, and we denote it by:

$$\int_I f = \sum_{j=1}^{\infty} c_j \lambda(J_j)$$

4.3 Theorem: Integral of Step Function Independent of Interval

We noted before that we can represent a step function using many different intervals and constants, but said that different representations don't affect the integral for the step function. This is formalised in the following theorem:

Let c_j, d_j be real numbers. Let J_j, K_j be bounded intervals for any $j \in \mathbb{N}$. Assuming that the following series converge:

$$\sum_{j=1}^{n} |c_j| \lambda(J_j) \qquad \sum_{j=1}^{n} |d_j| \lambda(K_j)$$

if we also have that:

$$\sum_{j=1}^{n} c_j \mathcal{X}_{J_j}(x) = \sum_{j=1}^{n} d_j \mathcal{X}_{K_j}(x)$$

for any x for which:

$$\sum_{j=1}^{n} |c_j| \mathcal{X}_{J_j}(x) < \infty \qquad \sum_{j=1}^{n} |d_j| \mathcal{X}_{K_j}(x) < \infty$$

Then we have:

$$\sum_{j=1}^{n} c_j \lambda(J_j) = \sum_{j=1}^{n} d_j \lambda(K_j)$$

In other words, the integral of a step function defined in 2 distinct ways is equal. [Theorem 4.1]

Proof is quite complicated, and left in the advanced section of the notes.

4.4 Corollary: Step Functions are Lebesgue Integrable

*Let ϕ be a **step function**, such that $\phi : \mathbb{R} \to \mathbb{R}$.
Then, ϕ is **Lebesgue Integrable**.*

Proof. Recall the definition of a step function:

$$\phi(x) = \begin{cases} 0, & x < x_0 \;\; or \;\; x > x_n \\ c_j, & x \in (x_j, x_{j+1}) \end{cases}$$

Further, recall we could express the step function as:

$$\phi(x) = \sum_{j=1}^{n} c_j \mathcal{X}_{J_j}(x)$$

It is easy to see that for $j > n$, we will have $\phi(x) = 0$. Since the sum is of finitely many terms, and each interval J_j is bounded, we are guaranteed that:

$$\sum_{j=1}^{\infty} |c_j| \lambda(J_j) < \infty$$

and that for any $x \in \mathbb{R}$:

$$\sum_{j=1}^{\infty} |c_j| \mathcal{X}_{J_j}(x) < \infty$$

Lastly, since indeed

$$\phi(x) = \sum_{j=1}^{\infty} c_j \mathcal{X}_{J_j}(x)$$

it must be the case that:

$$\int \phi(x) = \sum_{j=1}^{\infty} c_j \lambda(J_j) = \sum_{j=1}^{n} c_j \lambda(J_j)$$

Thus, Lebesgue Integrability coincides with the definition of Integrability for Step Functions

□

4.5 Properties of Lebesgue Integrals

The following are all part of Theorem 4.2

4.5.1 Theorem: Linearity of Lebesgue Integral

*Let $\alpha, \beta \in \mathbb{R}$. Moreover, let f, g be **Lebesgue Integrable** functions. Then, $\alpha f + \alpha g$ is also **Lebesgue Integrable**, and:*

$$\int_I \alpha f + \beta g = \alpha \int_I f + \beta \int_I g$$

Proof: Linearity of Lebesgue Integral. From the definition of Lebesgue Integrability, we know that for the interval I, since f and g are Lebesgue Integrable, we can find c_j, d_j and $J_j, K_j \subset I$ such that:

$$\sum_{j=1}^{\infty} |c_j|\lambda(J_j) < \infty \qquad \sum_{j=1}^{\infty} |d_j|\lambda(K_j) < \infty$$

and:

$$f(x) = \sum_{j=1}^{\infty} c_j \mathcal{X}_{J_j}(x) \qquad g(x) = \sum_{j=1}^{\infty} d_j \mathcal{X}_{K_j}(x)$$

holds for all $x \in I$ where both series are absolutely convergent.

Using this, we want to show that $\alpha f + \beta g$ are Lebesgue Integrable.

The first step is to show that there exist $b_j, I_j \subset I$ such that:

$$(\alpha f + \beta g)(x) = \sum_{j=1}^{\infty} b_j \mathcal{X}_{I_j}(x)$$

Doing this is fairly easy. We can use f for even j, and g for odd j. More specifically, we can define I_j and b_j such that:

$$I_j = \begin{cases} J_{\frac{j+1}{2}}, & j \text{ is odd} \\ K_{\frac{j}{2}}, & j \text{ is even} \end{cases}$$

$$b_j = \begin{cases} \alpha c_{\frac{j+1}{2}}, & j \text{ is odd} \\ \beta d_{\frac{j}{2}}, & j \text{ is even} \end{cases}$$

We can indeed show that:

$$(\alpha f + \beta g)(x) = \sum_{j=1}^{\infty} b_j \mathcal{X}_{I_j}(x)$$

since:

$$\begin{aligned} \sum_{j=1}^{\infty} b_j \mathcal{X}_{I_j}(x) &= \sum_{j=1}^{\infty} \alpha c_j \mathcal{X}_{J_j}(x) + \sum_{j=1}^{\infty} \beta d_j \mathcal{X}_{K_j}(x) \\ &= \alpha \sum_{j=1}^{\infty} c_j \mathcal{X}_{J_j}(x) + \beta \sum_{j=1}^{\infty} d_j \mathcal{X}_{K_j}(x) \\ &= \alpha f(x) + \beta g(x) \\ &= (\alpha f + \beta g)(x) \end{aligned}$$

and this holds for any x for which $\alpha \sum_{j=1}^{\infty} c_j \mathcal{X}_{J_j}(x)$ and $\beta \sum_{j=1}^{\infty} d_j \mathcal{X}_{K_j}(x)$ are absolutely convergent, by the work at the start of the proof.

Lastly, we know that $\sum_{j=1}^{\infty} |b_j|\lambda(I_j) < \infty$ since:

$$\begin{aligned} \sum_{j=1}^{\infty} |b_j|\lambda(I_j) &= \sum_{j=1}^{\infty} |\alpha c_j|\lambda(J_j) + \sum_{j=1}^{\infty} |\beta d_j|\lambda(K_j) \\ &= |\alpha| \sum_{j=1}^{\infty} |c_j|\lambda(J_j) + |\beta| \sum_{j=1}^{\infty} |d_j|\lambda(K_j) \\ &< \infty \end{aligned}$$

by the work at the start of the proof.

Thus, by the definition of Lebesgue Integrability, $\alpha f + \beta g$ is Lebesgue Integrable.

□

4.5.2 Theorem: Positivity of Lebesgue Integral

If $f \geq 0$ on I, then $\int_I f \geq 0$.
If $f \geq g$ on I, then $\int_I f \geq \int_I g$.

Proof: Positivity of Lebesgue Integrable Functions. The proof for the first part is a bit complex, and can be found in the notes.

The second part follows directly from the first part. Since $f \geq g$, then define:

$$h = f - g$$

Then, $h \geq 0$, so from the first part:

$$\int_I h \geq 0$$

From linearity of Lebesgue Integrals:

$$\int_I h \geq 0 \implies \int_I f - \int_I g \geq 0$$

from which the result follows.

□

4.5.3 Theorem: Lebesgue Integral of Absolute Value

If f is ***integrable****, then $|f|$ is* ***integrable*** *on I, and:*

$$\left| \int_I f \right| \leq \int_I |f|$$

Proof: Absolute Value of Lebesgue Integrable Function. Again, the proof that $|f|$ is integrable is quite complex, and can be found in the notes.

Once we know $|f|$ is integrable, we note that:

$$-|f| \leq f \leq |f|$$

These are all integrable, so by positivity:

$$-\int_I |f| \leq \int_I f \leq \int_I |f|$$

Which is precisely the definition of:

$$\left|\int_I f\right| \leq \int_I |f|$$

as required.

□

4.5.4 Theorem: Lebesgue Integral of Max/Min

> *If f, g are **integrable**, then both $\max\{f, g\}$ and $\min\{f, g\}$ are integrable.*

Proof: Lebesgue Integrability of Max/Min of Functions. Firstly, we know that:

$$\max\{f, 0\} = \frac{f + |f|}{2}$$

so $\max\{f, 0\}$ is integrable by linearity and by integrability of absolute value.

But then, notice that:

$$\max\{f, g\} = \max\{f - g, 0\} + g$$

so from the above, $\max\{f, g\}$ is integrable.

But then, $\min\{f, g\} = -\max\{-f, -g\}$, so $\min\{f, g\}$ is also integrable.

□

4.5.5 Theorem: Lebesgue Integrability of Function Products

> *Let f, g be **integrable**.*
> *If one of f, g is **bounded** then the product fg is **integrable** on I.*

4.5.6 Theorem: Bounded Functions and Lebesgue Integrability

> *Let f, g be **integrable**.*
> *If $f \geq 0$ with $\int_I f = 0$ then any function h such that $0 \leq h \leq f$ on I is **integrable** on I.*

5 Exercises

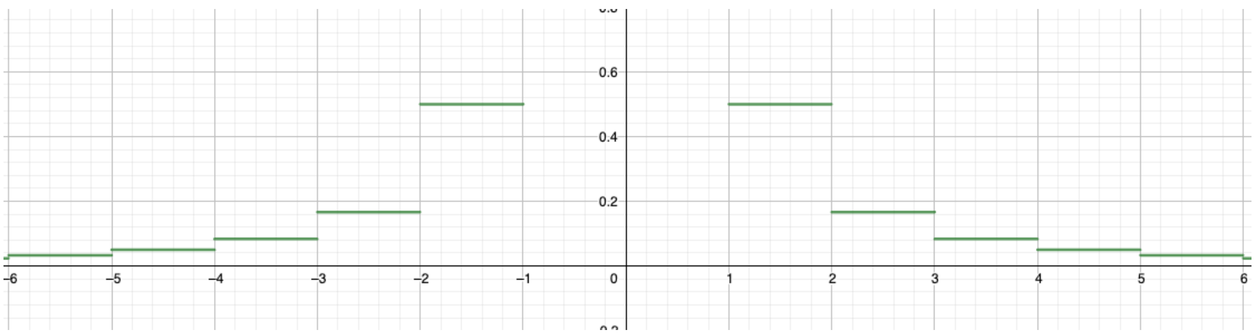
1. **Let $[x]$ denote the integer part of a number $x \in \mathbb{R}$. Define**

$$f(x) = \frac{1}{[x][x+1]}$$

for $x \geq 1$. Show that f is Lebesgue Integrable on the interval $[1, \infty)$.

We can write f using $c_j = \frac{1}{j(j+1)}$ and $\mathcal{X}_{[j,j+1)}$:

$$f(x) = \sum_{j=1}^{\infty} \frac{1}{j(j+1)} \mathcal{X}_{[j,j+1)}(x)$$



and this is true for all $x \geq 1$ (we don't need to check for the absolute convergence of $\sum_{j=1}^{\infty} \frac{1}{j(j+1)} \mathcal{X}_{[j,j+1)}(x)$ because each c_j is positive, so if the series converges, it converges absolutley).

Moreover, we know that:

$$\sum_{j=1}^{\infty} \left| \frac{1}{j(j+1)} \right| \lambda([j, j+1)) = \sum_{j=1}^{\infty} \frac{1}{j(j+1)} = 1 < \infty$$

Thus, it follows that on $[1, \infty)$, $f(x)$ is Lebesgue Integrable, and:

$$\int_I f(x) = \sum_{j=1}^{\infty} \frac{1}{j(j+1)} \lambda([j, j+1)) = \sum_{j=1}^{\infty} \frac{1}{j(j+1)} = 1$$

2. **Let I be an interval, and $E \subset I$ be a countable set. Show that $\mathcal{X}_E$ is integrable, and that** $\int_I X_E = 0$

Since E is countable, we can list each of its elements:

$$E = \{e_1, e_2, e_3, \ldots\}$$

We can then express $\mathcal{X}_E$ as an infinite series:

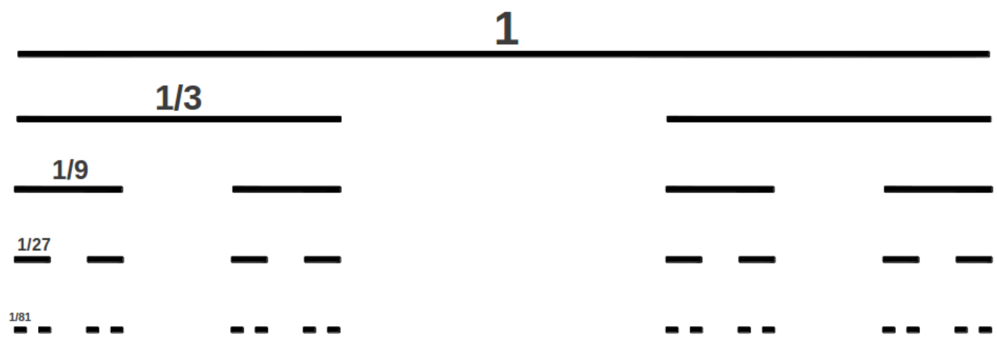
$$\mathcal{X}_E(x) = \sum_{j=1}^{\infty} \mathcal{X}_{\{e_j\}}(x)$$

where each $\mathcal{X}_{\{e_j\}}(x) = 1$ whenever $x = e_j$.

But then, from the integrability of the characteristic function:

$$\int_I \mathcal{X}_E = \int_I \sum_{j=1}^{\infty} \mathcal{X}_{\{e_j\}}(x) = \sum_{j=1}^{\infty} \lambda(\{e_j\}) = 0$$

3. **The Cantor Set C is defined as the set resulting from extracting the middle third out of $[0,1]$, and doing so iteratively**



C is uncountable. Show that $\mathcal{X}_C$ is integrable on $[0,1]$ or $\mathbb{R}$?

We would like to write:

$$\mathcal{X}_C = \sum_{j=1}^{\infty} c_j \mathcal{X}_{J_j}$$

Let F_j denote the set resulting from applying the iterative procedure j times. For example:

$$F_0 = [0,1]$$

$$F_1 = [0,1] - (1/3, 2/3) = F_0 - (1/3, 2/3)$$
$$F_2 = F_1 - (1/9, 2/9) - (7, 9, 8/9)$$

Overall, we can see that at each F_j, we are removing 2^{j-1} intervals of length 3^{-j}. In other words, F_j must be made up 2^j non-overlapping, closed intervals of length 3^{-j}.

Using all this, we notice that:

$$\mathcal{X}_{F_0} = \mathcal{X}_{[0,1]}$$
$$\mathcal{X}_{F_1} = \mathcal{X}_{[0,1]} - \mathcal{X}_{(1/3,2/3)}$$

(think that if x is in both $[0,1]$ and $(1/3, 2/3)$, $\mathcal{X}_{F_1} = 0$, and 1 otherwise, as expected)

If we continuously apply this, we get:

$$\mathcal{X}_C(x) = \mathcal{X}_{[0,1]}(x) - \sum_{j=1}^{\infty} \mathcal{X}_{J_j}(x)$$

where:

$$J_2 = (1/3, 2/3)$$

$$J_3 = (1/9, 2/9)$$
$$J_4 = (7/9, 8/9)$$

and inductively J_5, J_6, J_7, J_8 will be four open intervals of length $\frac{1}{3^3}$.

We expect that if the integral exists, then:

$$\int \mathcal{X}_C = \lambda([0,1]) - \sum_{j=1}^{\infty} \lambda(J_j)$$

If we consider the absolute convergence of the above, this depends on the absolute convergence of the series. In other words, we consider:

$$\begin{aligned}\sum_{j=1}^{\infty} |c_j|\lambda(J_j) &= \sum_{j=1}^{\infty} \lambda(J_j) \\ &= \frac{1}{3} + 2\left(\frac{1}{3^2}\right) + 2^2\left(\frac{1}{3^3}\right) + \dots \\ &= \sum_{j=1}^{\infty} 2^{j-1}\frac{1}{3^j} \\ &= \frac{1}{3}\sum_{j=1}^{\infty}\left(\frac{2}{3}\right)^{j-1}\end{aligned}$$

which is a convergent geometric series.

All of the above implies that $\mathcal{X}_C$ is integrable, and:

$$\int \mathcal{X}_C = 1 - \frac{1}{3}\sum_{j=1}^{\infty}\left(\frac{2}{3}\right)^{j-1} = 1 - \frac{1}{3}(3) = 0$$

and this is true for any $x \in \mathbb{R}$

If $\int \mathcal{X}_E$ is 0, then E is said to be a set of **measure zero**. Thus, all countable sets have measure zero, and the Cantor Set is an example of an uncountable set with measure zero.

4. **Let $f(x) = [x]$ for all $x \in \mathbb{R}$. Compute the following integrals:**

   (a)

$$\int_{(0,5)} f$$

   It is easy to see that $\forall x \in (0,5)$:

$$f(x) = \sum_{i=1}^{4} i\mathcal{X}_{[i,i+1)}(x)$$

   (we don't need to consider $[0,1)$, since in that case $\mathcal{X}_{[i,i+1)}(x)$ is just 0).

   We can then compute the integral by using linearity:

$$\int_{(0,5)} f = \int \sum_{i=1}^{4} i\mathcal{X}_{[i,i+1)}(x) = \sum_{i=1}^{4} i\lambda([i,i+1)) = \sum_{i=1}^{4} i = 10$$

(b)

$$\int_{(-\frac{7}{3},\frac{12}{5}))} f$$

Notice that:

$$-\frac{7}{3} = -2.\dot{3}$$

and

$$\frac{12}{5} = 2.4$$

So we can express f as (using $[x]$ as the floor function):

$$f(x) = -3 \times \mathcal{X}_{(-\frac{7}{3},-2)} + \sum_{i=-2}^{1} i\mathcal{X}_{(i,i+1]}(x) + 2 \times \mathcal{X}_{(2,\frac{12}{5})}$$

So:

$$\int_{(-\frac{7}{3},\frac{12}{5}))} f = -3\left(\frac{1}{3}\right) + (-2) + (-1) + 0 + 1 + 2\left(\frac{2}{5}\right) = -\frac{11}{5}$$

5. **Show that if $n \in \mathbb{Z}$ and $f(x) = [nx]^2$, for all $x \in \mathbb{R}$ then:**

$$\int_{(0,1)} f = \frac{1}{n}\sum_{j=1}^{n-1} j^2$$

We want to express f in the form:

$$f(x) = \sum_{j=1}^{\infty} c_j \mathcal{X}_{J_j}$$

Lets consider how the function looks like for different values of n on the interval $(0, 1)$:

- if $n = 0$, $f(x) = 0$
- if $n = 1$, $f(x) = [x]^2 = 0$
- if $n = 2$, $f(x) = [2x]^2$ so notice that we have:

$$f(x) = \begin{cases} 0, & x < \frac{1}{2} \\ 1, & x \geq \frac{1}{2} \end{cases}$$

- if $n = 3$, $f(x) = [3x]^2$ so notice that we have:

$$f(x) = \begin{cases} 0, & x < \frac{1}{3} \\ 1, & \frac{1}{3} \leq x < \frac{2}{3} \\ 4, & \frac{2}{3} \geq x \end{cases}$$

This means that, if we consider $n > 0$, we must have:

$$f(x) = \sum_{j=1}^{n-1} j^2 \times \mathcal{X}_{\left(\frac{j}{n},\frac{j+1}{n}\right]}$$

This is a finite sum, so we can compute the integral directly:

$$\int_{(0,1)} f = \sum_{j=1}^{n-1} j^2 \times \lambda\left(\left(\frac{j}{n}, \frac{j+1}{n}\right]\right) = \frac{1}{n}\sum_{j=1}^{n-1} j^2$$

3. Let $f(x) = \frac{1}{[x]^2}$ for all $x \geq 1$. Show that f is integrable on the interval $[1, \infty)$ and

$$\int_{[1,\infty)} f = \sum_{j=1}^{\infty} \frac{1}{j^2}.$$

Solution: We have

$$f(x) = \sum_{j=1}^{\infty} \frac{1}{j^2}\chi_{[j,j+1)}(x), \qquad \forall x \geq 1.$$

Since

$$\sum_{j=1}^{\infty} \left|\frac{1}{j^2}\right| \lambda([j, j+1)) = \sum_{j=1}^{\infty} \frac{1}{j^2}\lambda([j, j+1)) = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$$

we see that f is integrable on $[1.\infty)$ and its integral is

$$\int_{[1,\infty)} f = \sum_{j=1}^{\infty} \frac{1}{j^2}.$$

6 Workshop

This workshop covered an auxiliary topic: Uniform Continuity

1. **Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.**
   We know that it is continuous at a for all $a \in \mathbb{R}$.
   So, for every a, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.
   For $a > 1$ and $\varepsilon = 1$, find the best possible δ. Is this best possible δ independent of a?
   As a hint, draw the graph of the function, and include the horizontal lines $y = a^2 \pm 1$.

   > *I still have no idea what "best possible" δ means. As a course that takes marks off for failing to mention a theorem when justifying that a function is continuous, I find this hilarious.*

2. **Consider the same function, but now on $[0, 1]$. Prove that $\forall\varepsilon > 0$ if we take $\delta = \frac{\varepsilon}{2}$ we have that $|x - a| < \delta$ (where $x, a \in [0, 1]$) implies $|f(x) - f(a)| < \varepsilon$. In this case, the "best" δ can be taken to be independent of a.**

This works. Assume that $|x-a| < \frac{\varepsilon}{2}$. Then:

$$\begin{aligned} |f(x)-f(a)| &= |x^2-a^2| \\ &= |x-a||x+a| \\ &< \delta(|x|+|a|) \\ &= 2\delta \\ &= \varepsilon \end{aligned}$$

> *Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a function. f is* ***uniformly continuous*** *on I if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $x, y \in I$ and $|x-y| < \delta$ then $|f(x)-f(y)| < \varepsilon$*
> ***Uniform continuity*** *only makes sense when f is already continuous.*

3. **Let $f(x) = \frac{1}{x}$ on $(0, \infty)$. Is f uniformly continuous?**

> *If we negate the statement of uniform continuity, we get that f is not* ***uniformly continuous*** *if $\exists \varepsilon > 0$ such that $\forall \delta > 0$ we can find $x, y \in I$ such that $|x-y| < \delta$ but $|f(x)-f(y)| \geq \varepsilon$.*

This is false. $\forall \delta > 0$, pick $x \in (0,1)$ such that $x < \delta$, and define $y = \frac{x}{2}$. Then:

$$|x-y| = \left|x - \frac{x}{2}\right| = \left|\frac{x}{2}\right| < \frac{\delta}{2} < \delta$$

Now, consider:

$$|f(x)-f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{x} - \frac{2}{x}\right| = \frac{1}{x}$$

Now, since $x \in (0,1)$, then $\frac{1}{x} > 1$. Thus, if we set $\varepsilon = 1$, we indeed have that:

$$|f(x)-f(y)| \geq \varepsilon$$

and so, f isn't uniformly continuous.

> *In the solutions, they use sequences $x_n = \frac{1}{n}, y_n = \frac{1}{n+1}$, to show that $|f(x_n)-f(y_n)| = 1$ so that no matter the δ, $|f(x)-f(y)|$ won't be smaller than ε. However, the involvement of sequences makes me uneasy, since we haven't yet defined uniform continuity in terms of sequences.*

4. **Let $f(x) = \frac{1}{x}$ on $[a, \infty)$, where $a > 0$. Is f uniformly continuous?**

In this case, it works. Let $\delta > 0$, and assume that $x, y \in [a, \infty)$ such that:

$$|x-y| < \delta$$

Then:

$$\begin{aligned}|f(x)-f(y)| &= \left|\frac{1}{x}-\frac{1}{y}\right| \\ &= \left|\frac{y-x}{xy}\right| \\ &< \left|\frac{\delta}{xy}\right|\end{aligned}$$

Now, since $x, y \in [a, \infty)$, $xy \geq a^2 \implies \frac{1}{xy} \leq \frac{1}{a^2}$ so:

$$|f(x)-f(y)| < \frac{\delta}{a^2}$$

Thus, if $\forall \varepsilon > 0$ we set $\delta = a^{-2}\varepsilon$ then:

$$|x-y| < \delta \implies |f(x)-f(y)| < a^2\delta = \varepsilon$$

so f will be uniformly continuous.

5. **Let I be an open interval in $\mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ is differentiable, and its derivative f' is bounded on I. Prove that f is uniformly continuous on I.**

Let $x, y \in I$. Then, $[y, x]$ defines a closed interval, over which f is continuous, and (y, x) is an open interval over which f is differentiable. Then, $\exists c \in (y, x)$ such that, by the Mean Value Theorem:

$$f'(c) = \frac{f(x)-f(y)}{x-y} \implies |f(x)-f(y)| = |f'(c)||x-y|$$

Since the derivative is bounded, $\exists M$ such that:

$$|f(x)-f(y)| \leq M|x-y|$$

Then, if $\forall \varepsilon > 0$ we have $\delta = \frac{\varepsilon}{M}$ if $|x-y| < \delta$ we get that:

$$|f(x)-f(y)| < \varepsilon$$

so f will be uniformly continuous, as required.

6. **Show that $f(x) = \sin(x)$ is uniformly continuous on $\mathbb{R}$.**

In the solutions they simply quote the result above, which is fine, but giving all the details is more fun.

Notice, $\sin(x)$ is continuous and idfferentiable on $\mathbb{R}$, so the MVT applies on any interval $[y, x]$. Indeed, by MVT $\exists c \in (y, x)$ such that:

$$f'(c) = \frac{f(x)-f(y)}{x-y} \implies |f(x)-f(y)| = |f'(c)||x-y|$$

Since $f'(x) = \cos(x)$ we know that $|f'(c)| \leq 1$ so:

$$|\sin(x)-\sin(y)| \leq |x-y|$$

Then, if $|x-y| < \delta = \varepsilon$ we get that:

$$|\sin(x)-\sin(y)| < \varepsilon$$

so $\sin(x)$ is uniformly continuous on $\mathbb{R}$.

7. **Let I be an interval in $\mathbb{R}$. Prove that a continuous function $f : I \to \mathbb{R}$ is uniformly continuous on I *if and only if* whenever $s_n, t_n \in I$ are such that $|s_n - t_n| \to 0$, then $|f(s_n) - f(t_n)| \to 0$**

> *For the first part of the proof, we give identical proofs. For the second part, I use contradiction, whilst the solutions give direct proof.*

① **Uniform Continuity Implies Sequence Definition**

Assume that f is uniformly continuous. Then, $\forall \varepsilon > 0, \exists \delta > 0$ such that if $x, y \in I$ then:

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Now, consider sequences $s_n, t_n \in I$ such that:

$$|s_n - t_n| \to 0$$

By definition of convergence, this means that $\forall \delta > 0$ we can find a $N \in \mathbb{N}$ such that if $n \geq N$ then:

$$|s_n - t_n| < \delta$$

Hence, uniform continuity, and so we must have that:

$$|f(s_n) - f(t_n)| < \varepsilon \implies |f(s_n) - f(t_n)| \to 0$$

as required.

② **Sequence Definition Implies Uniform Continuity**

The solutions go by direct proof, and show that if f is not uniformly continuous, then the sequence definition doesn't follow.

Indeed, assume that f is continuous, but not uniformly continuous. Since f is not uniformly continuous, $\exists \varepsilon$ such that $\forall \delta = \frac{1}{n}$ we have that:

$$|s_n - t_n| < \delta \implies |f(s_n) - f(t_n)| \geq \varepsilon$$

But then,

$$|s_n - t_n| \to 0 \implies |f(s_n) - f(t_n)| \not\to 0$$

as required.

I proceeded by contradiction. Assume we have sequences $s_n, t_n \in I$ such that:

$$|s_n - t_n| \to 0 \implies |f(s_n) - f(t_n)| \to 0$$

but f is not uniformly continuous.

Then, $\exists \varepsilon > 0$ such that $\forall \delta > 0$ if $|x - y| < \delta$ then $|f(x) - f(y)| \geq \varepsilon$.

Now, this means that we can find $\varepsilon > 0$ such that $\forall \delta > 0$:

$$|s_n - t_n| < \delta \implies |f(s_n) - f(t_n)| \geq \varepsilon$$

which in particular means that:

$$|s_n - t_n| \to 0 \implies |f(s_n) - f(t_n)| \not\to 0$$

since $\varepsilon > 0$ (here we could have also used $\delta = \frac{1}{n}$ in the proof). This is a contradiction, and so, sequence definition implies uniform convergence.

> *Suppose $f : [a, b] \to \mathbb{R}$ is* ***continuous****. Then it is* ***uniformly continuous****.*
> *That is, any* ***continuous*** *function defined over a* ***closed, bounded*** *interval is automatically* ***uniformly continuous****.*

8. **Prove this theorem by arguing by contradiction, using the previous question, and the Bolzano-Weierstrass theorem.**

   Assume that this is false: assume that f is continuous over a closed interval, but that f is not uniformly continuous over said interval.

   Since f is not uniformly continuous, this means that, by the question above, there are sequences $s_n, t_n \in I$ such that:

   $$|s_n - t_n| \to 0 \implies |f(s_n) - f(t_n)| \not\to 0$$

   Now, since s_n, t_n are sequences over I, in particular they are bounded, so by Bolzano-Weierstrass, it follows that they have convergent subsequences, which converge on the interval:

   $$s_{n_k} \to s \in [a, b]$$

   $$t_{n_k} \to t \in [a, b]$$

   Now:

   $$|s_n - t_n| \to 0 \implies |s_{n_k} - t_{n_k}| \to 0$$

   In particular, this means that s_{n_k} and t_{n_k} must converge to the same value, and so $s = t$.

   Now, by continuity we have that:

   $$f(s_{n_k}) \to s \qquad f(t_{n_k}) \to t = s$$

   Hence, this means that:

   $$|f(s_{n_k}) - f(t_{n_k})| \to 0$$

   which is a contradiction.

   Hence, if f is continuous ove r abounded interval, f is uniformly continuous.

9. **Find an example of an $f : (0, 1) \to \mathbb{R}$ which is continuous, but not uniformly continuous. Where exactly did we use the fact that $[a, b]$ was a closed and bounded interval in the proof of the theorem?**

   We already saw that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$. We use Bolzano-Weierstrass because it allows us to find a subsequence which converges on a point inside the interval. The issue might arise if there is convergence at a or b.