Honours Analysis - Week 3 - Continuity, Differentiability, and Pointwise/Uniform Convergence of Functions

Antonio León Villares

October 2021

Contents

1 Continuity of Functions on $\mathbb{R}$

1.1 Defining Continuity of Functions

- **What is function continuity at a point?**
  - consider a function f:
$$f : dom(f) \to \mathbb{R}, \qquad dom(f) \subset \mathbb{R}$$
  - f is **continuous** at $a \in dom(f)$ if there is a **sequence** (x_n) with all its terms in $dom(f)$, such that:
$$\lim_{n\to\infty} x_n = a$$
  - if f is continuous at a, then
$$\lim_{n\to\infty} f(x_n) = f(a)$$
- **When is a function continuous on an interval?**
  - whenever f is continuous at every point of some interval $S \subset dom(f)$

1.2 Theorem: Epsilon-Delta Definition of Continuity

Let f be a function defined on $dom(f) \subset \mathbb{R}$, such that:

$$f : dom(f) \to \mathbb{R}$$

*Then, f is **continuous** at a point $a \in dom(f)$ **if and only if** $\forall \varepsilon > 0$, $\exists \delta > 0$ such that, **if** $x \in dom(f)$ **and** $|x - a| < \delta$, **then**:*

$$|f(x) - f(a)| < \varepsilon$$

This definition is equivalent to the one above. [Theorem 1.12]

Proof: Equivalence of Continuity Definitions. Assume that f is such that if $x \in dom(f)$ and $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$, for all $\varepsilon > 0$.

Take any sequence (x_n) in the domain of f such that $x_n \to a$. Then it follows by the definition of the limit that for any $\delta > 0$, we can always find some $N \in \mathbb{N}$ such that if $n > N$ then:

$$|x_n - a| < \delta$$

But this satisfies our assumption about f, so it must be the case that if $n > N$ we have:

$$|f(x_n) - f(a)| < \varepsilon$$

But then, it follows that $f(x_n) \to f(a)$. Hence, we have shown that if f satisfies the epsilon-delta condition, it satisfies the original definition of continuity.

Now, assume that continuity according to the original definition follows, but the epsilon-delta condition is not satisfied.

This means that $\exists \varepsilon > 0$ such that $\forall \delta > 0$, if $x \in dom(f)$ and $|x-a| < \delta$, then:

$$|f(x) - f(a)| \geq \varepsilon$$

What this means is that for each $n \in \mathbb{N}$, we have $x_n \to a$, but it can't be the case that $f(x_n) \to f(a)$ since $|f(x) - f(a)| \geq \varepsilon$. But this shows that f can't be continuous at a, which contradicts our initial assumption. □

1.3 Properties of Continuous Functions

- **Is the combination of continuous functions always continuous?**
  - in general yes. Let f, g be functions on the $\mathbb{R}$, continuous on the same interval. Let $\alpha \in \mathbb{R}$. Then, the following are continuous:
    * αf
    * $f + g$
    * fg

1.4 Theorem: Composition of Continuous Functions is Continuous

If $A, B \subseteq \mathbb{R}$, and:

$$f : A \to \mathbb{R}$$

$$g : B \to \mathbb{R}$$

and:

$$f(A) \subseteq B$$

then if f, g are continuous, $g \circ f : A \to \mathbb{R}$ is continuous.
In general, if f is continuous at a, and g is continuous at $f(a)$, then, $g \circ f$ is continuous at a. [Theorem 1.11]

Proof: Continuity of Composition at a Point. Since f is continuous at a, there exists a sequence (a_n) in $dom(f)$ such that $a_n \to a$ implies $f(a_n) \to f(a)$. Moreover, g is continuous at $f(a)$, which implies that $g(f(a_n)) \to g(f(a))$ (here $f(a_n)$ is our sequence in $dom(g)$). But then we have shown that if $a_n \to a$, then $g(f(a_n)) \to g(f(a))$, which means that $g \circ f$ must be continuous at a.

For more examples, this Stack Exchange post. □

1.5 Theorem: Bolzano's Theorem

*Bolzano's Theorem states that if a function is **continuous** on a **closed interval**, and at some point of the interval the function **changes signs**, then the function must have been equal to 0 at some point.*

> *Let f be a **continuous** function on the **closed** interval $[a, b]$, and suppose that $f(a)f(b) < 0$ (that is, $f(a)$ and $f(b)$ have opposite signs). Then, $\exists c \in (a, b)$ such that $f(c) = 0$. [Theorem 1.13]*

We give 2 proofs: one from the notes, and one from Milefoot.

Proof: Bolzano's Theorem (Notes). Assume that $f(a) < 0, f(b) > 0$ (the other case is analogous).

Define the set S as:

$$S = \{x | x \in [a, b], f(x) \leq 0\}$$

Clearly $a \in S$, as $f(a) < 0$, so S is **non-empty**. Moreover, S is **bounded above** by b. Thus, by the Completeness Axiom, $\exists c \in \mathbb{R}$ such that $supS = c$.

We claim that $f(c) = 0$

To prove this, we first show that $a < c < b$ (so $c \in (a, b)$). Since f is continuous on $[a, b]$, we know that if x gets arbitrarily close to a, then $f(x)$ will be arbitrarily close to $f(a)$. In other words, if $\varepsilon = \frac{|f(a)|}{2}$, we can always find some $\delta > 0$ such that if $x \in [a, a + \delta)$, then:

$$|f(x) - f(a)| < \varepsilon = \frac{|f(a)|}{2}$$

But then:

$$f(a) - \frac{|f(a)|}{2} < f(x) < f(a) + \frac{|f(a)|}{2} \implies f(x) < \frac{f(a)}{2} < 0$$

since $f(a) + \frac{|f(a)|}{2} = \frac{f(a)}{2}$, as $f(a)$ is negative, but $|f(a)|$ is positive. But then we are saying that if $x \in [a, a + \delta)$, then $f(x) < 0$. Thus, we have shown that there are elements in S greater than a, so in particular the supremum must be greater than a:

$$c \geq a + \delta > a$$

Using similar arguments, we can show that $c \leq b - \delta < b$. Thus, it follows that $a < c < b$ as required.

Now, since f is continuous on $[a, b]$ it is continuous on c, so in particular, by the definition of continuity, for all $\varepsilon > 0$, we can find a $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

Now, let $x_1 \in S \cap (c - \delta, c)$. Since c is the supremum of S, by the approximation property of the supremum, there must exist at least one such x_1. Thus, we have $f(x_1) \leq 0$.

Similarly, let $x_2 \in S \cap (c, c + \delta)$. Since c is the supremum of S, and $x_2 > c$, it follows that $f(x_2) > 0$, since $x_2 \notin S$.

Since $|f(x) - f(c)| < \varepsilon$, then $f(x) - \varepsilon < f(c) < f(x) + \varepsilon$. Then, using x_1 and x_2, and the fact that $f(x_1) \leq 0$ and $f(x_2) > 0$, it follows that:

$$-\varepsilon < f(x_2) - \varepsilon < f(c) < f(x_1) + \varepsilon < \varepsilon$$

Thus, we have shown that $|f(c)| < \varepsilon$ for any ε, which means that $f(c) = 0$, as required.

□

Proof: Bolzanos Theorem (Milefoot). As above, assume that $f(a) < 0, f(b) > 0$ (the other case is analogous).

Define the set S as:

$$S = \{x \mid x \in [a, b], f(x) \leq 0\}$$

Clearly $a \in S$, as $f(a) < 0$, so S is **non-empty**. Moreover, S is **bounded above** by b. Thus, by the Completeness Axiom, $\exists c \in \mathbb{R}$ such that $supS = c$.

There are 3 possibilities:

- $f(c) > 0$
- $f(c) = 0$
- $f(c) < 0$

We consider each of these separately.

Assume $f(c) > 0$. From continuity, we can fix $\varepsilon_1 = \frac{f(c)}{2}$. Then, we can find some $\delta_1 > 0$, such that if $|x - c| < \delta_1$, then:

$$|f(x) - f(c)| < \frac{f(c)}{2}$$

But this then implies that:

$$\frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}$$

which means that if $x \in (c - \delta_1, c + \delta_1)$, $f(x) > 0$. This however contradicts the fact that c is the supremum of S, as for example $c - \frac{\delta_1}{2}$ is less than c, and still an upper bound of S (as $f\left(c - \frac{\delta_1}{2}\right) > 0$).

From similar arguments, it is easy to see that if $f(c) < 0$, then if $x \in (c - \delta_2, c + \delta_2$, we'd find that $f(x) < 0$, which would contradict the fact that c is the supremum, as $c + \frac{\delta_2}{2}$ is an element larger than c, which is still in S.

It must thus be the case that $f(c) = 0$, as required.

If $f(a) > 0$ and $f(b) < 0$, we can just define the function $g(x) = -f(x)$, where g is continuous on $[a, b]$, $g(a) < 0$, $g(b) > 0$, from which it follows that we can find some $c \in (a, b)$ such that $g(c) = 0$. But then, $f(c) = -g(c) = 0$, as required.

□

1.6 Corollary: The Intermediate Value Theorem

The Intermediate Value Theorem can be thought of as a corollary of ***Bolzano's Theorem*** *(or viceversa). It states that if a function is continuous on some interval, then the function must attain every value between the bounds of the interval. The following proof is also taken from Milefoot. In the notes, the Intermediate Value Theorem is taken as Bolzano's Theorem.*

> *Let f be a* ***continuous*** *function on the* ***closed interval*** *$[a, b]$. If $L \in \mathbb{R}$, and L is strictly between $f(a)$ and $f(b)$, then $\exists c \in (a, b)$, such that $f(c) = L$. [Not in Notes]*

Proof: The Intermediate Value Theorem. Let f be a continuous function on $[a, b]$. Let $L \in \mathbb{R}$, such that L is strictly between $f(a)$ and $f(b)$.

Define the function $g(x) = f(x) - L$. g is continuous on $[a, b]$. Since $L \neq f(a)$ and $L \neq f(b)$, and L is strictly greater than one of them, and strictly smaller than the other, it must be the case that $g(a)g(b) < 0$.

But then, by Bolzano's Theorem, $\exists c \in \mathbb{R}$ such that $c \in (a, b)$, and $g(c) = 0$.

But if $g(c) = 0$, this then implies:

$$f(c) - L = 0 \implies f(c) = L$$

as required.

□

1.7 Theorem: The Extreme Value Theorem

The Extreme Value Theorem says that any continuous function on a closed, bounded interval attains its maximum and minimum on said interval.

*Let $a < b$ be **real** numbers.*
Let

$$f : [a, b] \to \mathbb{R}$$

*be a **continuous** function on $[a, b]$.*
Then $\exists c, d \in [a, b]$ such that:

$$f(c) = inf\{f(x) \mid x \in [a, b]\}$$

$$f(d) = sup\{f(x) \mid x \in [a, b]\}$$

*Thus, f is **bounded** on the interval $[a, b]$. More importantly, it attains its **minimum** value at some $c \in [a, b]$, and it attains its **maximum** value at some $d \in [a, b]$. [Theorem 1.14]*

The Extreme Value Theorem. We rely on the Bolzano-Weierstrass Theorem:

*Every **bounded** sequence of real numbers has a **convergent subsequence**. [Theorem 1.5]*

Moreover, we only prove the existence of the point d where the maximum is attained, as the case for the minimum is analogous.

Consider the set of all values of the continuous function f:

$$S = \{f(x) \mid x \in [a, b]\}$$

Clearly the set is non-empty.

Moreover, the set can either be bounder or unbounded.

If the set is **bounded**, then $\exists M \in \mathbb{R}$, such that $supS = M$. But then, by the approximation property of the supremum, for any $n \in \mathbb{N}$ it is easy for us to find a sequence of points $(d_n), d_n \in [a, b]$, such that:

$$M - \frac{1}{n} < f(d_n) \leq M$$

But then it follows that:

$$\lim_{n \to \infty} f(d_n) = M$$

Alternatively, if S were **unbounded**, then for any $n \in \mathbb{N}$ it is possible for us to construct $d_n \in [a, b]$ such that $f(d_n) > n$, which then implies that:

$$\lim_{n \to \infty} f(d_n) = \infty$$

Either way, we are capable of constructing a sequence (d_n), which is bounded (as $\forall n \in \mathbb{N}, d_n \in [a, b]$).

Then, by the **Bolzano-Weierstrass Theorem**, we can construct a subsequence (d_{n_k}) which converges to some $d \in [a, b]$. We claim that d is the point at which the maximum of f is attained.

Since f is continuous on $[a, b]$, f is continuous at d, so $\forall \varepsilon > 0$, we can always find some $\delta > 0$, such that if $|x - d| < \delta$, then $|f(x) - f(d)| < \varepsilon$.

Moreover, since d_{n_k} converges to d, $\forall \delta > 0$, there exists some $N \in \mathbb{N}$, such that $\forall n_k > N$, we have:

$$|d_{n_k} - d| < \delta$$

But using this, it follows from continuity that:

$$|f(d_{n_k}) - f(d)| < 1$$

where we have set $\varepsilon = 1$, and this is valid by setting $x = d_{n_k}$. But this then means that:

$$f(d_{n_k}) < f(d) + 1$$

Thus, we have shown that $f(d_{n_k})$ is bounded. But then, it can't be the case that $\lim_{n \to \infty} f(d_n) = \infty$, so we must have $\lim_{n \to \infty} f(d_n) = M$.

Now, by continuity, since we have a sequence (d_{n_k}) such that $d_{n_k} \to d$, it must be the case that:

$$f(d_{n_k}) \to f(d)$$

But if a sequence converges, all its subsequences converge to the same point. Since $\lim_{n \to \infty} f(d_n) = M$, then also:

$$f(d_{n_k}) \to M$$

From the uniqueness of the limit, we then have:

$$f(d) = M$$

as required.

$\square$

2 Differentiability of Functions on $\mathbb{R}$

- **When is a function differentiable at a point?**
  - f is differentiable at a if $\exists \delta > 0$ such that $(a-\delta, a+\delta) \subset dom(f)$ and the limit:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

  - the value of this limit is denoted as $f'(a)$.

The following are from FPM:

- ***Differentiability at a Point***: *let $f : I \to \mathbb{R}$ be a function, and let I be an open interval. If $x_0 \in I$, then f is differentiable at x_0 if the following limit is defined:*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

  Alternatively, if the following limit is defined:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

  - *we denote the value of the limit with $f'(x_0)$, the derivative of f at x_0*
- ***Right-Hand Derivative***:
  $f'(x_0^+) = \lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}, \ f : [x_0, b) \to \mathbb{R}$
- ***Left-Hand Derivative***:
  $f'(x_0^-) = \lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}, \ f : (a, x_0] \to \mathbb{R}$
- ***Derivative as One-Sided Limits***: *f is differentiable at x_0 if $f'(x_0^+) = f'(x_0^-)$*
- ***Continuity and Differentiability***: *if f is differentiable at x_0, for $x_0 \in I$, where I is an open interval, then f is continuous at x_0*

3 Uniform Convergence

3.1 Pointwise Convergence of Sequences of Functions

- **When does a sequence of functions converge pointwise?**
  - let E be a non-empty subset of $\mathbb{R}$
  - consider a sequence of functions:

$$f_n : E \to \mathbb{R}$$

- this sequence **converges pointwise** on E **if and only if**

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for each $x \in E$.

- that is, f_n **converges pointwise** on E **if and only if** $\forall \varepsilon > 0, \forall x \in E$ there is $N \in \mathbb{N}$ (which might depend on ε **and** x) such that, if $n \geq N$, then:

$$|f_n(x) - f(x)| < \varepsilon$$

- **If f_n are continuous, and $f_n \to f$ pointwise, is f continuous?**
  - Not necessarily. For example, let $f_n(x) = x^n$ on $[0, 1]$. Clearly, if $x = 1$, $\lim_{n\to\infty} f_n(x) = 1$. But if $x \in [0, 1)$, then $\lim_{n\to\infty} f_n(x) = 0$. Thus, each f_n is continuous, but f isn't.

Remark. The pointwise limit of continuous (respectively, differentiable) functions is not necessarily continuous (respectively, differentiable).

Proof. Let $f_n(x) = x^n$ and set

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1. \end{cases}$$

Then $f_n \to f$ pointwise on $[0, 1]$, each f_n is continuous and differentiable on $[0, 1]$, but f is neither differentiable nor continuous at $x = 1$. ■

Remark. The pointwise limit of integrable functions is not necessarily integrable.

Proof. Set

$$f_n(x) = \begin{cases} 1 & x = \frac{p}{m} \in \mathbb{Q}, \text{ if } x \text{ can be written in a reduced form, where } m \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

for $n \in \mathbb{N}$ and

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \to f$ pointwise on $[0, 1]$, each f_n is integrable on $[0, 1]$ (with integral zero), but f is not integrable on $[0, 1]$. ■

Remark. There exist differentiable functions f_n and f such that $f_n \to f$ pointwise on $[0,1]$ but

$$\lim_{n\to\infty} f_n'(x) \neq \left(\lim_{n\to\infty} f_n(x)\right)' \tag{4}$$

for $x = 1$.

Proof. Let $fn_(x) = \frac{x^n}{n}$ and set $f(x) = 0$. Then $f_n \to f$ pointwise on $[0,1]$, each f_n is differentiable with $f_n'(x) = x^{n-1}$. Thus the left side of (4) is 1 at $x = 1$ but the right side of (4) is zero. ■

Remark. There exist continuous functions f_n and f such that $f_n \to f$ pointwise on $[0,1]$ but

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right) dx. \tag{5}$$

Proof. Let $f_1(x) = 1$ and, for $n > 1$, let f_n be a sequence of functions whose graphs are triangles with bases $^2/_n$ and altitudes n (see Figure 1). By the point-slope form, formulas for these f_n's can be given by

$$f_n(x) = \begin{cases} n^2x & 0 \le x < {}^1/_n \\ 2n - n^2x & {}^1/_n \le x < {}^2/_n \\ 0 & {}^2/_n \le x \le 1. \end{cases}$$

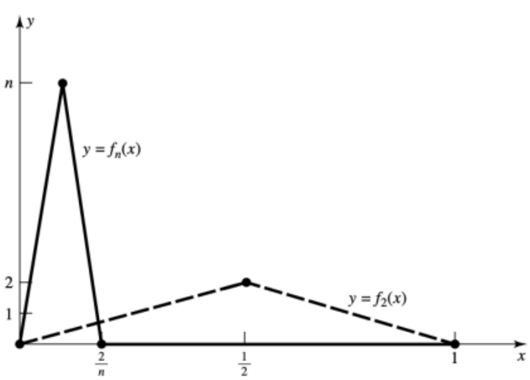

Figure 1

Then $f_n \to 0$ pointwise on $[0,1]$ and, since the area of a triangle is one-half base times altitude, $\int_0^1 f_n(x)dx = 1$ for all $n \in \mathbb{N}$. Thus, the left side of (5) is 1, but the right side is zero. ■

3.2 Uniform Convergence of Sequences of Functions

Pointwise convergence is a ***weak*** *form of convergence, so we define a stronger form.*

- **What is uniform convergence of functions?**

- let E be a non-empty subset of $\mathbb{R}$.
- consider a sequence of functions:

$$f_n : E \to \mathbb{R}$$

- this sequence **converges uniformly** on E to a function f **if and only if** $\forall \varepsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n \geq N$, then:

$$|f_n(x) - f(x)| < \varepsilon$$

  and this is true for any $x \in E$

- **What is the key difference between uniform and pointwise convergence?**
  - in **pointwise convergence**, N could depend on **both** x and ε, so for different x we could use different N
  - **uniform convergence** requires that one N is valid *for any* x
- **How can we visualise uniform and pointwise convergence?**
  - if a sequence of functions are uniformly convergent, they are all around the limit f, plus minus ε
  - for pointwise convergent sequences, this only happens eventually

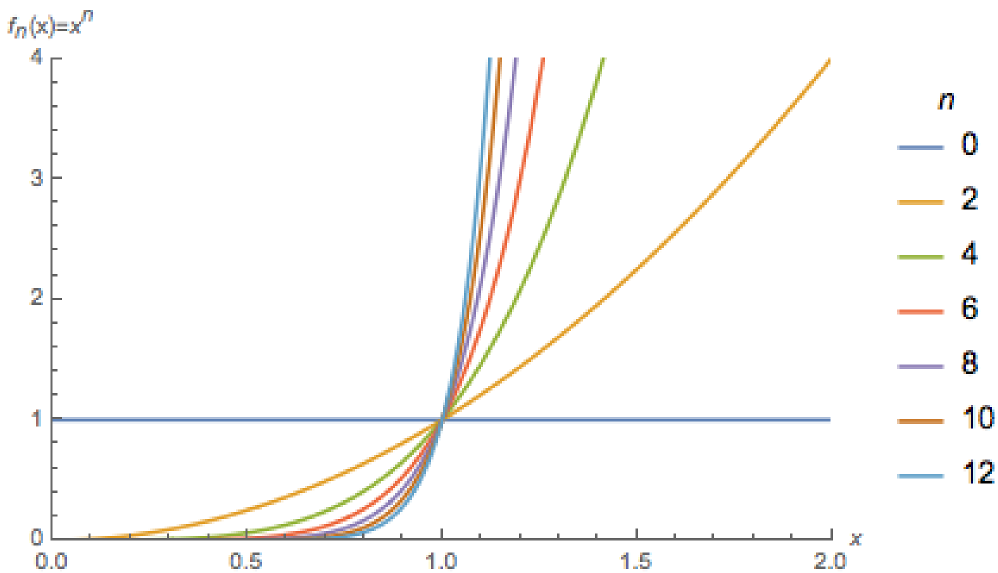

Figure 1: Pointwise convergent: eventually they get close together

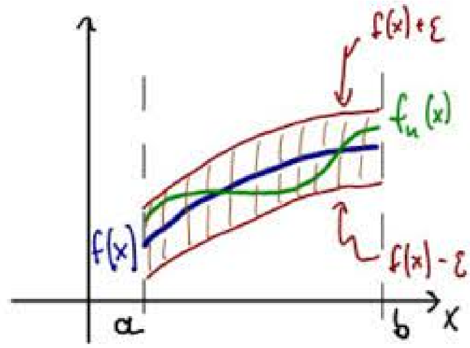

Figure 2: Uniform convergent: functions always within a bound

3.3 Proposition: Equivalent Definitions of Uniform Convergence

Let $f_n : E \to \mathbb{R}$ be a sequence of functions. Let $f : E \to \mathbb{R}$ be a function. Then, the following are equivalent:

1. $f_n \to f$ *uniformly on* E
2. $\sup_{x \in E} |f_n(x) - f(x)| \to 0$ *as* $n \to \infty$
   - *in other words,* $\forall \varepsilon > 0$ *we can find some* $N \in \mathbb{N}$ *such that if* $n \geq N$*, then:*
   $$\sup_{x \in E} |f_n(x) - f(x)| < \varepsilon$$
   - *here,* $\sup_{x \in E} |f_n(x) - f(x)|$ *is the sequence formed by:*
     - *for* $n = 1$*, consider the supremum of* $|f_1(x) - f(x)|$ *over all values of* x
     - *for* $n = 2$*, consider the supremum of* $|f_2(x) - f(x)|$ *over all values of* x
     - *...*
3. *there exists a sequence* $a_n \to 0$ *such that for all* $x \in E$, $|f_n(x) - f(x)| < a_n$

[Proposition 2.1]

Proof. • notice, 1 and 2 are equivalent, because 2 is essentially using the definition of uniform convergence: the only difference is that uniform convergence uses $\forall x \in E$ in the definition, whilst in 2, this is incorporated by considering the supremum, over all x. Clearly, if the supremum of $|f_n(x) - f(x)|$ is less than ε, then all elements will be less than ε, as required.

- notice, 2 and 3 are equivalent, because $a_n \to 0$, so saying that $|f_n(x) - f(x)| < a_n$ is essentially the same as saying that $\sup_{x \in E} |f_n(x) - f(x)| \to 0$

□

3.4 Theorem: Uniform Convergence of Continuous Functions

*Let E be a **non-empty** subset of $\mathbb{R}$, and suppose $f_n \to f$ **uniformly** on E as $n \to \infty$.*
*If each f_n is **continuous** at some $x_0 \in E$, then f is **continuous** at $x_0 \in E$. [Theorem 2.1]*

Proof: Continuity Uniform Convergence. Since f_n is uniformly convergent, we know that $\forall \varepsilon > 0$, we can pick $N \in \mathbb{N}$ such that if $n \geq N$, then:

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}$$

Moreover, since f_N is continuous at x_0, pick $\delta > 0$ such that if $|x - x_0| < \delta$:

$$|f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3}$$

Finally, suppose that $|x - x_0| < \delta$. Then, by the Triangle Inequality:

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

where the first and third terms are less than $\frac{\varepsilon}{3}$ because of uniform convergence, and the second term is less than $\frac{\varepsilon}{3}$ from continuity.

Thus, f must be continuous at x_0, as required. □

4 Exercises

1. **Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies:**

$$|f(x) - f(y)| \leq |x - y|^\alpha$$

   (a) **Show that if $\alpha > 1$, f is constant**

f is constant if $f' = 0$. Using the definition of differentiability:

$$\begin{aligned}
|f'(x)| &= \left| \lim_{y \to x} \frac{f(x) - f(y)}{x - y} \right| \\
&\leq \lim_{y \to x} | \frac{f(x) - f(y)|}{|x - y|} \\
&\leq \lim_{y \to x} | \frac{|x - y|^\alpha}{|x - y|} \\
&= \lim_{y \to x} |x - y|^{\alpha - 1} \\
&= 0
\end{aligned}$$

since $\alpha > 1$

(b) **Is the same conclusion valid if $\alpha = 1$?**

If $\alpha = 1$, then $\lim_{y \to x} |x - y|^{\alpha - 1} = 1$, and so, we can't tell whether f is constant or not.

2. (a) **Prove that $\frac{x}{n} \to 0$ uniformly as $n \to \infty$ on any closed interval $[a, b]$**

Let $f_n(x) = \frac{x}{n}$. We want to show that, $\forall \varepsilon > 0$ we can find some large $N \in \mathbb{N}$ such that if $n \geq N$:

$$|f_n(x) - 0| = \left| \frac{x}{n} \right| < \varepsilon$$

Notice, since $x \in [a, b]$, then $|x|$ must be bounded above by either $|a|$ or $|b|$. But then, let $N = \frac{\max\{|a|, |b|\}}{\varepsilon}$. If $n \geq N$:

$$|f_n(x) - 0| = \left| \frac{x}{n} \right| = \frac{|x|}{n} \leq \frac{\max\{|a|, |b|\}}{n} \leq \frac{\max\{|a|, |b|\}}{N} \leq \varepsilon$$

as required.

(b) **Prove that $\frac{1}{nx} \to 0$ pointwise but not uniformly as $n \to \infty$ on $[0, 1]$**

Pick $x \in (0, 1)$. Clearly, $\frac{1}{nx} \to 0$: for any x, if n gets sufficiently larger than $N(x, \varepsilon)$, f_n will converge to 0, so it is pointwise convergent.

However, it is not uniformly convergent. For that, we require that $\forall \varepsilon > 0$, we can find some $N \in \mathbb{N}$ such that for any $n \geq N$:

$$\left| \frac{1}{nx} - 0 \right| < \varepsilon$$

If f_n were uniformly convergent, the above should apply to any ε, so pick $\varepsilon = 1$. Then, we should have that if $n \geq N$, for all x:

$$\left| \frac{1}{nx} \right| < 1$$

But this is clearly false, as $\forall n \in \mathbb{N}$ we can pick $x = \frac{1}{n}$, and the above won't be satisfied.

Thus, f_n is pointwise convergent, but not uniformly convergent.

3. **A sequence of functions f_n is said to be *uniformly bounded* on a set E if there is an $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in E$ and all $n \in \mathbb{N}$.**

   Suppose that for each $n \in \mathbb{N}$, $f_n : E \to \mathbb{R}$ is bounded. If $f_n \to f$ uniformly on E, prove that f is a bounded function, and that f_n is uniformly bounded on E.

   Since f_n converges uniformly, there exists some $N \in \mathbb{N}$, such that:

   $$|f_n(x) - f(x)| < 1$$

   for every $x \in E$.

   This then means that each $f_1, f_2, \ldots, f_N$ is bounded, say by some M:

   $$M = \max_{1 \leq i \leq N} \underset{x \in E}{sup} |f_n(x)|$$

   M must be finite, since its the maximum out of a finite set of values. In particular $|f_N \leq M|$, so using the Triangle Inequality:

   $$|f(x)| = |f(x) - f_N(x) + f_N(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 1 + M$$

   Thus, f bounded. Taking $n \geq N$:

   $$|f_n(x)| = |f_n(x) - f(x) + f(x)| \leq |f_n(x) - f(x)| + |f(x)| < 1 + 1 + M = 2 + M$$

   Hence, each $f_n(x)$ is uniformly bounded by $2 + M$.

5 Workshop

1. **True or false.**
   (a) **The series $\sum_{n=1}^{\infty} a_n$ converges *if and only if* the sequence (a_n) is convergent**
   This is false. $a_n = \frac{1}{n}$ is a convergent sequence, but $\sum_{n=1}^{\infty} a_n$ is the Harmonic Series, which diverges. Another alternative (from solutions) is $a_n = 1$, which converges to 1, but $\sum_{n=1}^{\infty} a_n$ clearly diverges.
   (b) **The series $\sum_{n=1}^{\infty} a_n$ converges *if and only if* $\lim_{n\to\infty} a_n = 0$**
   This is false. $a_n = \frac{1}{n}$ is a convergent sequence, but $\sum_{n=1}^{\infty} a_n$ is the Harmonic Series, which diverges.
   (c) **The series $\sum_{n=1}^{\infty} a_n$ converges *if and only if* $\sum_{n=1}^{\infty} |a_n|$ converges**
   This is false. If $a_n = \frac{(-1)^n}{n}$, then $\sum_{n=1}^{\infty} a_n$ converges by the alternating series test, but $\sum_{n=1}^{\infty} |a_n|$ is the Harmonic Series, which diverges.

2. **Under what condition on p does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?**

   Whenever $p > 1$ (p-series test)

3. **State the ratio test, and explain how it is related to geometric series.**

If a sequence (a_n) satisfies:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r < 1$$

then the series:

$$\sum_n a_n$$

converges.
*If $r = 1$, the test is **inconclusive**.*
If $r > 1$, the series diverges.

This is related to geometric series because if we assume the hypothessis, it can be shown by induction that when n is large enough:

$$|a_n| \leq As^n$$

where $r < s < 1$. The test follows by applying the Series Comparison Test alongside the fact that As^n is a geometric series.

4. **Suppose that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove that $\sum_{n=1}^{\infty} |a_n|^p$ converges for all $p \geq 1$.**

My first instinct with this was "I have no idea". My second instinct was to use the ratio test, which worked. The solutions have a very nice alternative nonetheless.

From the solutions: since $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} |a_n|$ converges. In particular, this means that:

$$\lim_{n\to\infty} |a_n| = 0$$

which means that $\exists N$ such that if $n \geq N$:

$$|a_n| < 1$$

(this is just using the definition of sequence convergence)

Hence, if $n \geq N$, and since $p > 1$:

$$|a_n|^p < |a_n|$$

Hence, we have that:

$$\sum_{n=N}^{\infty} |a_n|^p < \sum_{n=N}^{\infty} |a_n|$$

so by the series comparison test, $\sum_{n=1}^{\infty} |a_n|^p$ converges, as required.

My solution: just apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{|a_{n+1}|^p}{|a_n|^p}\right| = \left(\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}\right)^p$$

But the absolute convergence of $\sum_{n=1}^{\infty} a_n$ tells us that $\sum_{n=1}^{\infty} |a_n|$ converges, so in particular, if we apply the ratio test, $\exists r < 1$ such that:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = r$$

So then we have that:

$$\lim_{n\to\infty} \left| \frac{|a_{n+1}|^p}{|a_n|^p} \right| = r^p$$

Since $r < 1$ and $p > 1$, we have that $r^p < 1$, so it follows that $\sum_{n=1}^{\infty} |a_n|^p$ converges by the ratio test.

5. **Suppose that $\sum_{n=1}^{\infty} a_n$ converges conditionally. Prove that $\sum_{n=1}^{\infty} n^p a_n$ diverges for all $p > 1$.**

> *For this I tried the ratio test again, and failed. For these sort of questions, contradiction is the way to go, since there are many ways for assumptions to fail.*

Lets assume that for $p > 1$ the series $\sum_{n=1}^{\infty} n^p a_n$ converges. This then means that:

$$\lim_{n\to\infty} n^p a_n = 0$$

But now recall that any convergent sequence is bounded, so in particular $\exists C \in \mathbb{R}$ such that:

$$|n^p a_n| \leq C \implies |a_n| \leq \frac{C}{n^p}$$

But then we must have:

$$\sum_{n=1}^{\infty} |a_n| \leq C \sum_{n=1}^{\infty} \frac{1}{n^p}$$

Since $p > 1$, by the p-series test the series $C\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges, so in particular by the series comparison test, $\sum_{n=1}^{\infty} |a_n|$ converges, which contradicts the fact that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent. Hence, if $p > 1$, the series $\sum_{n=1}^{\infty} n^p a_n$ must diverge.

6. **Let $f(x) = x^2$ when x is rational, and $f(x) = 0$ when x is irrational. Discuss the continuity and differentiability of f.**

> *The key to this question (at least what allowed me to get to an answer) was to notice that $f(x) = x^2$ evaluates to 0 precisely when $x = 0$. This hints at the fact that continuity could be an issue at all points except 0.*

Consider $x = a \neq 0$. We claim that $f(x)$ is not continuous at a. To show this, it is sufficient to find a sequence such that $x_n \to a$ but $f(x_n) \not\to f(a)$.

Assume that a is an irrational. Then, define a sequence, with each element x_n being the decimal expansion (to n decimal places) of a. Clearly:

$$x_n \to a$$

But then, each x_n is rational, so:

$$f(x_n) = x_n^2$$

By properties of the limit:

$$x_n^2 \to a^2$$

Thus:

$$x_n \to a \implies f(x_n) \to a^2$$

Since a is irrational, in particular $a \neq 0$ and $a \neq 1$, so $a^2 \neq a$, and so, f is not continuous at any irrational.

Now, assume that a is rational. Consider the sequence:

$$x_n = a - \frac{1}{\pi^n}$$

Each term is clearly irrational, and:

$$x_n \to a$$

But then:

$$f(x_n) = 0 \implies f(x_n) \to 0$$

Since $a \neq 0$, it is clear that wehn a is rational, f is not continuous.

Now, let $a = 0$. Consider the second definition of continuity, and assume that $\forall \varepsilon > 0$ we have a $\delta > 0$ such that:

$$|x - 0| = |x| < \delta$$

In particular, define $\delta = \sqrt{\varepsilon}$. Now, consider:

$$|f(x) - f(0)| = |f(x)|$$

Now, by definition, $f(x) \leq x^2$, since $f(x) = x^2$ or $f(x) = 0$, and $\forall x \in \mathbb{R}, 0 \leq x^2$. But then:

$$|f(x)| \leq |x|^2 < \delta^2 = \varepsilon$$

Hence, it follows that f is continuous at $x = 0$.

Now, recall that if a function is differentiable at a point, it is continuous there. Taking the contrapositive tells us that f can't be differentiable at any point, other than potentially 0. From definition, f is differentiable at $x = 0$ if the following limit exists:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$$

But now notice that:

$$\frac{f(x)}{x} \leq \frac{x^2}{x} = x$$

so:

$$\lim_{x \to 0} \frac{f(x)}{x} \leq \lim_{x \to 0} x = 0$$

By Squeeze Theorem, it must be the case that $\lim_{x \to 0} \frac{f(x)}{x} = 0$, thus, the limit exists, and the derivative of $f(x)$ at 0 is 0.

7. **True or false.**

   (a) **If f is continuous on $[a, b]$ and $J = f([a, b])$, then J is a closed, bounded interval**

   By the extreme value theorem, since f is continuous and defined over an interval $[a, b]$, it follows that f attains a maximum and a minimum on said interval, so J will be bounded.

   The intermediate value theorem then says that f, being a continuous function on a closed, bounded interval, attains each value in J, so J will be closed.

   Hence, J is a closed, bounded interval, so the statement is true.

(b) **If f, g is continuous on $[a, b]$ and $f(a) < g(a), f(b) > g(b)$ then $\exists c \in (a, b)$ such that $f(c) = g(c)$**
Define a function $h(x) = f(x) - g(x)$. Then, $h(x)$ is also defined on $[a, b]$, and:

$$h(a) < 0 \qquad h(b) > 0$$

In particular, Bolzano's Theorem tells us that since h is continuous (sum of 2 functions) over a closed, bounded interval, then $\exists c \in (a, b)$ such that:

$$h(c) = 0 \implies f(c) - g(c) = 0 \implies f(c) = g(c)$$

Hence, the statement is true.

(c) **Suppose that f, g are defined and finite valued on an open interval I containing a point a. Assume also that f is continuous at a and that $f(a) \neq 0$. Then, g is continuous at a *if and only if* fg is continuous at a**
The key here is that the **product** of **continuous** functions at a point is **continuous** at said point.

Proof. Let f, g be continuous at a. This means that $\forall \varepsilon_f, \varepsilon_g > 0$, we can find $\delta_f, \delta_g > 0$ such that:

$$|x - a| < \delta_f \implies |f(x) - f(a)| < \varepsilon_f$$

$$|x - a| < \delta_g \implies |g(x) - f(a)| < \varepsilon_g$$

Now, assume that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that:

$$|x - a| < \delta$$

If we use $\delta = \min\{\delta_f, \delta_g\}$, then we get access to the continuity conditions for both f, g. Then, consider

$$\begin{aligned} |f(x)g(x) - f(a)g(a)| &= |f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)| \\ &= |f(x)(g(x) - g(a)) + g(a)(f(x) - f(a))| \\ &\leq |f(x)||g(x) - g(a)| + |g(a)||f(x) - f(a)| \\ &< |f(x)|\varepsilon_g + |g(a)|\varepsilon_f \end{aligned}$$

Now, we can pick:

$$\varepsilon_f = \frac{\varepsilon}{2|g(a)|}$$

(here we assume that $g(a)$ is non-zero; if it is, we can just use $\varepsilon_f = \frac{\varepsilon}{2|g(a)|+1}$).

Moreover, by continuity we can bound $f(x)$. If $|x - a| < \delta_f$ then:

$$|f(x)| < |f(a)| + \varepsilon_f$$

So we can define:

$$\varepsilon_g = \frac{\varepsilon}{2(|f(a)| + \varepsilon_f)}$$

Hence, if $|x - a| < \delta$:

$$|f(x)g(x) - f(a)g(a)| < |f(x)|\varepsilon_g + |g(a)|\varepsilon_f < (|f(a)| + \varepsilon_f)\frac{\varepsilon}{2(|f(a)| + \varepsilon_f)} + |g(a)|\frac{\varepsilon}{2|g(a)|} < \varepsilon$$

so fg is continuous at $x = a$. □

This tells us that if g is continuous at a, since f is continuous at a, then fg is continuous at a. Now, assume fg is continuous at a. Notice, $\frac{1}{f}$ is continuous, since $f(a) \neq 0$, and $\frac{1}{f}$ is the composition of 2 functions which are continuous at a (namely $f(x)$ and $\frac{1}{x}$). Then, the following is a product of continuous functions at a:

$$(f(x)g(x))\frac{1}{f(x)} = g(x)$$

so g is continuous at a. Hence, the claim is true.

8. **State carefully the mean value theorem for a function $f : [0,1] \to \mathbb{R}$.**

Suppose that f is continuous on $[0,1]$ and differentiable on $(0,1)$. Then, $\exists c \in (0,1)$ such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = f(1) - f(0)$$

9. **Let $f : (0,1) \to \mathbb{R}$ be a function, and let $a \in (0,1)$. Match each statement from Group A with an equivalent statement from Group B.**

- Group A
  - $\forall \varepsilon > 0, \exists \delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$
  - $\forall \varepsilon > 0, \forall \delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$
  - $\exists \varepsilon > 0$ such that $\forall \delta > 0, |x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$
  - $\exists \varepsilon > 0$ and $\exists \delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$
  - $\forall \delta > 0, \exists \varepsilon > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$
  - $\exists \delta > 0$ such that $\forall \varepsilon > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$
- Group B
  - f is continuous at a
  - f is bounded on $(0,1)$
  - f is constant on $(0,1)$
  - There is some neighbourhood of a on which f is bounded
  - There is some neighbourhood of a on which f is constant

- "f is continuous at a" is equivalent to "$\forall \varepsilon > 0, \exists \delta > 0$ such that $|x-a| < \delta$ implies $|f(x)-f(a)| < \varepsilon$": it's the definition
- "f is bounded on $(0,1)$" is equivalent to "$\exists \varepsilon > 0$ such that $\forall \delta > 0, |x-a| < \delta$ implies $|f(x)-f(a)| < \varepsilon$" and to "$\forall \delta > 0, \exists \varepsilon > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$": in the first one we are saying that there exists a positive number such that no matter how close or far x gets from a, $f(x)$ is always within a certain distance of $f(a)$, which is the same as saying that for any $x \in (0,1)$, $f(x)$ must be bounded
- "f is constant on $(0,1)$" is equivalent to "$\forall \varepsilon > 0, \forall \delta > 0$ such that $|x-a| < \delta$ implies $|f(x)-f(a)| < \varepsilon$": we are saying that no matter how close or far x is from a, $f(x)$ can get as close (or as far) as it wants from $f(x)$
- "There is some neighbourhood of a on which f is bounded" is equivalent to "$\exists \varepsilon > 0$ and $\exists \delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$": we are saying that when x is a certain distance away from a, then $f(x)$ will be at a given distance from $f(a)$ for any such x
- "There is some neighbourhood of a on which f is constant" is equivalent to "$\exists \delta > 0$ such that $\forall \varepsilon > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$": we are saying that there exists a certain distance, such that if x is within said distance of a, then $f(x)$ can be as close as we want to $f(a)$