Honours Analysis - Week 4 - Uniform Convergence of Sequences and Series of Functions

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Contents

This shows the power of uniform convergence. If $f_n \to f$ pointwise, it is not always the case that the sequence $\int f_n dx$ converges to $\int f$. The same thing is true when taking derivatives. However, if $f_n \to f$ uniformly, this is no longer the case.

1 Uniform Convergence and Calculus

1.1 Theorem: The Mean Value Theorem

*If f is **continuous** on $[\alpha, \beta]$ and **differentiable** on (α, β), then there exists some $\gamma \in (\alpha, \beta)$, such that:*

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} = f'(\gamma)$$

1.2 Theorem: Integrability and Uniform Convergence

*Suppose that $f_n \to f$ **uniformly** on a **closed** interval $[a, b]$. If each f_n is **integrable** on $[a, b]$, then so is f, and:*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right) dx = \int_a^b f(x)dx$$

In fact, $\forall x \in [a, b]$:

$$\lim_{n\to\infty} \int_a^x f_n(t)dt = \int_a^x \left(\lim_{n\to\infty} f_n(t)\right) dt$$

[Theorem 2.2]

This is proved later on, when we properly define integration.

1.3 Theorem: Differentiability and Uniform Convergence

*Let (a, b) be an **open, bounded** interval. If $f_n : (a, b) \to \mathbb{R}$, and:*

1. *f_n **converges** at some $x_0 \in (a, b)$ (so $\lim_{n\to\infty} f_n(x_0)$ exists)*
2. *each f_n is **differentiable** on (a, b) (so f_n' exists $\forall x \in (a, b)$)*
3. *f_n' **converges uniformly** on (a, b)*

*then, it follows that f_n also **converges uniformly** on (a, b), and moreover, for each $x \in (a, b)$:*

$$\lim_{n\to\infty} f_n'(x) = \frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right)$$

[Theorem 2.3]

Notice that we have defined the differentiability condition on an open interval (a, b). This is because this allows us to use 2 sided limits for derivatives. If we had a closed interval, we would need to use one-sided limits. If the intervals are unbounded, it is possible that f_n might not converge uniformly.

Proof: Differentiability and Uniform Convergence. This proof has 2 parts. Firstly, we use the assumptions to prove that f_n converges uniformly on (a, b). Then, we prove that:

$$\lim_{n\to\infty} f_n'(x) = \frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right)$$

for any $x \in (a, b)$

We claim that f_n converges uniformly on (a, b). For that, define a sequence of functions g_n, such that:

$$g_n(x) = \begin{cases} \frac{f_n(x)-f_n(c)}{x-c}, & x \neq c \\ f_n'(c), & x = c \end{cases}$$

where $c \in (a, b)$. g_n is always defined on (a, b), since f_n' is defined on all of (a, b).

Using the above definition, notice that, for any $x \in (a, b)$:

$$f_n(x) = f_n(c) + (x - c)g_n(x)$$

since if $x = c$, we get:

$$f_n(c) = f_n(c)$$

and if $x \neq c$:

$$f_n(x) = f_n(c) + (x-c)g_n(x) = f_n(c) + (x-c)\frac{f_n(x) - f_n(c)}{x-c} = f_n(x)$$

Thus, it follows that if $g_n(x)$ is uniformly convergent on (a, b), so will $f_n(x)$. This is because we know that f_n converges at x_0, so if we choose $c = x_0$, we know then that:

$$f_n(x) = f_n(x_0) + (x - x_0)g_n(x)$$

so the uniform convergence of g_n implies the uniform convergence of f_n.

To show that g_n is uniformly convergent, we use the **Mean Value Theorem**. Now, let:

- $x \in (a, b)$
- $n, m \in \mathbb{N}$
- some γ on the open interval defined x and c (so $\gamma \neq x \neq c$)

We can define a function $\Phi(x) = f_n(x) - f_m(x)$, defined on the open interval defined by x and c, such that, by the Mean Value Theorem:

$$\frac{\Phi(x) - \Phi(c)}{x-c} = \Phi'(\gamma)$$

But then we get that:

$$\begin{aligned}
& \frac{\Phi(x) - \Phi(c)}{x-c} = \Phi'(\gamma) \\
\implies \quad & \frac{f_n(x) - f_m(x) - (f_n(c) - f_m(c))}{x-c} = f_n'(\gamma) - f_m'(\gamma) \\
\implies \quad & \frac{(f_n(x) - f_n(c)) - (f_m(x) - f_m(c))}{x-c} = f_n'(\gamma) - f_m'(\gamma) \\
\implies \quad & g_n(x) - g_m(x) = f_n'(\gamma) - f_m'(\gamma)
\end{aligned}$$

Now, we use the assumption that f_n' converges uniformly on (a, b). In particular, this means that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n, m \geq N$:

$$|f_n'(x) - f_m'(x)| < \varepsilon$$

from which it follows that:

$$|g_n(x) - g_m(x)| < \varepsilon$$

too, given $x \in (a, b), x \neq c$. If $x = c$, from the definition of g_n, we know that $g_n(c) = f_n'(c)$, so the above still applies.

Thus, we have shown that g_n is a Cauchy Sequence, and so it converges. Since the N is independent of x, it follows that g_n converges uniformly on (a, b). So from the above, f_n also converges uniformly on (a, b).

We now show that:

$$\lim_{n\to\infty} f_n'(x) = \frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right)$$

Pick some $c \in (a, b)$. Let:

$$\lim_{n\to\infty} f_n(x) = f(x)$$

$$\lim_{n\to\infty} g_n(x) = g(x)$$

We claim that $\forall c \in (a,b)$:

$$\lim_{n\to\infty} f_n'(c) = f'(c)$$

If $x = c$, we know that $g_n(c) = f_n'(c)$. Thus:

$$\lim_{n\to\infty} f_n'(c) = \lim_{n\to\infty} g_n(c) = g(c)$$

If $x \neq c$, we know that:

$$g_n(x) = \frac{f_n(x) - f_n(c)}{x - c}$$

so:

$$g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{f_n(x) - f_n(c)}{x - c} = \frac{f(x) - f(c)}{x - c}$$

But then recall that by the definition of the derivative:

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

so:

$$f'(c) = \lim_{x\to c} g(x) = g(c)$$

since each g_n is continuous at c, and so g is also continuous there.

But then we have shown that:

$$\lim_{n\to\infty} f_n'(c) = g(c)$$

$$f'(c) = g(c)$$

so it follows that:

$$\lim_{n\to\infty} f_n'(c) = f'(c)$$

as required.

□

2 Uniform Convergence of Series of Functions

2.1 Defining Convergence of Series of Functions

Consider a sequence of functions f_n defined on some set E. Consider the sequence of partial sums:

$$s_n(x) = \sum_{k=1}^{n} f_k(x)$$

for any $x \in E, n \in \mathbb{N}$

- **When does a series of functions converge pointwise?**
  - the series $\sum_{k=1}^{\infty} f_k(x)$ **converges pointwise** on E if and only if $s_n(x)$ converges **pointwise** on E as $n \to \infty$
- **When does a series of functions converge uniformly?**
  - the series $\sum_{k=1}^{\infty} f_k(x)$ **converges uniformly** on E if and only if $s_n(x)$ converges **uniformly** on E as $n \to \infty$
- **When does a series of functions converge absolutely (pointwise)?**
  - the series $\sum_{k=1}^{\infty} f_k(x)$ **converges absoluteley (pointwise)** on E if and only if $\sum_{k=1}^{\infty} |f_k(x)|$ converges for each $x \in E$

2.2 Properties of Uniformly Convergent Series of Functions

Convergence of series depends on convergence of sequences; thus, we can use the results about convergence of function sequences to derive the following results about function series.

2.2.1 Theorem: Continuity of Series

Let E be a non-empty subset of $\mathbb{R}$. and let f_n be a sequence of real functions defined on E.
*Then, if each f_n is **continuous** at some $x_0 \in E$, and if*

$$f = \sum_{k=1}^{\infty} f_k$$

***converges uniformly** on E, then f is also **continuous** at $x_0 \in E$.*
[Theorem 2.4]

2.2.2 Theorem: Term-by-Term Integration of Function Series

Let E be a non-empty subset of $\mathbb{R}$. and let f_n be a sequence of real functions defined on E.
*Then, suppose that E is a **closed** interval $[a, b]$, and each f_n is **integrable** on $[a, b]$. If*

$$f = \sum_{k=1}^{\infty} f_k$$

***converges uniformly** on E, then f is **integrable** on E and:*

$$\int_a^b \sum_{k=1}^{\infty} f_k(x)dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)dx$$

[Theorem 2.4]

2.2.3 Theorem: Term-by-Term Differentiation of Function Series

Let E be a non-empty subset of $\mathbb{R}$. and let f_n be a sequence of real functions defined on E.
Then, suppose that E is an ***open, bounded*** *interval. If:*

- *each f_n is* ***differentiable*** *on E*
- *$\sum_{k=1}^{\infty} f_k(x_0)$* ***converges*** *for some $x_0 \in E$*
- *$g = \sum_{k=1}^{\infty} f'_k$* ***converges uniformly*** *on E*

then $f = \sum_{k=1}^{\infty} f_k$ ***converges uniformly*** *on E, and is* ***differentiable****, such that for any $x \in E$:*

$$f'(x) = \left(\sum_{k=1}^{\infty} f_k(x)\right)' = \sum_{k=1}^{\infty} f'_k(x) = g(x)$$

2.3 Theorem: The Weierstrass M-Test

Sometimes it is inconvenient to have to derive convergence ofa series from the sequence of partial sums. We can use the ***Weierstrass M-Test*** *(the best convergence test alongside the ratio test).*

Let E be a non-empty subset of $\mathbb{R}$. Let

$$f_k : E \to \mathbb{R}$$

and suppose that $\exists M_k \geq 0$ such that:

$$\sum_{k=0}^{\infty} M_k < \infty$$

If $|f_k(x)| \leq M_k$ for all $k \in \mathbb{N}$, and $x \in E$, then:

$$f = \sum_{k=0}^{\infty} f_k$$

converges ***absolutely*** *and* ***uniformly*** *on E.*
This is a ***sufficient*** *but not* ***necessary*** *condition for convergence: failing the M-test doesn't imply divergence. [Theorem 2.5]*

Proof: Weierstrass M-Test. Since $\sum_{k=0}^{\infty} M_k$ converges, it must follow the Cauchy Criterion. In other words $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ :\ \forall m \geq n \geq N$, such that:

$$\left|\sum_{k=n}^{m} M_k\right| < \varepsilon$$

But then:

$$\left|\sum_{k=n}^{m} f_k(x)\right| \leq \sum_{k=n}^{m} |f_k(x)| \leq \sum_{k=n}^{m} M_k < \varepsilon$$

From which it follows that:

- the partial sums of $\sum_{k=0}^{\infty} f_k(x)$ are uniformly Cauchy
- the partial sums of $\sum_{k=n}^{m} |f_k(x)|$ are Cauchy

Thus, the series must converge absolutely and uniformly on E, as required.

□

3 Exercises

3.1 Uniform Convergence

1. **Prove that the following limits exist, and evaluate them**:

(a) $\lim\limits_{n\to\infty} \int_1^3 \frac{nx^{99}+5}{x^3+nx^{66}} dx$

We claim that $\frac{nx^{99}+5}{x^3+nx^{66}} \to x^{33}$ uniformly on $[1,3]$. To see why, $\forall\varepsilon > 0$, let $N = \frac{5}{\varepsilon}$. Then, if $n \geq N$, and noticing that $1 \leq x \leq 3$:

$$\begin{aligned}
\left|\frac{nx^{99}+5}{x^3+nx^{66}} - x^{33}\right| &< \left|\frac{nx^{99}+5}{nx^{66}} - x^{33}\right| \quad \textit{(since } x^3 + nx^{66} > nx^{66}, \forall x > 0\textit{)} \\
&= \left|\frac{nx^{99}+5-nx^{99}}{nx^{66}}\right| \\
&= \left|\frac{5}{nx^{66}}\right| \\
&= \frac{5}{nx^{66}} \\
&\leq \frac{5}{n} \quad \textit{(since } 1 \leq x \textit{ so } 1 \leq x^{66}\textit{)} \\
&< \frac{5}{N} \\
&= \frac{5}{\frac{5}{\varepsilon}} \\
&= \varepsilon
\end{aligned}$$

Thus, it follows that $\frac{nx^{99}+5}{x^3+nx^{66}} \to x^3$ uniformly on $[1,3]$. But then:

$$\lim_{n\to\infty} \int_1^3 \frac{nx^{99}+5}{x^3+nx^{66}} dx = \int_1^3 x^{33} dx = \frac{1}{34}\left(3^{34}-1\right)$$

(b) $\lim\limits_{n\to\infty} \int_0^2 e^{\frac{x^2}{n}} dx$

We claim that $e^{\frac{x^2}{n}} \to e^0 = 1$ uniformly on $[0,2]$. To see why, $\forall\varepsilon > 0$, let $N = \frac{4}{\ln(\varepsilon+1)}$. Then, if

$n \geq N$, and noticing that $0 \leq x \leq 2$:

$$\begin{aligned}
\left| e^{\frac{x^2}{n}} - 1 \right| &\leq |e^{\frac{4}{n}} - 1| \quad \textit{(since } 0 \leq x \leq 2\textit{:)} \\
&\leq |e^{\frac{4}{N}} - 1| \\
&= |e^{\frac{4}{\frac{4}{\ln(\varepsilon+1)}}} - 1| \\
&= |e^{\ln(\varepsilon+1)} - 1| \\
&= |\varepsilon + 1 - 1| \\
&= \varepsilon
\end{aligned}$$

Thus, it follows that $e^{\frac{x^2}{n}} \to e^0 = 1$ uniformly on $[0, 2]$. But then:

$$\lim_{n\to\infty} \int_0^2 e^{\frac{x^2}{n}} dx = \int_0^2 1 dx = 2$$

(c) $\lim_{n\to\infty} \int_0^3 \sqrt{\sin\left(\frac{x}{n}\right) + x + 1} dx$

We claim that $\sqrt{\sin\left(\frac{x}{n}\right) + x + 1} \to \sqrt{x+1}$ uniformly on $[0, 3]$. To see why, $\forall \varepsilon > 0$, let $N = \frac{3}{2\varepsilon}$. Then, if $n \geq N$, and noticing that $0 \leq x \leq 3$:

$$\begin{aligned}
\left| \sqrt{\sin\left(\frac{x}{n}\right) + x + 1} - \sqrt{x+1} \right| &= \left| \frac{\sin\left(\frac{x}{n}\right) + x + 1 - (x+1)}{\sqrt{\sin\left(\frac{x}{n}\right) + x + 1} + \sqrt{x+1}} \right| \\
&= \left| \frac{\sin\left(\frac{x}{n}\right)}{\sqrt{\sin\left(\frac{x}{n}\right) + x + 1} + \sqrt{x+1}} \right| \\
&< \frac{\left|\sin\left(\frac{x}{n}\right)\right|}{2} \quad \textit{(since } \sqrt{\sin\left(\frac{x}{n}\right) + x + 1} + \sqrt{x+1} \geq \sqrt{1} + \sqrt{1} = 2\textit{)} \\
&\leq \frac{\left|\frac{x}{n}\right|}{2} \quad \textit{(since } |\sin(x)| \leq |x| \textit{ by the Mean Value Theorem)} \\
&= \frac{|x|}{2n} \\
&\leq \frac{3}{2n} \quad \textit{(since } x \leq 3\textit{)} \\
&\leq \frac{3}{2N} \\
&= \frac{3}{2\left(\frac{3}{2\varepsilon}\right)} \\
&= \varepsilon
\end{aligned}$$

Thus, it follows that $\sqrt{\sin\left(\frac{x}{n}\right) + x + 1} \to \sqrt{x+1}$ uniformly on $[0, 3]$. But then:

$$\lim_{n\to\infty} \int_0^3 \sqrt{\sin\left(\frac{x}{n}\right) + x + 1} dx = \int_0^3 \sqrt{x+1} dx = \frac{14}{3}$$

2. **Suppose that $b > a > 0$. Prove that:**

$$\lim_{n\to\infty} \int_a^b \left(1 + \frac{x}{n}\right)^n e^{-x} dx = b - 1$$

The exponential function e^x can be defined as:

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

To see why, we can take the logs of both sides:

$$\lim_{n\to\infty} n \ln \left(1 + \frac{x}{n}\right)$$

If we use L'Hopital's Rule or the Binomial Theorem, it can be shown that the limit above converges to x (see here).

Thus, it follows that $\left(1 + \frac{x}{n}\right)^n e^{-x}$ converges uniformly on $[a, b]$ to 1, from which the result follows.

3. **Let f, g be continuous on $[a, b]$, with $|g(x)| > 0$ for $x \in [a, b]$. Suppose that $f_n \to f, g_n \to g$ uniformly on $[a, b]$.**

   (a) **Prove that $\frac{1}{g_n}$ is defined for large n, and show that $\frac{f_n}{g_n} \to \frac{f}{g}$ uniformly on $[a, b]$**

   Firstly, we prove that g_n is always non-zero. To do so, notice that since g is continuous, by the Extreme Value Theorem, it must attain its minimum on $[a, b]$. Moreover, since $|g| > 0$, this minimum must be non-zero. Call it m. f, g must attain their maxima on the interval aswell. Call the largest of the maxima $M > 0$. But then, since f_n, g_n converge uniformly, they must be uniformly bounded, so for any $n \in \mathbb{N}$ and for all $x \in [a, b]$:

$$0 < m \leq |g| \leq M$$

$$|f| \leq M$$

$$|g_n| \leq M$$

$$|f_n| \leq M$$

   From the definition of uniform convergence, we know that for some $N \in \mathbb{N}$, if $n \geq N$:

$$|g_n(x) - g(x)| < \frac{m}{2}$$

   since $\frac{m}{2}$. But then:

$$g(x) - \frac{m}{2} < g_n(x) < g(x) + \frac{m}{2}$$

   But notice that the minimum of $g(x)$ is m, so $g(x) - \frac{m}{2} > 0$. In other words, $\forall n \geq N$ and for all $x \in [a, b]$:

$$0 < g_n(x)$$

   Thus, $\frac{1}{g_n}$ must be well defined, as required.

Now, letting $n \geq N$, we consider:

$$\begin{aligned}
\left|\frac{f_n}{g_n} - \frac{f}{g}\right| &= \left|\frac{f_n g - f g_n}{g_n g}\right| \\
&= \left|\frac{f_n g - fg + fg - f g_n}{g_n g}\right| \\
&= \left|\frac{g(f_n - f) - f(g_n - g)}{g_n g}\right| \\
&\leq \frac{1}{|g_n g|}\left(|g||f_n - f| + |f||g_n - g|\right) \\
&\leq \frac{2}{m^2}\left(M|f_n - f| + M\frac{m}{2}\right)
\end{aligned}$$

If we require $\frac{2}{m^2}\left(M|f_n - f| + M\frac{m}{2}\right) < \varepsilon$, we just need to pick $N_2 \in \mathbb{N}$ such that if $n \geq N_2$:

$$|f_n - f| < \frac{m^2}{2M}\varepsilon - \frac{m}{2}$$

(In the solutions, they use $\frac{2}{m^2}\left(M|f_n - f| + M|g_n - g|\right) < \varepsilon$ which leads to finding $N_2 \in \mathbb{N}$ such that if $n \geq N_2$:

$$|f_n - f| < \frac{\varepsilon m^2}{4M}$$

$$|g_n - g| < \frac{\varepsilon m^2}{4M}$$

The videos have a better proof, but I am currently very tired.

(b) **Show that a) is false if we have an open interval** (a, b)

Consider the interval $(0, 1)$. Let $f_n(x) = \frac{1}{n}, g_n(x) = x$. It is easy to see that $f_n \to 0$ and $g_n \to x$ uniformly. However,

$$\frac{f_n}{g_n} = \frac{1}{nx}$$

which we showed converges pointwise to 0 on $(0, 1)$.

As a reminder:

Pick $x \in (0, 1)$. Clearly, $\frac{1}{nx} \to 0$: for any x, if n gets sufficiently larger than $N(x, \varepsilon)$, f_n will converge to 0, so it is pointwise convergent.

However, it is not uniformly convergent. For that, we require that $\forall \varepsilon > 0$, we can find some $N \in \mathbb{N}$ such that for any $n \geq N$:

$$\left|\frac{1}{nx} - 0\right| < \varepsilon$$

If f_n were uniformly convergent, the above should apply to any ε, so pick $\varepsilon = 1$. Then, we should have that if $n \geq N$, for all x:

$$\left|\frac{1}{nx}\right| < 1$$

But this is clearly false, as $\forall n \in \mathbb{N}$ we can pick $x = \frac{1}{n}$, and the above won't be satisfied.

Thus, f_n is pointwise convergent, but not uniformly convergent.

3.2 Series Convergence

1. (a) **Prove that** $\sum_{k=1}^{\infty} \sin\left(\frac{x}{k^2}\right)$ **converges uniformly on any bounded interval of** $\mathbb{R}$

   We can use the M-Test. Consider:
   $$\left|\sin\left(\frac{x}{k^2}\right)\right|$$
   Recall, for any $y \in \mathbb{R}$:
   $$|sin(y)| \leq |y|$$
   So it follows that $\forall k \in \mathbb{R}$:
   $$\left|\sin\left(\frac{x}{k^2}\right)\right| < \frac{|x|}{k^2}$$
   Moreover, since x is in a bounded interval (say $[a, b]$):
   $$\frac{|x|}{k^2} < \frac{\max\{|a|, |b|\}}{k^2}$$
   If we let $M_k = \frac{\max\{|a|,|b|\}}{k^2}$, and consider:
   $$\sum_{k=1}^{\infty} M_k$$
   clearly it must converge by the p-series test, with $p = 2$.

   Thus, by the Weierstrass M-Test, $\sin\left(\frac{x}{k^2}\right)$ converges absolutely and uniformly on any bounded interval $[a, b]$.

   (b) **Prove that** $\sum_{k=1}^{\infty} e^{-kx}$ **converges uniformly on any closed subinterval of** $(0, \infty)$

   Consider some interval $[a, \infty)$. with $a > 0$. It is easy to see that:
   $$0 \leq e^{-kx} \leq e^{-ka}$$
   Letting $M_k = e^{-ka}$, we can see that $\sum_{k=1}^{\infty} M_k$ converges (we can use the ratio test to see that $\left|\frac{M_{k+1}}{M_k}\right| = e^{-a} < 1, \forall a > 0$). Thus, the series converges absolutely and uniformly on any interval $[a, \infty)$ for $a > 0$.

2. **Let**
   $$f(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$
   Prove that
   $$\int_0^{\pi/2} f(x)dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$$
   The key is to be able to show that $\frac{\cos(kx)}{k^2}$ converges uniformly. Using that, we can easily integrate $f(x)$, by using the infinite series.

   Notice that:
   $$\left|\frac{\cos(kx)}{k^2}\right| < \frac{1}{k^2}$$
   If $M_k = \frac{1}{k^2}$, $\sum_{k=1}^{\infty} M_k$ converges by the p-series test, so by Weierstrass M-Test, $\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$ converges uniformly and absolutely on $\mathbb{R}$.

Thus, we can integrate the series term by term:

$$\begin{aligned}\int_0^{\pi/2} f(x)dx &= \int_0^{\pi/2}\sum_{k=1}^{\infty}\frac{\cos(kx)}{k^2}\\ &= \sum_{k=1}^{\infty}\int_0^{\pi/2}\frac{\cos(kx)}{k^2}\\ &= \sum_{k=1}^{\infty}\left[\frac{\sin(kx)}{k^3}\right]_0^{\frac{\pi}{2}}\\ &= \sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\pi}{2}\right)}{k^3}\end{aligned}$$

Now,consider the first terms of the summand (ignoring $k = 0$):

$$\begin{aligned} k=1 &\longrightarrow \frac{\sin\left(\frac{\pi}{2}\right)}{1^3} = \frac{1}{1^3}\\ k=2 &\longrightarrow \frac{\sin(\pi)}{2^3} = 0\\ k=3 &\longrightarrow \frac{\sin\left(\frac{3\pi}{2}\right)}{3^3} = -\frac{1}{3^3}\\ k=4 &\longrightarrow \frac{\sin(2\pi)}{4^3} = 0\\ k=5 &\longrightarrow \frac{\sin\left(\frac{5\pi}{2}\right)}{5^3} = \frac{1}{5^3}\end{aligned}$$

In other words:

$$\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\pi}{2}\right)}{k^3} = \sum_{k=0}^{\infty}\frac{(-1)^3}{(2k+1)^3}$$

as required.

3. **Show that**

$$\sum_{k=1}^{\infty}\frac{1}{k}\sin\left(\frac{x}{k+1}\right)$$

converges uniformly on each bounded interval in $\mathbb{R}$ to a differentiable function which satisfies:

$$|f(x)| \le |x| \qquad |f'(x)| \le 1$$

Firstly, we show that the series converges uniformly on any bounded interval $[a, b]$ of the reals.

Let $c = \max\{|a|, |b|\}$. Consider:

$$\left|\frac{1}{k}\sin\left(\frac{x}{k+1}\right)\right| = \frac{1}{k}\left|\sin\left(\frac{x}{k+1}\right)\right| \le \frac{|x|}{k(k+1)} \le \frac{c}{k(k+1)} = \frac{c}{k^2+k} < \frac{c}{k^2}$$

Letting $M_k = \frac{c}{k^2}$, $\sum_{k=1}^{\infty} M_k$ converges by the p-series test, so by the Weierstrass M-Test, $\frac{1}{k}\sin\left(\frac{x}{k+1}\right)$ converes uniformly and absolutely on $[a, b]$.

We now determine whether $f(x) = \frac{1}{k}\sin\left(\frac{x}{k+1}\right)$ is a differentiable function.

Each term is differentiable, since sin is differentiable on all the reals.

We know that on any closed interval $\sum_{k=1}^{\infty} \frac{1}{k}\sin\left(\frac{x}{k+1}\right)$ converges uniformly.

Lastly, we consider the convergence of:

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}\sin\left(\frac{x}{k+1}\right)\right)' = \sum_{k=1}^{\infty}\frac{\cos\left(\frac{x}{k+1}\right)}{k(k+1)}$$

Again, by the M-Test, this converges, since:

$$\left|\frac{\cos\left(\frac{x}{k+1}\right)}{k(k+1)}\right| \leq \frac{1}{k(k+1)}$$

and $\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty}\frac{1}{k(k+1)}$ converges, since it is a telescoping series:

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k(k+1)} &= \sum_{k=1}^{\infty}\frac{1}{k} - \frac{1}{k+1}\\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \ldots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1}\\ &= 1 - \frac{1}{n+1}\end{aligned}$$

So the limit of partial sums goes to 1. In other words:

$$\sum_{k=1}^{\infty} M_k = 1$$

Thus, it follows that, from all the above, f is differentaible, and moreover:

$$f'(x) = \sum_{k=1}^{\infty}\frac{\cos\left(\frac{x}{k+1}\right)}{k(k+1)}$$

Moreover:

$$|f'(x)| \leq \sum_{k=1}^{\infty}\left|\frac{\cos\left(\frac{x}{k+1}\right)}{k(k+1)}\right| \leq \sum_{k=1}^{\infty} M_k = 1$$

so:

$$|f'(x)| \leq 1$$

Now we need to establish that $|f(x)| \leq |x|$. We employ the Mean Value Theorem, on some interval $[0, x]$:

$$|f(x) - f(0)| = |f'(c)||x - 0| \leq |x|$$

Since $f(0) = 0$, it follows that $|f(x)| \leq |x|$ as required.

4. **Prove that the geometric series:**

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

converges uniformly on any closed subinterval of $(-1, 1)$

Take $[a, b]$ to be the closed subinterval of $(-1, 1)$. Let $c = \max\{|a|, |b|\}$ Then, it follows that:

$$|x^k| = |x|^k \leq c^k$$

since by construction $0 \leq c < 1$, if $M_k = c_k$, then $\sum_{k=0}^{\infty} M_k$ must be convergent, since we have a geometric series with common ratio less than 1.

To show that:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

we'd need to consider the sequence of partial sums. For example, here.

5. **Let**

$$E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

(a) **Show that the series defining** $E(x)$ **converges uniformly on any closed interval** $[a, b]$**.**

Again we apply the M-Test. Let $c = \max\{|a|, |b|\}$

$$\left|\frac{x^k}{k!}\right| = \frac{|x|^k}{k!} \leq \frac{c^k}{k!}$$

Letting $M_k = \frac{c^k}{k!}$, $\sum_{k=0}^{\infty} M_k$ converges by the Ratio Test. By the M-Test, the series converges absolutely and uniformly.

(b) **Show that**

$$\int_a^b E(x)dx = E(b) - E(a)$$

for any $a, b \in \mathbb{R}$**.**

Since the series converges uniformly, we can integrate term by term:

$$\begin{aligned}
\int_a^b E(x)dx &= \int_a^b \sum_{k=0}^{\infty} \frac{x^k}{k!} dx \\
&= \sum_{k=0}^{\infty} \int_a^b \frac{x^k}{k!} dx \\
&= \sum_{k=0}^{\infty} \left[\frac{x^{k+1}}{(k+1)!}\right]_a^b \\
&= \sum_{k=0}^{\infty} \left[\frac{b^{k+1}}{(k+1)!} - \frac{a^{k+1}}{(k+1)!}\right] \\
&= \sum_{k=0}^{\infty} \frac{b^{k+1}}{(k+1)!} - \sum_{k=0}^{\infty} \frac{a^{k+1}}{(k+1)!}
\end{aligned}$$

If we define $m = k+1$:

$$\int_a^b E(x)dx = \sum_{m=1}^{\infty} \frac{b^m}{(m)!} - \sum_{m=1}^{\infty} \frac{a^m}{(m)!}$$

Moreover, notice that if $m = 0$, the first term of the series goes to 0, so:

$$\int_a^b E(x)dx = \sum_{m=0}^{\infty} \frac{b^m}{(m)!} - \sum_{m=0}^{\infty} \frac{a^m}{(m)!} = E(b) - E(a)$$

as required.

(c) **Show that E satisfies:**

$$E'(x) - E(x) = 0, \quad E(0) = 1$$

Again, from uniform convergence, we can differentiate elementwise *provided that certain conditions are met.*

Consider some open interval $(-a, a), a > 0$. Since $\frac{x^k}{k!}$ is a polynomial, it is differentiable on all $\mathbb{R}$, so in particular it is differentiable on $(-a, a)$.

Now, pick $x_0 = 0$. Clearly $x_0 \in (-a, a)$. Notice that:

$$E(x_0) = E(0) = 1$$

so $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges at x_0.

Finally, consider the series:

$$\sum_{k=0}^{\infty} \left(\frac{x^k}{k!}\right)'$$

It converges uniformly, since:

$$\begin{aligned}\sum_{k=0}^{\infty} \left(\frac{x^k}{k!}\right)' &= \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} \\ &= \sum_{m=0}^{\infty} \frac{x^m}{m!} \\ &= E(x)\end{aligned}$$

and $E(x)$ converges uniformly.

But by the above, it then means that:

$$E'(x) = \sum_{k=0}^{\infty} \left(\frac{x^k}{k!}\right)'$$

However, we have shown that $\sum_{k=0}^{\infty} \left(\frac{x^k}{k!}\right)' = E(x)$, so indeed:

$$E'(x) = E(x) \implies E'(x) - E(x) = 0$$

Moreover, we have seen that:

$$E(0) = 1$$

4 Workshop

1. **Let**

$$f_n(x) = \frac{xn^{\frac{1}{2}}}{1+nx^2}, \qquad x \in \mathbb{R}$$

Prove that f_n converges pointwise to the zero function. Is the convergence uniform over $\mathbb{R}$? As a hint, fix n, and think about $\sup_{x\in\mathbb{R}} |f_n(x)|$. Does this go to zero as $n \to \infty$?

We go from definition:

$$\begin{aligned}|f_n(x) - 0| &= \left|\frac{xn^{\frac{1}{2}}}{1+nx^2}\right| \\ &< \left|\frac{xn^{\frac{1}{2}}}{nx^2}\right| \\ &= \left|\frac{1}{n^{\frac{1}{2}}x}\right|\end{aligned}$$

Hence, $\forall \varepsilon > 0$ let $N = \frac{1}{x^2\varepsilon^2}$. Then, if $n \geq N$:

$$|f_n(x) - 0| = \left|\frac{1}{n^{\frac{1}{2}}x}\right| < \left|\frac{1}{N^{\frac{1}{2}}x}\right| = \left|\frac{1}{\frac{1}{x\varepsilon}x}\right| = \varepsilon$$

and so $f_n(x) \to 0$ pointwise.

> *As an alternative, in the solutions they simply note that:*
>
> $$|f_n(x)| \leq \left|\frac{xn^{\frac{1}{2}}}{nx^2}\right| = \frac{1}{n^{\frac{1}{2}}x} \to 0$$
>
> *by thinking of x as a constant.*

Now, recall that $f_n \to f$ uniformly is equivalent to showing that:

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \to 0$$

The key is to notice that we can rewrite $f_n(x)$ as:

$$f_n(x) = \frac{n^{\frac{1}{2}}}{\frac{1}{x} + nx}$$

Using calculus, we know that f_n achieves a local maximum when $\frac{1}{x} + nx$ is minimised, so we compute when this expression is minimised, by differentiating and equation to 0:

$$n - \frac{1}{x^2} = 0 \implies x = \frac{1}{n^{\frac{1}{2}}}$$

Thus, $\forall x \in \mathbb{R}$, if $x = \frac{1}{n^{\frac{1}{2}}}$, then $f_n(x)$ is maximised, so:

$$\sup_{x\in\mathbb{R}} |f_n(x) - 0| = f_n(n^{-\frac{1}{2}}) = \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n}} = \frac{1}{2}$$

so that

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \to \frac{1}{2}$$

Hence, f_n doesn't converge uniformly on $\mathbb{R}$.

2. **Let $f_n : [0,1) \to \mathbb{R}$ be defined by $f_n(x) = nx^n$. Show that $f_n \to 0$ pointwise, but $\int_0^1 f_n \to 1$. What does this demonstrate?**

> *The solutions are rather unhelpful in this question, since they show that $nx^n \to 0$ pointwise by citing FPM ... without citing any specific theorem or nugget of knowledge.*

To show that $nx^n \to 0$ pointwise, we will use the ratio test for sequences:

> *If (a_n) is a sequence of positive real numbers, such that:*
>
> $$\lim_{n\to\infty} \frac{a_{n+1}}{n} = L$$
>
> *and if $L < 1$, then the sequence (a_n) converges, and:*
>
> $$\lim_{n\to\infty} a_n = 0$$
>
> *For more on this: The Ratio Test for Sequences*

Now, since $x \in [0,1)$, for any x we have that nx^n is non-negative, so we compute:

$$\lim_{n\to\infty} \frac{(n+1)x^{n+1}}{nx^n} = x \lim_{n\to\infty} \frac{n+1}{n} = x$$

Indeed, $x < 1$, so it follows that for any $x \in [0,1)$, we have $nx^n \to 0$ pointwise on $[0,1)$.

We now consider the integral:

$$\int_0^1 nx^n \; dx = \left[\frac{n}{n+1}x^{n+1}\right]_0^1 = \frac{n}{n+1}$$

So it is easy to see that:

$$\lim_{n\to\infty} \int_0^1 nx^n \; dx = 1$$

Thus, we have shown that:

$$\lim_{n\to\infty} \int_0^1 nx^n \; dx \neq \int_0^1 \left(\lim_{n\to\infty} nx^n\right) \; dx$$

so it follows that f_n doesn't converge uniformly on $[0,1)$.

3. **Consider the sequence of functions on $\mathbb{R}$ given by:**

$$f_n(x) = \left(x - \frac{1}{n}\right)^2$$

Prove that it converges pointwise, and find the limit function. Is the convergence uniform on $\mathbb{R}$? Is the convergence uniform on bounded intervals?

In the solutions it is argued that $x - \frac{1}{n} \to x$ pointwise, so by properties of limits:

$$\lim_{n\to\infty}\left(x - \frac{1}{n}\right)^2 = \left(\lim_{n\to\infty} x - \frac{1}{n}\right)^2 = x^2$$

I do prefer to use the rigorous way, particularly since we will have to use it either way for uniform continuity.

We claim that $f_n \to x^2$ pointwise:

$$\begin{aligned}\left|\left(x - \frac{1}{n}\right)^2 - x^2\right| &= \left|x^2 - \frac{2x}{n} + \frac{1}{n^2} - x^2\right| \\ &= \left|\frac{1}{n^2} - \frac{2x}{n}\right| \\ &= \frac{\left|2x - \frac{1}{n}\right|}{n} \\ &\leq \frac{2|x| + \frac{1}{n}}{n} \\ &\leq \frac{2|x| + 1}{n}\end{aligned}$$

So if we pick $N > \frac{2|x|+1}{\varepsilon}$, we ensure that if $n \geq N$:

$$\left|\left(x - \frac{1}{n}\right)^2 - x^2\right| \leq \frac{2|x|+1}{n} \leq \frac{2|x|+1}{N} < \frac{2|x|+1}{\frac{2|x|+1}{\varepsilon}} = \varepsilon$$

so $f_n(x) \to x^2$ pointwise.

However, it doesn't converge uniformly on $\mathbb{R}$. Indeed, consider the sequence:

$$\sup_{x\in\mathbb{R}}\left|\left(x - \frac{1}{n}\right)^2 - x^2\right| = \sup_{x\in\mathbb{R}} \frac{\left|2x - \frac{1}{n}\right|}{n}$$

But notice, as x ranges over $\mathbb{R}$, $\frac{\left|2x - \frac{1}{n}\right|}{n}$ keeps getting bigger and bigger, so the supremum is not even defined, so certainly $\sup_{x\in\mathbb{R}}\left|\left(x - \frac{1}{n}\right)^2 - x^2\right| \not\to 0$, and so, $f_n(x)$ won't converge uniformly to x^2 on $\mathbb{R}$.

An alternative for the solutions is to consider:

$$|f_n(n) - n^2| = \frac{\left|2n - \frac{1}{n}\right|}{n} \geq 1$$

so $|f_n(n) - n^2| \not\to 0$ *(this corresponds to the third equivalence of uniform convergence, which states that a sequnece is uniformly convergent if we can bound* $|f_n(x) - f(x)|$ *with a sequence* (a_n) *such that* $a_n \to 0$ *for any* $x \in \mathbb{R}$*)*

If however we consider a bounded interval $I = [a, b]$, f_n does converge uniformly on I. Indeed, let $M = \max\{|a|, |b|\}$. Then, $\forall x \in I, |x| \leq M$ so it follows that:

$$\left|\left(x - \frac{1}{n}\right)^2 - x^2\right| \leq \frac{2|x| + 1}{n} \leq \frac{2M + 1}{n}$$

so for any $x \in [a, b]$, if we pick $N > \frac{2M+1}{\varepsilon}$ then whenever $n \geq N$:

$$\left|\left(x - \frac{1}{n}\right)^2 - x^2\right| < \varepsilon$$

so $f_n(x) \to x^2$ uniformly on $[a, b]$.

Otherwise just notice that:

$$0 \leq \sup_{x \in I} |f_n(x) - x^2| \leq \sup_{x \in I} \frac{2M + 1}{n} = \frac{2M + 1}{n}$$

So by squeeze theorem:

$$\sup_{x \in I} |f_n(x) - x^2| \to 0$$

4. **Let:**

$$f_n(x) = x - x^n$$

Prove that f_n converges pointwise on $[0, 1]$, and find the limit function. Is the convergence uniform on $[0, 1]$? Is the convergence uniform on $[0, 1)$?

Here we need to be careful: $f_n(x)$ *tends to different values depending on the value of* x*, and this determines how we think of its convergence.*

We claim $f_n(x) \to f(x)$ pointwise on $[0, 1]$, where:

$$f(x) = \begin{cases} x, & x \in [0, 1) \\ 0, & x = 1] \end{cases}$$

If $x = 1$, then:

$$x - x^n = x - x = 0, \qquad \forall n \in \mathbb{N}$$

so clearly $x - x^n \to 0$

Now consider $x \in [0, 1)$

> *In the solutions they just claim that since $x \in [0,1)$ then $x^n \to 0$, so $x - x^n \to x$. I give a proof from first principles.*

We claim that $x_n \to x$:

$$|x - x^n - x| = |x^n| = |x|^n = x^n$$

If $\varepsilon > 0$, then we pick $N > \log_x(\varepsilon)$, and if $n \geq N$ then:

$$x^n \leq x^N < x^{\log_x(\varepsilon)} = \varepsilon$$

so it follows that $x - x^n \to x$ pointwise on $[0, 1)$

The convergence won't be uniform on $[0, 1]$. This is because $f_n(x)$ is continuous over the whole interval (since x and x^n are continuous, but $f(x)$ is not continuous at $x = 1$.

Even if we consider the interval $[0, 1)$, $f_n(x)$ doesn't converge uniformly to x. Consider:

$$\sup_{x \in [0,1)} |f_n(x) - f(x)| = \sup_{x \in [0,1)} (x^n) = 1$$

so the sequence $\sup_{x \in [0,1)} |f_n(x) - f(x)|$ doesn't converge to 0.

> *To see why, let $A = \{x^n \mid x \in [0,1)\}$ where $n \in \mathbb{N}$. Then, we claim sup$(A) = 1$.*
> *Since $x < 1$ then $x^n < 1^n = 1$ so 1 is an upper bound for A.*
> *Now assume that sup$(A) = t < 1$. Then pick $\varepsilon > 0$ such that $t + \varepsilon \in [0,1)$ (for example $\varepsilon = \frac{1-t}{2}$). Then, $(t+\varepsilon)^n \in A$, but clearly $t < (t+\varepsilon)^n$, which contradicts the fact that t is a supremum.*
> *Thus, no supremum can exist smaller than 1, and so, sup$(A) = 1$.*

5. **Consider the sequence of functions defined on $[0, \infty)$ defined by:**

$$f_n(x) = \frac{x^n}{1 + x^n}$$

Prove that f_n converges pointwise, and find the limit function. Is the convergence uniform on $[0, \infty)$? Is the convergence uniform on bounded intervals of the form $[0, a)$?

We claim that $f_n(x) \to f$ pointwise on $[0, \infty)$ where:

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ \frac{1}{2}, & x = 1 \\ 1, & x > 1 \end{cases}$$

In the solutions this is dispatched in a fairly succint manner.
When $x \in [0,1)$, we have that $x^n \to 0$, so $\frac{x^n}{1+x^n} \to \frac{0}{1+0} = 0$.
When $x = 1$, we have that $\frac{x^n}{1+x^n} = \frac{1}{2} \to \frac{1}{2}$.
Finally, when $x \in (1,\infty)$, we get that $\frac{x^n}{1+x^n} = \frac{1}{x^{-n}+1} \to 1$ by using the fact that $x^{-n} \to 0$.
Hence, $\forall x \in [0,\infty)$ we get that $f_n(x) \to f(x)$.
Below I'll get into all the gory details, except for when $x = 1$, since that is identically explained.

We can fix $x \in [0,1)$, such that:

$$f(x) = 0$$

Then, we consider:

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \frac{x^n}{1+x^n} \right| \\ &= \frac{x^n}{1+x^n} \\ &\leq x^n \end{aligned}$$

where in the last step we have made use of the fact that:

$$x \geq 0 \implies 1 + x^n \geq 1$$

If $x = 0$, then we get that $|f_n(x) - f(x)| \leq 0 < \varepsilon$ for any $\varepsilon > 0$ and for any $n \geq N \in \mathbb{N}$, so $f_n(x)$ converges pointwise to $f(x)$ at $x = 0$.

We thus consider the remaining cases in which $0 < x < 1$. But then, notice that since $0 < x < 1$, there exists a $y \in \mathbb{R}, y > 1$ such that:

$$x = \frac{1}{y}$$

But then, $\forall \varepsilon > 0$, if N is the smallest natural number larger than $\log_y\left(\frac{1}{\varepsilon}\right)$, if $n \geq N$, then:

$$\begin{aligned} |f_n(x) - f(x)| &\leq x^n \\ &= \frac{1}{y^n} \\ &\leq \frac{1}{y^N} \\ &= \frac{1}{y^{\log_y\left(\frac{1}{\varepsilon}\right)}} \\ &= \varepsilon \end{aligned}$$

So it follows that $f_n(x)$ is pointwise convergent to $f(x)$, given that $x \in [0,1)$

If we fix $x \in (1,\infty)$, then:

$$f(x) = 1$$

Then, consider:

$$\begin{aligned}|f_n(x) - f(x)| &= \left|\frac{x^n}{1+x^n} - 1\right| \\ &= \left|\frac{x^n}{1+x^n} - \frac{1+x^n}{1+x^n}\right| \\ &= \left|\frac{-1}{1+x^n}\right| \\ &= \frac{1}{1+x^n} \\ &< \frac{1}{x^n}\end{aligned}$$

where in the last step we have made use of the fact that:

$$x > 1 \implies 1 + x^n > x^n$$

But then, $\forall \varepsilon > 0$, if N is the smallest natural number larger than $\log_x\left(\frac{1}{\varepsilon}\right)$, if $n \geq N$, then:

$$\begin{aligned}|f_n(x) - f(x)| &< \frac{1}{x^n} \\ &\leq \frac{1}{x^N} \\ &= \frac{1}{x^{\log_x\left(\frac{1}{\varepsilon}\right)}} \\ &= \varepsilon\end{aligned}$$

So it follows that $f_n(x)$ is pointwise convergent to $f(x)$, given that $x \in (1, \infty)$

In other words, we have shown that, $\forall x \in [0, \infty)$, $f_n(x)$ converges pointwise to $f(x)$

However, by Theorem 2.1 of the notes, it is not the case that $f_n(x)$ converges uniformly to $f(x)$ on $[0, \infty)$, since on this interval each f_n is continuous at all points, but $f(x)$ is not continuous at all points of the interval, as $x = 1$ is a point of discontinuity. This can be seen by the fact:

$$\lim_{x \to 1^-} f(x) = 0 \neq 1 = \lim_{x \to 1^+} f(x)$$

Nonetheless, convergence will be uniform on bounded intervals of the form $[0, a)$, provided that $0 < a < 1$.

Indeed, if $a > 1$, then $f_n(x)$ will converge pointwise to $f(x)$, so we again have discontinuity. We thus need to consider the case $a \leq 1$. Notice, if $x \in [0, a)$ then $f(x) = 0$.

Consider:

$$\sup_{x \in [0,a)} |f_n(x) - f(x)| = \sup_{x \in [0,a)} \left|\frac{x^n}{1+x^n}\right| == \sup_{x \in [0,a)} \left|\frac{1}{\frac{1}{x^n} + 1}\right|$$

Now, on $[0, a)$, $\frac{1}{\frac{1}{x^n}+1}$ is an increasing function (since as x increases, $\frac{1}{x^n}$ gets smaller, so $\frac{1}{\frac{1}{x^n}+1}$ increases). Thus:

$$\sup_{x \in [0,a)} |f_n(x) - f(x)| = \frac{1}{\frac{1}{a^n} + 1}$$

If we allow for the possibility of $a = 1$, then:

$$\frac{1}{\frac{1}{a^n} + 1} \to \frac{1}{2} \not\to 0$$

so if $a \leq 1$, convergence won't be uniform.

However, if $a < 1$, we know that $a^n \to 0$:

$$\frac{1}{\frac{1}{a^n} + 1} \to 0$$

Thus, on the interval $[0, a)$ with $a < 1$ we have that $f_n(x)$ is uniformly convergent.

6. **Let:**

$$f_n(x) = nx(1 - x^2)^n, \qquad x \in [0, 1]$$

Prove that f_n converges pointwise on $[0, 1]$ and find the limit function. Is the convergence uniform on $[0, 1]$? As a hint, consider the integrals $\int_0^1 f_n$. Is the convergence uniform on $[a, 1]$, where $0 < a < 1$?

Notice, if $x = 1$, then $f_n(x) = 0$. If $x \in [0, 1)$, then $1 - x^2 \in [0, 1)$, so $(1 - x^2)^n \to 0$ and so $f_n(x) \to 0$. Hence, $f_n(x) \to 0$ pointwise on $[0, 1]$.

We claim that $f_n(x) \to 0$:

$$|nx(1 - x^2)^n - 0| = nx(1 - x^2)^n \leq n(1 - x^2)^n$$

But if we let $y = 1 - x^2$, this is just ny^n, which we showed above converges pointwise to 0. Thus, $\forall \varepsilon > 0$, if $n \geq N$, we have $ny^n < \varepsilon$ and so:

$$|nx(1 - x^2)^n - 0| < \varepsilon$$

so $nx(1 - x^2)^n \to 0$ pointwise on $[0, 1]$.

To check if this convergence is uniform, we use integrals:

$$\int_0^1 nx(1 - x^2)^n \, dx$$

if we apply the substitution $y = 1 - x^2$ then $\frac{dy}{dx} = -2x$ so:

$$\int_0^1 nx(1 - x^2)^n \, dx = \frac{n}{2} \int_0^1 y^n \, dy = \frac{n}{2(n+1)} \left[y^{n+1}\right]_0^1 = \frac{n}{2(n+1)}$$

But then notice that:

$$\lim_{n \to \infty} \int_0^1 nx(1 - x^2)^n \, dx = \frac{1}{2}$$

but:

$$\int_0^1 \lim_{n \to \infty} (nx(1 - x^2)^n) \, dx = \int_0^1 0 \, dx = 0$$

Thus, f_n isn't uniformly convergent on $[0, 1]$.

We now consider the interval $[a, 1]$ where $0 < a < 1$.

Important to note: up to now we have been using Theorem 2.2 of the notes, whereby the assumption of uniform continuity implies the fact that the limit of the integrals is the integral of the limits. However, this theorem is not sufficient to prove uniform continuity, just to disprove it.
In this case, we would have:

$$\int_a^1 nx(1-x^2)^n\, dx = \frac{n}{2}\int_0^{1-a^2} y^n\, dy = \frac{n}{2(n+1)}\left[y^{n+1}\right]_0^{1-a^2} = \frac{n}{2(n+1)}(1-a^2)^n \to 0$$

since $1-a^2 < 1$. Hence, Theorem 2.2 won't help here.

For uniform continuity we go from definition. Notice, if $x \in [a,1]$, then $1-x^2 \leq 1-a^2$. Then, we have that we can bound $|f_n(x) - 0|$ by the sequence:

$$n(1-a^2)^n$$

where we have used the fact that $\forall x \in [a,1], n(1-a^2)^n \geq nx(1-a^2)$.
Moreover, since $n(1-a^2)^n \to 0$ for any x, it follows that $f_n(x)$ converges uniformly on $[a,1]$.

7. **Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions which converges uniformly to a function $f : \mathbb{R} \to \mathbb{R}$. Let x_n be a sequence of real numbers which converge to $x \in \mathbb{R}$. Show that $f_n(x_n) \to f(x)$.**

This one is direct from the solutions; I tried using only the sequence definition, but for some reason (not explained since the homework feedback is trash) my method wasn't correct. The proof is rather simple so it is quite shameful that I could come up with it.

We first note that f will be continuous, since each f_n is continuous, and $f_n \to f$ uniformly.

By uniform continuity, we have that $\forall \varepsilon > 0$ and $\forall y \in \mathbb{R}$ we have that $\exists N \in \mathbb{N}$ such that if $n \geq N$:

$$|f_n(y) - f(y)| < \frac{\varepsilon}{2}$$

In particular, this must be true for $y = x_n$:

$$|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{2}$$

Moreover, by continuity of f, since $x_n \to x$ we know that $f(x_n) \to f(x)$ so in particular, $\exists M \in \mathbb{N}$ such that if $n \geq M$:

$$|f(x_n) - f(x)| < \frac{\varepsilon}{2}$$

Now, let $n \geq \max\{N, M\}$. Then:

$$|f_n(x_n) - f(x)| = |f_n(x_n) - f(x_n) + f(x_n) - f(x)| <\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \varepsilon$$

so indeed:

$$f_n(x_n) \to f(x)$$