Honours Analysis - Week 5 - Power Series

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October 2021

Contents

Very extensive notes on power series
Notes by student from previous year

1 Defining Power Series

- **What is a power series?**
  - an infinite series of the form:
  $$\sum_{n=0}^{\infty} a_n(x-c)^n$$
  where:
    * a_n are the **coefficients** of the power series
    * c is the **centre** of the power series
- **What are key questions regarding power series?**
  - when is a power series **convergent** (if at all)?
  - if the power series is convergent, is its convergent function **differentiable**? How can we compute the derivative?
  - if the power series is convergent, is its convergent function **integrable**? How can we compute the integral?
  - if the power series is convergent, is its convergent function **continuous**?

2 The Radius of Convergence

2.1 Defining the Radius of Convergence

- **What is the radius of convergence?**
  - let
  $$\sum_{n=0}^{\infty} a_n(x-c)^n$$
  be a power series
  - its **radius of convergence** is:
  $$R = \sup\{r \geq 0, \quad a_n r^n \textit{ is bounded}\}$$
- **What values can the radius of convergence take?**
  - $\boldsymbol{R=0}$: if $a_n r^n$ is never bounded, no matter the value of r
  - $\boldsymbol{R} \in \mathbb{R}^+$: if $a_n r^n$ is always bounded for some $r > 0$
  - $\boldsymbol{R=\infty}$: if $a_n r^n$ is always bounded, no matter the value of r

2.2 Importance of the Radius of Convergence

Intuitively, the radius of convergence is a real number which tells us for which values of x a given power series is convergent. We formalise this in the next theorem.

Consider the power series:

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

Then, let R is the radius of convergence of the power series:

- *if $|x-c| < R$, the power series* ***converges absolutely***
- *if $|x-c| > R$, the power series* ***diverges***
- *if $|x-c| = R$, the power series can* ***converge*** *or* ***diverge***

In other words:

- *if $R = 0$, the power series can only converge at $x = c$*
- *if $R = \infty$, the power series converges $\forall x \in \mathbb{R}$*

[Theorem 3.1]

Proof: Power Series and Radius of Convergence. Let:

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

be a power series with radius of convergence R. Here we consider only $0 < R < \infty$, but the cases $R = 0$ or $R = \infty$ are easy to check by inspection.

Firstly, assume that $|x-c| < R$. We can then find some $\rho \in \mathbb{R}$ such that:

$$|x-c| < \rho < R$$

Moreover, by the definition of radius of convergence:

$$\rho \in \{r \geq 0, \quad a_n r^n \text{ is bounded}\}$$

so in particular it follows that the sequence $(a_n \rho^n)$ is bounded, say:

$$\forall n \in \mathbb{N} \qquad |a_n \rho^n| \leq M$$

for some $M \in \mathbb{R}$. But then, consider the sequence $|a_n||x-c|^n$:

$$\begin{aligned} |a_n||x-c|^n &= |a_n|\rho^n \left(\frac{|x-c|}{\rho}\right)^n, \qquad \left(\textit{multiplying by } \frac{\rho^n}{\rho^n}\right) \\ &\leq M \left(\frac{|x-c|}{\rho}\right)^n \end{aligned}$$

But then, if we take summations of both sides of the inequality:

$$\sum_{n=0}^{\infty} |a_n||x-c|^n \leq \sum_{n=0}^{\infty} M \left(\frac{|x-c|}{\rho} \right)^n$$

The RHS is a geometric series, with common ratio $\frac{|x-c|}{\rho} < 1$ (by construction), so it must converge. Thus, by the comparison test $\sum_{n=0}^{\infty} |a_n||x-c|^n$ converges as well, and so, our original power series converges absolutely.

Now assume that $|x-c| > R$. We can do a similar treatment as above, selecting some μ such that $|x-c| > \mu > R$, which implies that $a_n \mu^n$ is unbounded from the definition of the Radius of Convergence. There are three cases: each a_n In other words, $|a_n \mu^n| \geq K$ for some $K \in \mathbb{R}$. Finally:

$$\begin{aligned} |a_n||x-c|^n &= |a_n|\mu^n \left(\frac{|x-c|}{\mu} \right)^n, \qquad \left(\textit{multiplying by } \frac{\mu^n}{\mu^n} \right) \\ &\geq K \left(\frac{|x-c|}{\rho} \right)^n \end{aligned}$$

Since $|x-c| > \mu$, the RHS will be ever increasing, which implies that the term $a_n(x-c)^n$ will be unbounded, so its series can't converge.

□

2.3 Power Series at Limits of Radius of Convergence

We have said that if $|x-c| = R$, we don't know whether a power series converges or diverges. We illustrate by using an example.

2.3.1 Interval of Convergence Doesn't Contain Radius of Convergence

Consider the power series:

$$\sum_{n=0}^{\infty} x^n$$

It is quite easy to see that for convergence we required $|x| < 1$, as this is a geometric series. This means that, since $a_n = 1$:

$$1 = \sup\{r \geq 0, \quad r^n \textit{ is bounded}\}$$

(again, easy to see, as r^n is bounded if and only if $r \leq 1$). If we let $x = 1$, $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 1$ which diverges. Similarly, if $x = -1$ then $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (-1)^n$ which again diverges. Thus, $\sum_{n=0}^{\infty} x^n$ converges on the interval $(-1, 1)$.

2.3.2 Interval of Convergence Partially Contains Radius of Convergence

Consider the power series:

$$\sum_{n=0}^{\infty} \frac{x^n}{n}$$

Using $a_n = \frac{1}{n}$:

$$R = \sup\{r \geq 0, \quad \frac{r^n}{n} \textit{ is bounded}\}$$

Informally, we require $r \leq 1$, as otherwise r^n will grow exponentially, which is "faster" than the polynomial growth of n. Thus, again, $R = 1$. If we let $x = 1$, $\sum_{n=0}^{\infty} \frac{x^n}{n} = \sum_{n=0}^{\infty} \frac{1}{n}$ which diverges (Harmonic Series). Similarly, if $x = -1$ then $\sum_{n=0}^{\infty} \frac{x^n}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ which converges (Alternating Series Test). Thus, $\sum_{n=0}^{\infty} \frac{x^n}{n}$ converges on the interval $[-1, 1)$.

2.3.3 Interval of Convergence Contains Radius of Convergence

Consider the power series:

$$\sum_{n=0}^{\infty} \frac{x^n}{n^2}$$

Using $a_n = \frac{1}{n^2}$:

$$R = \sup\{r \geq 0, \quad \frac{r^n}{n^2} \text{ is bounded}\}$$

Informally, we require $r \leq 1$, as otherwise r^n will grow exponentially, which is "faster" than the polynomial growth of n^2. Thus, again, $R = 1$. If we let $x = 1$, $\sum_{n=0}^{\infty} \frac{x^n}{n^2} = \sum_{n=0}^{\infty} \frac{1}{n^2}$ which converges (p-Series Test). Similarly, if $x = -1$ then $\sum_{n=0}^{\infty} \frac{x^n}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$ which converges (Alternating Series Test). Thus, $\sum_{n=0}^{\infty} \frac{x^n}{n}$ converges on the interval $[-1, 1]$.

> *Indeed, we had 3 power series, with the exact same radius of convergence (can be computed by the ratio test), but their* ***interval of convergence*** *varied. We could've also found the radii of convergence by using the Ratio Test, but this was an alternative way, direct from the definition.*

2.4 Computing the Radius of Convergence

> *Let:*
>
> $$\sum_{n=0}^{\infty} a_n (x-c)^n$$
>
> *be a power series with radius of convergence R. Then:*
>
> - *if* $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ *converges:*
>
> $$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = R$$
>
> - *if* $\lim_{n\to\infty} |a_n|^{-\frac{1}{n}}$ *converges:*
>
> $$\lim_{n\to\infty} |a_n|^{-\frac{1}{n}} = R$$
>
> *In general, it is a fact that:*
>
> $$R = \lim_{n\to\infty} \inf_{k\geq n} |a_k|^{-\frac{1}{k}}$$
>
> *but this is less convenient to use than the formulations above.[Example 3.2]*

Proof 1: Using Convergence Tests. The above theorem is just a statement of the ratio and root tests for convergence of series.

By the ratio test, $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges if and only if:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n}\right| < 1$$

If we compute the limit:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n}\right| &= \lim_{n\to\infty}\left|\frac{a_{n+1}(x-c)}{a_n}\right| \\ &= |x-c|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\end{aligned}$$

If we let $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$, then for convergence we require:

$$L|x-c| < 1 \implies |x-c| < \frac{1}{L}$$

(assuming $L \neq 0$; if $L = 0$, then we'd get that the power series converges for any $x \in \mathbb{R}$)

In other words:

$$R = \frac{1}{L} \implies \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = R$$

by the properties of limits.

A similar procedure can be done with the root test, which states that a_n converges if:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$$

□

Proof 2: First Principles. Let $\rho = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$. We want to show that, if:

$$R = \sup\{r \geq 0, \quad a_n r^n \textit{ is bounded}\}$$

then $R = \rho$.

From $\rho = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$, it follows that $\forall\varepsilon > 0$, there exists some $N \geq \mathbb{N}$, such that if $n \geq N$:

$$\left|\left|\frac{a_n}{a_{n+1}}\right| - \rho\right| < \varepsilon$$

from which it follows that:

$$|a_{n+1}|(\rho-\varepsilon) < |a_n| < |a_{n+1}|(\rho+\varepsilon)$$

If we consider $|a_{n+1}|(\rho-\varepsilon) < |a_n|$, we can multiply through by $(r-\varepsilon)^n$, such that:

$$|a_{n+1}|(\rho-\varepsilon)^{n+1} < |a_n|(\rho-\varepsilon)^n$$

But then, it follows that the sequence $|a_n|(\rho - \varepsilon)^n$ is a **decreasing** sequence. Moreover, it must also be bounded. From this it follows that it must be the case that:

$$R \geq \rho - \varepsilon$$

since $r = \rho - \varepsilon$ means that $a_n r^n$ is bounded.

Similarly, if we consider $|a_{n+1}|(\rho + \varepsilon) > |a_n|$, we can multiply through by $(r - \varepsilon)^n$, such that:

$$|a_{n+1}|(\rho + \varepsilon)^{n+1} > |a_n|(\rho + \varepsilon)^n$$

But then, it follows that the sequence $|a_n|(\rho + \varepsilon)^n$ is an **increasing** sequence. In particular, it must be bounded from below, such that

$$|a_n|(\rho + \varepsilon)^n \geq M$$

for all $n \in \mathbb{N}$. This does not really help us: it is an increasing sequence, but it could still be bounded from above. Thus, consider:

$$|a_n|(\rho + 2\varepsilon)^n$$

In particular, notice that:

$$\begin{aligned} |a_n|(\rho + 2\varepsilon)^n &= |a_n|(\rho + \varepsilon)^n \left(\frac{\rho + 2\varepsilon}{\rho + \varepsilon}\right)^n \\ &\geq M \left(\frac{\rho + 2\varepsilon}{\rho + \varepsilon}\right)^n \end{aligned}$$

But then, $\frac{\rho+2\varepsilon}{\rho+\varepsilon} > 1$, so $M\left(\frac{\rho+2\varepsilon}{\rho+\varepsilon}\right)^n$ is an unbounded term, and thus, the sequence $|a_n|(\rho + 2\varepsilon)^n$ must also be unbounded. In particular, it thus means that:

$$\rho + 2\varepsilon \notin \{r \geq 0, \quad a_n r^n \textit{ is bounded}\}$$

and in particular, we must have $R \leq \rho + 2\varepsilon$.

But then it follows that:

$$\rho + \varepsilon \leq R \leq \rho + 2\varepsilon$$

But as ε is arbitrarily small, it must be the case that:

$$R = \rho = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

as required.

This proof comes from the videos. For the proof from the notes:

Let $\rho = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$. The for any $\varepsilon > 0$ there exists N such that for all $n \geq N$ we have

$$\left|\frac{a_n}{a_{n+1}}\right| \leq \rho + \varepsilon.$$

By induction

$$\left|\frac{a_n}{a_{n+k}}\right| \leq (\rho + \varepsilon)^k.$$

Multiplying by $|a_{n+k}|r^n$ we have

$$|a_n|r^n \leq |a_{n+k}|r^{n+k}\left(\frac{\rho+\varepsilon}{r}\right)^k.$$

If $(|a_n|r^n)$ is bounded (say by M) this implies

$$|a_n|r^n \leq M\left(\frac{\rho+\varepsilon}{r}\right)^k \to 0, \quad \text{for } r > \rho + \varepsilon,$$

as $k \to \infty$ which would imply that $a_n = 0$ for all $n \geq N$. This is a contradiction and hence $(|a_n|r^n)$ cannot be bounded for $r > \rho+\varepsilon$. From this the radius of convergence is $\leq \rho$ (as $\varepsilon > 0$ can be arbitrary small).

On the other hand we also have for all $n \geq N$:

$$\left|\frac{a_n}{a_{n+1}}\right| \geq \rho - \varepsilon.$$

From this

$$|a_n|(\rho-\varepsilon)^n \geq |a_{n+1}|(\rho-\varepsilon)^{n+1} \geq |a_{n+2}|(\rho-\varepsilon)^{n+2} \geq \dots.$$

Hence $|a_n|(\rho-\varepsilon)^n$ is bounded and the radius of convergence is therefore at least $\rho - \varepsilon$. Combining these two results we see that the radius of convergence is exactly ρ.

We follow the similar logic. Let $\rho = \lim_{n\to\infty} \frac{1}{|a_n|^{1/n}}$. For any $\varepsilon > 0$ there is N such that for all $n \geq N$ we have

$$|a_n|^{1/n}(\rho - \varepsilon) \leq 1$$

or equivalently

$$|a_n|(\rho-\varepsilon)^n \leq 1.$$

So the radius of convergence is at least $\rho - \varepsilon$.

Conversely, we have

$$1 \leq |a_n|^{1/n}(\rho+\varepsilon)$$

and therefore

$$\left(\frac{\rho+2\varepsilon}{\rho+\varepsilon}\right)^n \leq |a_n|(\rho+2\varepsilon)^n.$$

As the left hand side goes to ∞ as $n \to \infty$ we conclude that $|a_n|(\rho+2\varepsilon)^n$ is unbounded. Thus the radius of convergence must be less than $\rho+2\varepsilon$. Again combining these two results we see that the radius of convergence is exactly ρ.

□

3 Continuity, Differentiability and Integrability of Power Series

3.1 Theorem: Continuity of Power Series

Let $R > 0$, and $0 < r < R$. Consider the power series:

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

This series ***converges absolutely*** *and* ***uniformly*** *for $x \in [c-r, c+r]$ to a function $f(x)$.*
Moreover, f is a ***continuous*** *function for $x \in (c-R, c+R)$. [Theorem 3.2]*

Proof: Continuity of Power Series. We already showed in (2.2) that the power series will be absolutely convergent if $|x-c| < R$, so in particular it is absolutely convergent if $|x-c| \leq r$.

For uniform convergence we employ the Weirstress M-Test. Again from (2.2), recall that we showed that if $r < \rho < R$, then:

$$|a_n||x-c|^n \leq M \left(\frac{|x-c|}{\rho} \right)^n$$

From which it follows that:

$$|a_n||x-c|^n \leq M \left(\frac{r}{\rho} \right)^n$$

If we define $M_n = M \left(\frac{r}{\rho} \right)^n$, we notice that $r < \rho \implies \frac{r}{\rho} < 1$, and so it follows that $\sum M_n$ converges, as it is a geometric series. By the Weierstrass M-Test, our power series must converge uniformly on $[c-r, c+r]$.

Since the power series converges uniformly, and each $a_n(x-c)^n$ is continuous on $\mathbb{R}$, it then follows that $f(x)$ must also be continuous on $(c-R, c+R)$ (since we picked arbitrary r).

□

3.2 Lemma: Conservation of Radius of Convergence Under Elementwise Differentiation

The power series:

$$\sum_{n=1}^{\infty} a_n (x-c)^n$$

and

$$\sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$$

have the ***same radius of convergence****. [Lemma 3.1]*

Proof: Radius of Convergence - Elementwise Differentiation. We first notice that since $(x-c)$ is independent of n, we can simply consider the series:

$$\sum_{n=1}^{\infty} a_n(x-c)^n$$

and

$$\sum_{n=1}^{\infty} na_n(x-c)^n$$

(we have added a factor of $(x-c)$ to the second power series)

Let R_1 and R_2 be the respective radii of convergence for the series above. Now, it is easy to see that:

$$|a_n r^n| \leq |na_n r^n|$$

for any $n \in \mathbb{N}$. But then, it follows (intuitively) from the definition of the radius of convergence that $R_2 \leq R_1$ (the terms of the second series are "bigger", there's in principle a smaller chance that it'll converge).

We proceed by contradiction, assuming that $R_2 < R_1$. If this is the case, then we can find ρ, r such that:

$$R_2 < \rho < r < R_1$$

We now consider the values of $|na_n\rho^n|$:

$$\begin{aligned}|na_n\rho^n| &= n|a_n|\rho^n \\ &= n|a_n|\rho^n \times \left(\frac{r^n}{r^n}\right) \\ &= |a_n r^n| \times n\left(\frac{\rho}{r}\right)^n\end{aligned}$$

But now we notice that:

- since $r < R_1$, from the definition of the radius of convergence $a_n r^n$ is bounded, so there exists some M such that:

$$|a_n r^n| \leq M$$

- since $\frac{\rho}{r} < 1$, it follows that $n\left(\frac{\rho}{r}\right)^n \to 0$, so in particular $n\left(\frac{\rho}{r}\right)^n$ is eventually bounded

In other words, we have shown that $na_n\rho^n$ is bounded, which contradicts the fact that R_2 is the radius of convergence of the second series, since we have found $\rho > R_2$ such that $na_n\rho^n$ is bounded. In other words, it can't be the case that $R_2 < R_1$, so it follows that $R_1 = R_2$ as required.

□

3.3 Theorem: Differentiability of Power Series

Consider the power series:

$$\sum_{n=1}^{\infty} a_n (x-c)^n$$

with radius of convergence R. If $|x-c| < R$, the power series converges:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

*Then, $f(x)$ is **infinitely differentiable** on $x \in (c-R, c+R)$($|x-c| < R$), and for any such x:*

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}$$

*This power series also **converges uniformly and absolutely** on $[c-r, c+r]$ for some $0 < r < R$ (so its radius of convergence is also R). Moreover:*

$$a_n = \frac{f^{(n)}(c)}{n!}$$

[Theorem 3.3]

Proof: Differentiability of Power Series. We recall from last weak the Theorem on Differentiability for Uniformly Convergent Series:

*Suppose that E is an **open, bounded** interval. If:*

- *each f_n is differentiable on E*
- *$\sum_{k=1}^{\infty} f_k(x_0)$ converges for some $x_0 \in E$*
- *$g = \sum_{k=1}^{\infty} f'_k$ converges uniformly on E*

*then $f = \sum_{k=1}^{\infty} f_k$ converges **uniformly** on E, and is **differentiable**, such that for any $x \in E$:*

$$f'(x) = \left(\sum_{k=1}^{\infty} f_k(x)\right)' = \sum_{k=1}^{\infty} f'_k(x) = g(x)$$

Notice that:

- $a_n(x-c)^n$ is differentiable on $\mathbb{R}$, so it is differentiable on $(c-R, c+R)$
- $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges at $x = c, x \in (c-R, c+R)$
- in the previous lemma, we showed that $\sum_{n=0}^{\infty} a_n(x-c)^{n-1}$ has the same radius of convergence as $\sum_{n=0}^{\infty} a_n(x-c)^n$, so in particular it converges uniformly and absolutely given $x \in (c-R, c+R)$

Thus, we can apply the theorem, and it follows that:

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}$$

Clearly, we can repeatedly apply this, as we will be differentiating an nth degree polynomial, so it must be infinitely differentiable.

To prove the second part, we notice that (taking $0^0 = 1$):

$$f(c) = a_0$$

$$f'(c) = a_1$$

By repeatedly differentiating the power series, it is easy to see that indeed:

$$f^{(n)}(c) = a_n n!$$

from which the result follows.

□

3.4 Theorem: Integrability of Power Series

Consider the power series:

$$\sum_{n=1}^{\infty} a_n (x-c)^n$$

with radius of convergence R. If $|x-c| < R$, the power series converges:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

Then, $f(x)$ is Riemann Integrable $\forall a, b \in (c-R, c+R) (|x-c| < R)$, such that:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where:

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$$

This power series also ***converges uniformly and absolutely*** *on $[c-r, c+r]$ for some $0 < r < R$ (so its radius of convergence is also R).*

4 Analytic Functions

4.1 Defining Analytic Functions

- **What is an analytic function?**
  - consider a function f for $x \in (c-r, c+r)$
  - f is **analytic** if there exists a **power series** which converges to f for $x \in (c-r, c+r)$
  - an **analytic** function thus satisfies:
    * infinite differentiability
    * having a power series with terms:

$$a_n = \frac{f^{(n)}(c)}{n!}$$

4.2 Taylor's Theorem

Let f be a function which is k times differentiable, then:

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(c)(x-c)^n}{n!} + R_{n+1}(f, x, c)$$

*where R_{n+1} is **Taylor's Remainder**:*

$$R_{n+1}(f, x, c) = \frac{f^{(n+1)}(\xi)(x-\xi)^{n+1}}{(n+1)!}$$

where ξ is between c and x.

- **Can we use Taylor's Remainder to determine if a function is analytic?**
  - f will be analytic on $|x-c| < r$ is:

$$\lim_{n\to\infty} R_{n+1} = 0$$

- **Are all functions analytic?**

Not all infinitely differentiable functions are analytic. For example, let $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Then it can be proved that for all j, f is j times differentiable and that there exist polynomials q_j such that $f^{(j)}(x) = e^{-1/x^2} q_j\left(\frac{1}{x}\right)$ for $x \neq 0$, and $f^{(j)}(0) = 0$. We can apply Taylor's theorem to f to obtain that for all x and n we have

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)x^j}{j!} + R_{n+1}(f)(x,0) = R_{n+1}(f)(x,0)$$

since all the $f^{(j)}(0)$ are zero) where

$$R_{n+1}(f)(x,0) = \frac{f^{(n+1)}(\xi)x^{n+1}}{(n+1)!}.$$

The problem is that for no x other than zero do we have that $R_{n+1}(f)(x,0) \to 0$. This is because although $\frac{f^{(n+1)}(\xi)x^{n+1}}{(n+1)!}$ looks difficult to calculate for this particular f, we already know it is *precisely* e^{-1/x^2} for all n when $x \neq 0$, so it quite simply does not go to 0 as $n \to \infty$. At the end of the day, such a function as f simply does not admit a power series expansion in any interval $(-r, r)$.

5 Exercises

1. **Suppose that:**

$$\sum_{n=0}^{\infty} a_n x^n$$

has radius of convergence R.

(a) **Determine the radius of convergence of:**

$$\sum_{n=0}^{\infty} a_n x^{2n}$$

Since we know nothing about the value of each a_n, we proceed from the definition.

Since R is the radius of convergence fo the first series, we know that:

$$R = \sup\{r \geq 0, \quad a_n r^n \ \textit{is bounded}\}$$

Now, lets define a new variable $y = x^2$, and consider the power series:

$$\sum_{n=0}^{\infty} a_n y^n$$

Clearly, by the definition of radius of convergence, such a series also has radius of convergence R. But then, it must be the case that if $|y| < R$ then $\sum_{n=0}^{\infty} a_n y^n$ must be convergent. But:

$$|y| = |x^2| < R \implies |x| < \sqrt{R}$$

In other words, $\sum_{n=0}^{\infty} a_n y^n = \sum_{n=0}^{\infty} a_n x^{2n}$ must have radius of convergence $\sqrt{R}$.

(b) **Determine the radius of convergence of:**

$$\sum_{n=0}^{\infty} a_n^2 x^n$$

This could be done by simply applying the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}^2 x^{n+1}}{a_n^2 x^n}\right| = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^2 |x|$$

We know $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ must converge, since the original power series converges, say to L. Then:

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^2 |x| = L^2|x|$$

For convergence we require $L^2|x| < 1$ so:

$$|x| < \frac{1}{L^2}$$

but $R = \frac{1}{L}$, so $|x| < R^2$ for convergence.

Alternatively, we can prove going from the definition. Let S be the radius of convergence of $\sum_{n=0}^{\infty} a_n^2 x^n$. Then, from the definitions:

$$R = \sup\{r \geq 0, \quad a_n r^n \ \textit{is bounded}\}$$

$$S = \sup\{s \geq 0, \quad a_n^2 s^n \ \textit{is bounded}\}$$

Now, what if we define $s = m^2$. Then, we get that:

$$S = \sup\{m^2 \geq 0, \quad (a_n m^n)^2 \ \textit{is bounded}\}$$

But we know that $a_n m^n$ is bounded whenever $m < R$. In other words, if $m^2 < R^2$, $(a_n m^n)^2$ will be bounded. But $s = m^2$, so it follows that $S = R^2$.

2. **Suppose that $|a_k| \leq |b_k|$ for large k. Prove that if $\sum_{k=0}^{\infty} b_k x^k$ converges on an open interval I, then $\sum_{k=0}^{\infty} a_k x^k$ also converges on I. Is this true if I is a closed interval?**

Let R be the radius of convergence of $\sum_{k=0}^{\infty} b_k x^k$. Then, it must be the case that:

$$I \subset (-R, R)$$

Recall the definition of radius of convergence:

$$R = \sup\{r \geq 0, \quad b_k r^k \ \textit{is bounded}\}$$

But if b_k is bounded, so is $|b_k|$ so:

$$R = \sup\{r \geq 0, \quad |b_k| r^k \ \textit{is bounded}\}$$

Since $|a_k| \leq |b_k|$, and $|b_k| r^k$ is bounded, it must be the case that $|a_k| r^k$ is also bounded. In particular:

$$\{r \geq 0, \quad |b_k| r^k \ \textit{is bounded}\} \subseteq \{r \geq 0, \quad |a_k| r^k \ \textit{is bounded}\}$$

which follows by the fact that $|a_k| \leq |b_k|$ so any r which bounds $|b_k| r^k$ must also bound $|a_k| r^k$, but there might be r which bound $|a_k| r^k$ and not $|b_k| r^k$. If we then take the supremum of the sets, we get that:

$$R \leq \sup\{r \geq 0, \quad |a_k| r^k \ \textit{is bounded}\}$$

Hence, the radius of convergence of $\sum_{k=0}^{\infty} a_k x^k$ is at least as large as R. In other words, $\sum_{k=0}^{\infty} a_k x^k$, must also converge on I, since $I \subset (-R, R)$.

However, if the interval is closed, this might not be the case. In particular, we want to find b_k such that b_k converges at an endpoint, but a_k doesn't. For example, $\sum_{k=0}^{\infty} b_k x^k$ might converge on $(-R, R]$, but $\sum_{k=0}^{\infty} a_k x^k$ won't. The key is to exploit the absolute values in the assumption that $|a_k| \leq |b_k|$. It allows us to define b_k as an alternating sequence, which when summated is more likely to converge than its non-alternating counterpart. For example:

$$b_k = \frac{(-1)^k}{k}$$

$$a_k = \frac{1}{k}$$

These sequences do satisfy $|a_k| \leq |b_k|$. If we consider their power series:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k} x^k$$

$$\sum_{k=0}^{\infty} \frac{1}{k} x^k$$

The first power series has $R = 1$, with interval of covergence $(-1, 1]$, since if $x = 1$, we get the alternating Harmonic Series, which converges. However, the second series does **not** converge at $x = 1$, since it is just the Harmonic Series.

3. **The notes have a set of very interesting exercises in 3.3, which formalise properties of e^x, defined as a power series**

6 Workshop

1. **What is the radius of convergence of the power series:**

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

What is the interval of convergence I?

If we apply the ratio test:

$$\left| \frac{\frac{(-1)^{k+1} x^{2(k+1)+1}}{(2(k+1)+1)!}}{\frac{(-1)^k x^{2k+1}}{(2k+1)!}} \right| = \left| \frac{(-1)^{k+1} x^{2k+3} (2k+1)!}{(-1)^k x^{2k+1} (2k+3)!} \right| = \left| \frac{(-1) x^2}{(2k+2)(2k+3)} \right| = \frac{x^2}{(2k+2)(2k+3)}$$

Hence:

$$\lim_{k \to \infty} \frac{x^2}{(2k+2)(2k+3)} = 0$$

and this is independent of x. Hence, $R = \infty$ is the radius of convergence, and the interval of convergence is $\mathbb{R}$

2. **Define:**

$$S(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \qquad x \in I$$

Prove that S is differentiable on I and that for $x \in I$:

$$S'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$S(x)$ is a power series which converges $\forall x \in \mathbb{R}$, therefore it is infinitely differentiable on $\mathbb{R}$, with the resulting series being uniformly (and absolutely) convergent (Theorem 3.3). In particular, this means that the derivative of $S(x)$ can be obtained by term-by-term differentiation, and said derivative will also be a convergent power series $\forall x \in \mathbb{R}$.

We thus compute:

$$\begin{aligned} S'(x) &= \sum_{k=0}^{\infty} \frac{d}{dx}\left(\frac{(-1)^k x^{2k+1}}{(2k+1)!}\right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1) x^{2k}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \end{aligned}$$

3. **Define $C(x) := S'(x)$. Show that $C'(x) = -S(x)$. Prove that $C(x)^2 + S(x)^2 = 1$ for all x, and deduce that for all x we have $|S(x)| \leq 1$ and $|C(x)| \leq 1$.**

Again from Theorem 3.3, $C(x)$ is infinitely differentiable on $I = (-\infty, \infty)$, so can differentiate term-wise:

$$\begin{aligned} C'(x) &= \sum_{k=0}^{\infty} \left(\frac{(-1)^k x^{2k}}{(2k)!}\right)' \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (2k) x^{2k-1}}{(2k)!} \end{aligned}$$

Notice that for $k = 0$, $\frac{(-1)^k (2k) x^{2k}}{(2k)!} = 0$, so we can ignore the $k = 0$ index:

$$\begin{aligned} C'(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k (2k) x^{2k-1}}{(2k)!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k (2k) x^{2k-1}}{(2k)!} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{(2k-1)!} \end{aligned}$$

If we set $k = m + 1$, then:

$$C'(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+1} x^{2m+1}}{(2m+1)!}$$

m is just a variable, so for coherence, set it back to k (this doesn't change the sum):

$$C'(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{(2k+1)!}$$

We can extract a factor of -1, which results in:

$$C'(x) = -\sum_{k=0}^{\infty} \frac{(-1)^{k} x^{2k+1}}{(2k+1)!} = -S(x)$$

as required.

We now show that $C(x)^2 + S(x)^2 = 1$ for all x.

Since C and S are infinitely differentiable on I, the function:

$$f(x) = C(x)^2 + S(x)^2$$

is also differentiable on I. If we differentiate using the chain rule:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(C(x)^2 + S(x)^2\right) \\ &= 2C(x)C'(x) + 2S(x)S'(x) \\ &= 2C(x)(-S(x)) + 2S(x)C(x) \\ &= 0 \end{aligned}$$

where we have used the fact that $C'(x) = -S(x)$ and $C(x) := S'(x)$.

Moreover, $\forall x \in \mathbb{R}, f'(x) = 0$. Consider any interval $[a, b]$ (over which f is clearly continuous and differentiable). By the mean value theorem $\exists c \in (a, b)$:

$$f'(c) = 0 = \frac{f(b) - f(a)}{b - a} \implies f(b) = f(a)$$

Hence, over any interval, $f(x)$ is a constant function. Now, notice that $C(0) = 1$ and $S(0) = 0$. Hence:

$$f(0) = C(0)^2 + S(0)^2 = 1 + 0 = 1$$

Thus, it follows that $C(x)^2 + S(x)^2$ is constant $\forall x \in \mathbb{R}$, and at $x = 0$ it is 1, so it follows that $\forall x \in \mathbb{R}$, $C(x)^2 + S(x)^2 = 1$ as required.

Lastly, we show that $\forall x \in \mathbb{R}$, $|S(x)| \leq 1$ and $|C(x)| \leq 1$.

Since $C(x)^2$ and $S(x)^2$ are both non-negative for any $x \in \mathbb{R}$, and we have that $C(x)^2 + S(x)^2 = 1$, it must be the case that:

$$\begin{aligned} 0 \leq C(x)^2 \leq 1 \\ 0 \leq S(x)^2 \leq 1 \end{aligned}$$

Further notice that the square of any number is equal to the square of the absolute value of said number, so indeed:

$$\begin{aligned} 0 \leq |C(x)|^2 \leq 1 \\ 0 \leq |S(x)|^2 \leq 1 \end{aligned}$$

Lastly, since we are dealing with non-zero quantities, taking square roots preserves the inequality so:

$$0 \leq |C(x)| \leq 1$$

$$0 \leq |S(x)| \leq 1$$

as required.

4. **Prove that for all real x, y we have:**

$$S(x+y) = S(x)C(y) + C(x)S(y)$$

and

$$C(x+y) = C(x)C(y) - S(x)S(y)$$

Consider the function:

$$f(x) = (S(x+y) - (S(x)C(y) + C(x)S(y)))^2 + (C(x+y) - (C(x)C(y) - S(x)S(y)))^2$$

Since S, C are continuous and differentiable on $\mathbb{R}$, so is f.

If we expand out f:

$$\begin{aligned}
f(x) &= (S(x+y) - (S(x)C(y) + C(x)S(y)))^2 + (C(x+y) - (C(x)C(y) - S(x)S(y)))^2 \\
&= (S(x+y) - S(x)C(y) - C(x)S(y))^2 + (C(x+y) - C(x)C(y) + S(x)S(y))^2 \\
&= S(x+y)^2 - 2S(x+y)S(x)C(y) - 2S(x+y)C(x)S(y) + (S(x)C(y))^2 \\
&+ 2S(x)C(y)C(x)S(y) + (C(x)S(y))^2 \\
&+ C(x+y)^2 - 2C(x+y)C(x)C(y) + 2C(x+y)S(x)S(y) + (C(x)C(y))^2 \\
&- 2C(x)C(y)S(x)S(y) + (S(x)S(y))^2 \\
&= 1 - 2S(x+y)S(x)C(y) - 2S(x+y)C(x)S(y) \\
&- 2C(x+y)C(x)C(y) + 2C(x+y)S(x)S(y) \\
&+ S(y)^2(S(x)^2 + C(x)^2) + C(y)^2(C(x)^2 + S(x)^2) \\
&= 2 - 2S(x+y)S(x)C(y) - 2S(x+y)C(x)S(y) \\
&- 2C(x+y)C(x)C(y) + 2C(x+y)S(x)S(y)
\end{aligned}$$

If we take the derivative with respect to x, thinking of y as a constant, and recalling that $S' = C$ and $C' = -S$:

$$\begin{aligned}
f'(x) &= -2C(y)(C(x+y)S(x) + C(x)S(x+y)) \\
&- 2S(y)(C(x+y)C(x) - S(x)S(x+y)) \\
&- 2C(y)(-S(x+y)C(x) - S(x)C(x+y)) \\
&+ 2S(y)(S(x+y)S(x) + C(x)C(x+y)) \\
&= 0
\end{aligned}$$

By reasoning similar to above, using the Mean Value Theorem it follows that $f(x)$ is a constant function on $\mathbb{R}$. In particular, consider:

$$f(0) = (S(y) - (S(0)C(y) + C(0)S(y)))^2 + (C(y) - (C(0)C(y) - S(0)S(y)))^2 = (S(y) - S(y))^2 + (C(y) - C(y))^2 = 0$$

Since f is the sum of 2 non-negative functions, it follows that:

$$(S(y) - (S(0)C(y) + C(0)S(y)))^2 = 0 \implies S(x+y) = S(x)C(y) + C(x)S(y)$$

$$(C(y) - (C(0)C(y) - S(0)S(y)))^2 = 0 \implies C(x+y) = C(x)C(y) - S(x)S(y)$$

as required.

5. (a) **Prove that $S(x) > 0$ for $0 < x \leq \sqrt{6}$**

We know that:

$$S(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

We can group terms into pairs of positive and negative terms:

$$S(x) = \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \dots$$

Now, consider one such general pair of terms (using k even):

$$\frac{x^{2k+1}}{(2k+1)!} - \frac{x^{2k+3}}{(2k+3)!}$$

We can simplify:

$$\begin{aligned} & \frac{x^{2k+1}}{(2k+1)!} - \frac{x^{2k+3}}{(2k+3)!} \\ = \quad & x^{2k+1}\left(\frac{1}{(2k+1)!} - \frac{x^2}{(2k+3)!}\right) \\ = \quad & x^{2k+1}\left(\frac{(2k+2)(2k+3) - x^2}{(2k+3)!}\right) \end{aligned}$$

Thus, we can rewrite $S(x)$ as:

$$S(x) = \sum_{k=0}^{\infty} x^{2k+1}\left(\frac{(2k+2)(2k+3) - x^2}{(2k+3)!}\right)$$

If each term in this series is non-negative when $0 < x \leq \sqrt{6}$, and we have at least one positive term, then it is easy to see that for $0 < x \leq \sqrt{6}$, $S(x) > 0$.

Now, let $0 < x \leq \sqrt{6}$. Consider the term:

$$x^{2k+1}\left(\frac{(2k+2)(2k+3) - x^2}{(2k+3)!}\right)$$

For any $k \geq 0$ and since $x > 0$, x^{2k+1} and $(2k+3)!$ will always be positive. Thus, the sign of the term above is solely dependent on the value of:

$$(2k+2)(2k+3) - x^2$$

Notice that, if $k \geq 0$, $(2k+2)(2k+3) \geq 6$. Moreover, since $0 < x \leq \sqrt{6}$, $0 < x^2 \leq 6$. Using this, we get the following inequality:

$$(2k+2)(2k+3) - 6 \leq (2k+2)(2k+3) - x^2 < (2k+2)(2k+3) - 0$$

But since $(2k+2)(2k+3) \geq 6$, it follows that $(2k+2)(2k+3) - 6 \geq 0$, so:

$$0 \leq (2k+2)(2k+3) - x^2$$

for $k \geq 0$, $0 < x \leq \sqrt{6}$. In fact, $(2k+2)(2k+3) - 6 = 0$ if and only if $k = 0$; for $k > 0$, $(2k+2)(2k+3) - 6 > 0$. Overall, it follows that if $k = 0$:

$$(2k+2)(2k+3) - x^2 \geq 0 \implies x^{2k+1}\left(\frac{(2k+2)(2k+3) - x^2}{(2k+3)!}\right) \geq 0$$

and if $k > 0$:

$$(2k+2)(2k+3) - x^2 > 0 \implies x^{2k+1}\left(\frac{(2k+2)(2k+3)-x^2}{(2k+3)!}\right) > 0$$

for any $0 < x \leq \sqrt{6}$. But then it follows that if $0 < x \leq \sqrt{6}$, each term being added in our modified $S(x)$ will be strictly positive, except possibly for the first term (if $x = \sqrt{6}$, the term at $k = 0$ will be 0, but this will be the only such term to be 0), so it follows that:

$$S(x) > 0$$

as required.

(b) **Prove that $C(x) > 0$ for $0 \leq x \leq \sqrt{2}$**

We know that:

$$C(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

We can group terms into pairs of positive and negative terms:

$$C(x) = \left(1 - \frac{x^2}{2!}\right) + \left(\frac{x^4}{4!} - \frac{x^6}{6!}\right) + \ldots$$

Now, consider one such general pair of terms (using k even):

$$\frac{x^{2k}}{(2k)!} - \frac{x^{2k+2}}{(2k+2)!}$$

We can simplify:

$$\begin{aligned} & \frac{x^{2k}}{(2k)!} - \frac{x^{2k+2}}{(2k+2)!} \\ = \; & x^{2k}\left(\frac{1}{(2k)!} - \frac{x^2}{(2k+2)!}\right) \\ = \; & x^{2k}\left(\frac{(2k+1)(2k+2)-x^2}{(2k+2)!}\right) \end{aligned}$$

Thus, we can rewrite $C(x)$ as:

$$C(x) = \sum_{k=0}^{\infty} x^{2k}\left(\frac{(2k+1)(2k+2)-x^2}{(2k+2)!}\right)$$

If each term in this series is non-negative when $0 \leq x \leq \sqrt{2}$, and we have at least one positive term, then it is easy to see that for $0 \leq x \leq \sqrt{2}$, $C(x) > 0$.

Now, let $0 \leq x \leq \sqrt{2}$. Consider the term:

$$x^{2k}\left(\frac{(2k+1)(2k+2)-x^2}{(2k+2)!}\right)$$

For any $k \geq 0$ and since $x \geq 0$, x^{2k} and $(2k+2)!$ will always be non-negative. Thus, the sign of the term above is solely dependent on the value of:

$$(2k+1)(2k+2) - x^2$$

Notice that, if $k \geq 0$, $(2k+1)(2k+2) \geq 2$. Moreover, since $0 \leq x \leq \sqrt{2}$, $0 \leq x^2 \leq 2$. Using this, we get the following inequality:

$$(2k+1)(2k+2) - 2 \leq (2k+1)(2k+2) - x^2 \leq (2k+1)(2k+2) - 0$$

But since $(2k+1)(2k+2) \geq 2$, it follows that $(2k+1)(2k+2) - 2 \geq 0$, so:

$$0 \leq (2k+1)(2k+2) - x^2$$

for $k \geq 0$, $0 \leq x \leq \sqrt{2}$. In fact, $(2k+1)(2k+2) - 2 = 0$ if and only if $k = 0$; for $k > 0$, $(2k+1)(2k+2) - 2 > 0$. Overall, it follows that if $k = 0$:

$$(2k+1)(2k+2) - x^2 \geq 0 \implies x^{2k}\left(\frac{(2k+1)(2k+2) - x^2}{(2k+2)!}\right) \geq 0$$

and if $k > 0$:

$$(2k+1)(2k+2) - x^2 > 0 \implies x^{2k}\left(\frac{(2k+1)(2k+2) - x^2}{(2k+2)!}\right) > 0$$

for any $0 \leq x \leq \sqrt{2}$. But then it follows that if $0 \leq x \leq \sqrt{2}$, each term being added in our modified $C(x)$ will be strictly positive, except possibly for the first term (if $x = \sqrt{2}$, the term at $k = 0$ will be 0, but this will be the only such term to be 0), so it follows that:

$$C(x) > 0$$

as required.

(c) **Prove that for $0 \leq x \leq \sqrt{56}$, if:**

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} < 0$$

then $C(x) < 0$, and deduce that $C(\frac{8}{5}) < 0$.

We know that:

$$C(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Lets consider the terms in the summation for which $k \geq 3$:

$$-\frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \ldots$$

As we have done before, we can group these in negative-positive term pairs, like so:

$$\left(\frac{x^8}{8!} - \frac{x^6}{6!}\right) + \left(\frac{x^{12}}{12!} - \frac{x^10}{10!}\right) + \ldots$$

If we consider a general term of this summation, for $k \geq 3$ (and k odd):

$$\frac{x^{2k+2}}{(2k+2)!} - \frac{x^{2k}}{(2k)!}$$

We can simplify:

$$\begin{aligned}
& \frac{x^{2k+2}}{(2k+2)!} - \frac{x^{2k}}{(2k)!} \\
= \quad & x^{2k}\left(\frac{x^2}{(2k+2)!} - \frac{1}{(2k)!}\right) \\
= \quad & x^{2k}\left(\frac{x^2 - (2k+1)(2k+2)}{(2k+2)!}\right)
\end{aligned}$$

Thus, we can rewrite $C(x)$ as:

$$C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \sum_{k=3}^{\infty} x^{2k} \left(\frac{x^2 - (2k+1)(2k+2)}{(2k+2)!} \right)$$

If each term in the series is non-positive when $0 \leq x \leq \sqrt{56}$, and we have at least one negative term, then it is easy to see that for $0 \leq x \leq \sqrt{56}$, $C(x) < 0$, since we are assuming that $1 - \frac{x^2}{2!} + \frac{x^4}{4!} < 0$.

Now, let $0 \leq x \leq \sqrt{56}$. Consider the term:

$$x^{2k} \left(\frac{x^2 - (2k+1)(2k+2)}{(2k+2)!} \right)$$

For any $k \geq 3$ and since $x \geq 0$, x^{2k} and $(2k+2)!$ will always be non-negative. Thus, the sign of the term above is solely dependent on the value of:

$$x^2 - (2k+1)(2k+2)$$

Notice that, if $k \geq 3$, $(2k+1)(2k+2) \geq 56$. Moreover, since $0 \leq x \leq \sqrt{56}$, $0 \leq x^2 \leq 56$. Using this, we get the following inequality:

$$0 - (2k+1)(2k+2) \leq x^2 - (2k+1)(2k+2) \leq 56 - (2k+1)(2k+2)$$

But since $(2k+1)(2k+2) \geq 56$, it follows that $56 - (2k+1)(2k+2) \leq 0$, so:

$$x^2 - (2k+1)(2k+2) \leq 0$$

for $k \geq 3$, $0 < x \leq \sqrt{56}$. In fact, $56 - (2k+1)(2k+2) = 0$ if and only if $k = 3$; for $k > 3$, $56 - (2k+1)(2k+2) < 0$. Overall, it follows that if $k = 3$:

$$x^2 - (2k+1)(2k+2) \leq 0 \implies x^{2k} \left(\frac{x^2 - (2k+1)(2k+2)}{(2k+2)!} \right) \leq 0$$

and if $k > 3$:

$$x^2 - (2k+1)(2k+2) < 0 \implies x^{2k} \left(\frac{x^2 - (2k+1)(2k+2)}{(2k+2)!} \right) < 0$$

for any $0 \leq x \leq \sqrt{56}$. But then it follows that if $0 \leq x \leq \sqrt{56}$, each term being added in our modified $C(x)$ will be stricly negative, except possibly for the first term (if $x = \sqrt{56}$, the term at $k = 3$ will be 0, but this will be the only such term to be 0), so it follows that:

$$\sum_{k=3}^{\infty} x^{2k} \left(\frac{x^2 - (2k+1)(2k+2)}{(2k+2)!} \right) < 0$$

Moreover, since:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} < 0$$

So it follows that:

$$C(x) < 0$$

as required.

Finally, notice that:

$$0 < \frac{8}{5} < \frac{10}{5} = 2$$

Moreover,

$$7^2 < 56 \implies 7 < \sqrt{56}$$

Thus, it follows that:

$$0 < \frac{8}{5} < 2 < 7 < \sqrt{56}$$

so in particular:

$$0 \leq \frac{8}{5} \leq \sqrt{56}$$

Now, we compute:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

using $x = \frac{8}{5}$:

$$1 - \frac{8^2}{5^2 \times 2} + \frac{8^4}{5^4 \times 24} = -\frac{13}{1875}$$

Hence, from the work above, it follows that:

$$C\left(\frac{8}{5}\right) < 0$$

as required.

6. **Deduce that there is a unique number $\frac{\omega}{2}$ satisfying $\sqrt{2} < \frac{\omega}{2} < \frac{8}{5}$ such that:**

$$C\left(\frac{\omega}{2}\right) = 0$$

Further show that:

$$S\left(\frac{\omega}{2}\right) = 1$$

Notice above we have shown that $C(\sqrt{2}) < 0$ and $C\left(\frac{8}{5}\right) > 0$. C is continuous on $\mathbb{R}$, so in particular it is continuous on the closed, bounded interval $\left[\sqrt{2}, \frac{8}{5}\right]$. Thus, we can apply Bolzano's Theorem (intermediate value theorem in the notes) to see that $\exists \alpha \in \left(\sqrt{2}, \frac{8}{5}\right)$ such that:

$$C(\alpha) = 0$$

Moreover, we have that:

$$C(\alpha)^2 + S(\alpha)^2 = 1 \implies S(\alpha)^2 = 1$$

Notice, since $\sqrt{2} < \alpha < \frac{8}{5}$ we have that:

$$\alpha < \sqrt{6}$$

So that $S(\alpha) > 0$. Hence, we must have that $S(\alpha) = 1$.

Here $\alpha = \frac{\omega}{2}$.

7. **Prove that for all x:**

$$S\left(x + \frac{\omega}{2}\right) = C(x)$$

$$S\left(x + \omega\right) = -S(x)$$

$$S\left(x + \frac{3\omega}{2}\right) = -C(x)$$

$$S(x + 2\omega) = S(x)$$

We just need to apply the identities which we saw above:

$$S(x+y) = S(x)C(y) + C(x)S(y)$$

$$C(x+y) = C(x)C(y) - S(x)S(y)$$

Indeed:

$$S\left(x+\frac{\omega}{2}\right) = S(x)C(\alpha) + C(x)S(\alpha) = C(x)$$

$$\begin{aligned} S\left(x+\omega\right) &= S(x)C(2\alpha) + C(x)S(2\alpha) \\ &= S(x)C(\alpha+\alpha) + C(x)S(\alpha+\alpha) \\ &= S(x)C(\alpha+\alpha) + C(x)C(\alpha) \\ &= S(x)(C(\alpha)C(\alpha) - S(\alpha)S(\alpha)) \\ &= -S(x) \end{aligned}$$

$$\begin{aligned} S\left(x+\frac{3\omega}{2}\right) &= S(x)C(3\alpha) + C(x)S(3\alpha) \\ &= S(x)C(\alpha+2\alpha) + C(x)S(\alpha+2\alpha) \\ &= S(x)(C(\alpha)C(2\alpha) - S(\alpha)S(2\alpha)) - C(x)S(\alpha) \\ &= S(x)(-C(\alpha)) - C(x)S(\alpha) \\ &= -C(x) \end{aligned}$$

$$\begin{aligned} S\left(x+2\omega\right) &= S(x)C(4\alpha) + C(x)S(4\alpha) \\ &= S(x)C(2\alpha+2\alpha) - C(x)S(2\alpha) \\ &= S(x)(C(2\alpha)C(2\alpha) - S(2\alpha)S(2\alpha)) - C(x)C(\alpha) \\ &= S(x)(C(2\alpha))^2 \\ &= S(x)(C(\alpha)C(\alpha) - S(\alpha)S(\alpha))^2 \\ &= S(x)(-1)^2 \\ &= S(x) \end{aligned}$$