Honours Analysis - Week 2 - Sequences and Series of Real Numbers

Antonio León Villares

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Contents

1 The Bolzano-Weirstrass Theorem **2**
1.1 Defining Subsequences . . . 2
1.2 Theorem: Bolzano-Weierstrass Theorem . . . 2

2 Cauchy Sequences **4**
2.1 Defining Cauchy Sequences . . . 4
2.2 Theorem: Cauchy If and Only If Convergent . . . 4

3 Limit Superior and Limit Inferior **5**
3.1 Defining the Limit Superior and Limit Inferior . . . 5
3.2 Intuition of Existence for the Limit Superior and Limit Inferior . . . 6
3.3 LimSup and LimInf vs Limit of a Sequence . . . 7

4 Infinite Series of Real Numbers **7**
4.1 Convergence of a Series . . . 7
4.2 Theorem: Cauchy Criterion for Series . . . 8
4.3 Defining Absolute Convergence . . . 8
4.3.1 Theorem: Properties of Absolute Convergence - Convergence and Rearrangements . . 8
4.4 Conditionally Convergent Series . . . 10

5 Exercises **10**

6 Workshop **12**

1 The Bolzano-Weirstrass Theorem

1.1 Defining Subsequences

- **What is a subsequence?**
  - let $(x_n)_{n\in\mathbb{N}}$ be a sequence
  - a **subsequence** of this is a **sequence** $(x_{n_k})_{k\in\mathbb{N}}$, where we require:

$$n_1 < n_2 < n_3 < \ldots < n_k < \ldots$$

1.2 Theorem: Bolzano-Weierstrass Theorem

*Every **bounded** sequence of real numbers has a **convergent subsequence**. [Theorem 1.5]*

Proof: Bolzano-Weierstrass Theorem. The idea of the proof is the following.

1. A sequence is bounded, so all its terms are bounded in an interval of the reals
2. We can create subintervals, each with infinitely many terms
3. The infinite intersection of these subintervals will have a single term x (by the Compactness of a Closed Interval, Week 1)
4. We claim that x will be the limit of some subsequence
5. To construct the subsequence, we take an element from each subinterval, and show that the terms in the subsequence get arbitrarily close to x

Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. In particular, define the interval of finite length:

$$I_0 = [a, b]$$

where a is a lower bound of x_n, and b is an upper bound of x_n.

We now claim that given some **non-empty, closed, bounded** interval of the reals (with infinitely many terms of x_n), we can always find a sub-interval which is also non-empty, closed and bounded, and also contains infinitely many x_n. We can do so by induction: given our interval I_0 satisfying the above properties, we can construct n sub-intervals $I_1, I_2, \ldots, I_n$, such that $I_1 \supset I_2 \supset \ldots \supset I_n$ and each is non-empty, closed and bounded, and contains infinitely many terms from x_n.

(1) **Base Case: n = 1**

We begin with our initial interval, I_0. Then, we can split it into 2 subintervals:

$$I' = \left[a, \frac{a+b}{2}\right]$$

$$I' = \left[\frac{a+b}{2}, b\right]$$

But since I_0 contains infinitely many terms, at least one of I' or I'' must also contain infinitely many terms. Call whichever it is I_1. Notice that I_1 is non-empty, closed and bounded, so the case $n = 1$ is true.

(2) **Inductive Hypothesis: n = k**

Assume the claim is true for $n = k$: we have constructed k intervals, such that $I_1 \supset I_2 \supset \ldots \supset I_k$, and each of them are non-empty, closed and bounded.

(3) **Inductive Step: n = k+1**

By the inductive hypothesis, in particular we have a non-empty, closed and bounded I_k. Now, we can construct the interval I_{k+1}, such that it is also non-empty, closed and bounded. Since it has infinitely many terms of x_n, splitting I_k into 2 subintervals at the middle guarantees that at least one of them has infinitely many terms. Call this I_{k+1}. Certainly, I_{k+1} has infinitely may terms, it is non-empty, closed and bounded.

Thus, we have constructed a sequence of intervals $(I_n)_{n\in\mathbb{N}}$ which are closed, non-empty and bounded, have infinitely many terms, and:

$$I_1 \supset I_2 \supset \ldots \supset I_n \supset \ldots$$

Moreover, since to get to I_{k+1} we split I_k into 2, it follows that:

$$\lambda(I_k) = \frac{b-a}{2^k}$$

so in particular:

$$\lim_{n\to\infty} \lambda(I_n) = 0$$

But then, by the Nested Interval Property, it follows that $\exists x \in I_0$ such that:

$$\bigcap_{n=1}^{\infty} x_n = x$$

In other words, x belongs to all sub-intervals. We claim that x is the limit of some subsequence.

To see how, consider the following. Pick $x_1 = x_{n_1} \in I_0$. For the next element, since I_1 has infinitely many terms, we can pick $n_2 > 1 = n_1$, such that $x_{n_2} \in I_1$. Since I_2 has infinitely many terms, we know there exists some $n_3 > n_2$, such that $x_{n_3} \in I_2$. Proceeding like this, at any interval I_k, we can always find some $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in I_k$.

Now, consider the distance between the elements in our subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and the number x. If this difference goes to 0, then it clearly means that the subsequence converges to x. Now, $x_{n_k} \in I_{k-1}$, and $x \in I_{k-1}$ aswell, since x is in the intersection of all sub-intervals. But then, it follows that:

$$|x_{n_k} - x| \leq \frac{b-a}{2^{k-1}}$$

(this is the case in which x is at one endpoint of the interval, and x_{n_k} is at the other endpoint).

But we know that

$$\lim_{k\to\infty} \frac{b-a}{2^{k-1}} = 0$$

which means that the terms x_{n_k} can get arbitrarily close to x, so in particular:

$$\lim_{k\to\infty} x_{n_k} = x$$

This proves that there must exist at least some subsequence x_{n_k} of x_n which converges, given that x_n is bounded.

□

2 Cauchy Sequences

2.1 Defining Cauchy Sequences

- **When is a sequence Cauchy?**
  - let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers
  - x_n is a **Cauchy Sequence** if $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ :\ \forall n, m \geq N$ we have that:

$$|x_n - x_m| < \varepsilon$$

2.2 Theorem: Cauchy If and Only If Convergent

*Let (x_n) be a **convergent sequence** of **real** numbers.*
*Then, (x_n) is a **Cauchy** Sequence **if and only if** (x_n) is **convergent**.*
[Theorem 1.3, 1.4]

Proof: If Convergent, Then Cauchy. Assume $(x_n)_{n\in\mathbb{N}}$ is convergent. Then, $\exists x \in \mathbb{R}$, such that $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ :\ \forall n > N$ we have that:

$$|x_n - x| < \frac{\varepsilon}{2}$$

But then, letting $n, m > N$, and using the triangle inequality:

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &\leq |x_n - x| + |x - x_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

So it follows that x_n is Cauchy. □

Proof: If Cauchy, Then Convergent. Assume $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Firstly, (x_n) is bounded. We know that $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$:

$$|x_n - x_m| < \varepsilon$$

In particular, by the Reverse Triangle Inequality:

$$|x_n| - |x_m| \leq |x_n - x_m| < \varepsilon \implies |x_n| - |x_m| < \varepsilon$$

But then, letting $m = N$, we know that x_N is a constant, and moreover,

$$|x_n| < \varepsilon + |x_N|$$

Since ε is an arbitrary, positive constant, setting $\varepsilon = 1$ shows that each x_n is bounded:

$$|x_n| \leq \max\{x_1, x_2, \ldots, x_{N-1}, 1 + |x_N|\}$$

But then, the *Bolzano-Weierstrass Theorem* applies to (x_n), so there exists some subsequence x_{n_k}, such that, for some $a \in \mathbb{R}$, $x_{n_k} \to a$.

We now recall the definitions of Cauchy and Convergent Sequences. Let $\varepsilon > 0$. Then:

- since (x_n) is Cauchy, we can pick $N_1 \in \mathbb{N}$ such that $\forall n, m \geq N_1$:

$$|x_n - x_m| < \frac{\varepsilon}{2}$$

- since $x_{n_k} \to a$, $\exists N_2 \in \mathbb{N}$ such that $\forall k \geq N_2$:

$$|x_{n_k} - a| < \frac{\varepsilon}{2}$$

Then, pick n_k, such that $k \geq N_2$ and $n_k \geq N_1$. If $n \geq N_1$:

$$\begin{aligned} |x_n - a| &= |x_n - x_{n_k} + x_{n_k} - a| \\ &\leq |x_n - x_{n_k}| + |x_{n_k} - a| \end{aligned}$$

Now, since $n, n_k \geq N_1$, it follows that $|x_n - x_{n_k}| < \frac{\varepsilon}{2}$ by the first bullet point. Moreover, since $k \geq N_2$, by the second bullet point $|x_{n_k} - a| < \frac{\varepsilon}{2}$. Thus:

$$|x_n - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

so it follows that x_n converges, and $x_n \to a$.

□

3 Limit Superior and Limit Inferior

3.1 Defining the Limit Superior and Limit Inferior

- **What is the intuitive idea of a limit superior? And a limit inferior?**
  - there are sequences which don't converge, but still have a convergent flavour
  - we know from Bolzano-Weierstrass that any bounded sequence has a convergence subsequence
  - for example, $a_n = (-1)^n$ oscillates permanently between 1 and -1, so there is some notion that 1 and -1 are "pseudo-limits" of the sequence
  - in particular, the limit superior refers to the largest possible limit that a sequence can have, so:

    $$\limsup_{n \to \infty} a_n = 1$$

    (i.e out of all the convergent subsequences, what is the largest possible limit)
  - the limit inferior refers to the smallest possible limit that a sequence can have, so:

    $$\liminf_{n \to \infty} a_n = -1$$

- **How are the limit inferior and limit superior defined?**
  - let $(x_n)_{n \in \mathbb{N}}$ is a bounded sequence of real numbers
  - the **limit superior** is:

    $$\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \left(\sup_{k \geq n} x_k \right)$$

– the **limit inferior** is:

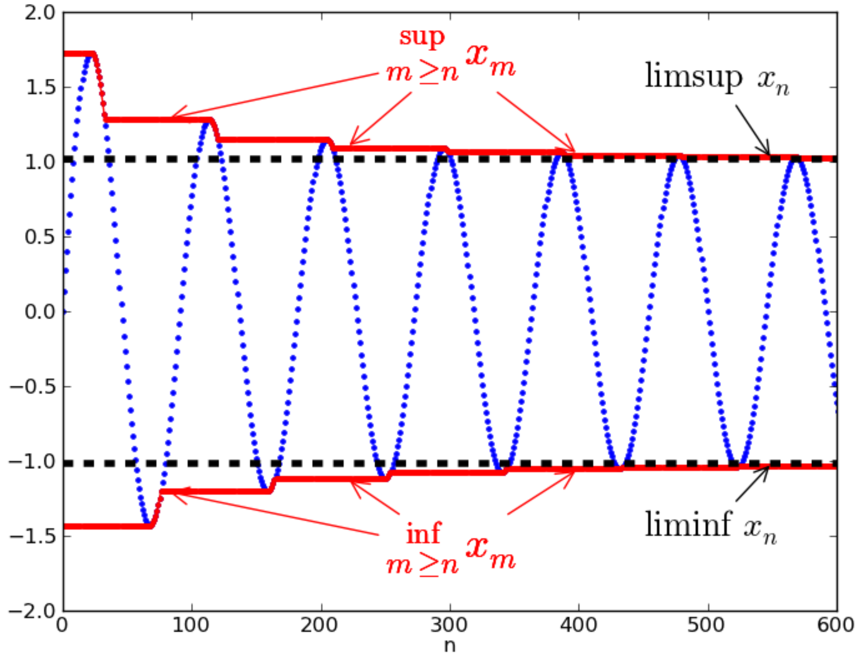
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$$

- **What are the limit superior and limit inferior of an unbounded sequence?**
  – if (x_n) is unbounded above, then by convention:

$$\limsup_{n\to\infty} x_n = +\infty$$

  – if (x_n) is unbounded below, then by convention:

$$\liminf_{n\to\infty} x_n = -\infty$$



3.2 Intuition of Existence for the Limit Superior and Limit Inferior

Lets consider the definition of limit superior, in particular the sequence $\sup_{k\geq n} x_k$:

$$\begin{aligned} k = 1 &\longrightarrow \sup\{x_1, x_2, \ldots\} \\ k = 2 &\longrightarrow \sup\{x_2, x_3, \ldots\} \\ k = 3 &\longrightarrow \sup\{x_3, x_4, \ldots\} \\ &\vdots \end{aligned}$$

In other words, as n increases, we are considering the supremum of a sequence with less terms. What this means is that the sequence $\sup_{k\geq n} x_k$ is **monotonically decreasing** (for example, if the sequence x_n were increasing, $x_1 > x_2 > \ldots$, then as n increases, the supremum is being computed over terms which are getting smaller).

Similarly, it can be seen that $\inf_{k \geq n} x_k$ is **monotonically increasing** (at each step we potentially remove the smallest item of the sequence).

The key here is the **Monotone Convergence Theorem**: (x_n) is bounded, so in particular both $\sup_{k \geq n} x_k$ and $\inf_{k \geq n} x_k$ are *bounded*. Moreover, they are also *monotone*. Thus, it follows that the sequences are convergent, and thus, we guarantee that the limit superior and the limit inferior of a bounded sequence always exist.

3.3 LimSup and LimInf vs Limit of a Sequence

- **Is the LimSup larger than the LimInf?**
  - Yes, we always have:
$$\limsup_{n \to \infty} x_n \geq \liminf_{n \to \infty} x_n$$
- **Under which conditions are the LimSup and LimInf equal?**
  - for this we have a theorem:

> *Let (x_n) be a sequence of real numbers. Then, (x_n) converges **if and only if**:*
> $$\limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n$$
> *and* $\limsup, \liminf \in \mathbb{R}$. *[Theorem 1.6]*

4 Infinite Series of Real Numbers

4.1 Convergence of a Series

- **What does it mean for a series to converge?**
  - let (a_n) be a sequence, and let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series
  - consider the **partial sums** of the series:
$$s_n = \sum_{k=1}^{n} a_k$$
  - S **converges** if the **sequence of partial sums** converges:
$$\exists s \in \mathbb{R} \; : \; \lim_{n \to \infty} s_n = s$$
  - we then define:
$$S = \sum_{k=1}^{\infty} a_k = s$$

4.2 Theorem: Cauchy Criterion for Series

Let $S = \sum_{k=1}^{\infty} a_k$ be a series.
Then, S converges ***if and only if*** $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ :\ \forall m \geq n \geq N,$

$$\left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon$$

[Theorem 1.7]

Proof: Cauchy Criterion For Series. Notice that $\sum_{k=n+1}^{m} a_k = s_m - s_n$, so the Cauchy Criterion is just saying that S converges if and only if:

$$|s_m - s_n| < \varepsilon$$

This condition this implies that the sequence of partial sums is Cauchy, and thus, converges. □

4.3 Defining Absolute Convergence

- **When is a series absolutely convergent?**
  - let $S = \sum_{k=1}^{\infty} a_k$ be a series
  - S is **absolutely convergence** if:

    $$\sum_{k=1}^{\infty} |a_k|$$

    is also a convergent sequence
- **When is a series conditionally convergent?**
  - if $S = \sum_{k=1}^{\infty} a_k$ but $S = \sum_{k=1}^{\infty} |a_k|$ doesn't

4.3.1 Theorem: Properties of Absolute Convergence - Convergence and Rearrangements

Let $S = \sum_{k=1}^{\infty} a_k$ be an ***absolutely convergent series****. Then:*

1. *The series S is* ***convergent***
2. *Let $z : \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{k=1}^{\infty} a_{z(k)}$ is* ***convergent****, and:*

   $$\sum_{k=1}^{\infty} a_{z(k)} = \sum_{k=1}^{\infty} a_k$$

   The series $\sum_{k=1}^{\infty} a_{z(k)}$ is known as a ***rearrangement****, and it is constructed by adding terms in the sequence in a different order than the order of the sequence. [Theorem 1.8]*

Proof: Absolute Convergence Implies Convergence. If we can prove that $s_n = \sum_{k=1}^{n} a_k$ is Cauchy, we will have proven convergence.

Let $A = \sum_{k=1}^{\infty} |a_k|$. From absolute convergence, it thus follows that $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}$ such that $\forall n \geq N$:

$$\left| A - \sum_{k=1}^{n} |a_k| \right| = A - \sum_{k=1}^{n} |a_k| = \sum_{k=n+1}^{\infty} |a_k| < \varepsilon$$

But then, without loss of generality, we can assume that for any $m \in \mathbb{N}$ such that $n \leq m$:

$$\begin{aligned} |s_n - s_m| &= |\sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k| \\ &= |\sum_{k=n+1}^{m} a_k| \\ &\leq \sum_{k=n+1}^{m} |a_k| \qquad \textit{(from triangle inequality)} \\ &\leq \sum_{k=n+1}^{\infty} |a_k| \\ &< \varepsilon \end{aligned}$$

Thus, the sequence of partial sums is Cauchy, and thus, $\sum_{k=1}^{\infty} a_k$ is convergent.

□

Proof: Absolute Convergence Doesn't Affect Rearrangement. Let $A = \sum_{k=1}^{\infty} |a_k|$. From absolute convergence, it thus follows that $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}$ such that $\forall n \geq N$:

$$\left| A - \sum_{k=1}^{n} |a_k| \right| = A - \sum_{k=1}^{n} |a_k| = \sum_{k=n+1}^{\infty} |a_k| < \varepsilon$$

Define the partial sums of the series of a_n and the rearrangement $a_{z(n)}$:

$$s_n = \sum_{k=1}^{n} a_k$$

$$t_n = \sum_{i=1}^{n} a_{z(i)}$$

Now, consider:

$$|s_n - t_n| = \left| \sum_{k=1}^{n} a_k - \sum_{i=1}^{n} a_{z(i)} \right|$$

Finally, let $M \geq N$ be a natural number such that if $n \geq M$

$$S_n = \{z(1), z(2), \ldots, z(n)\}$$

contains the subset $\{1, 2, \ldots, N\}$.

Then, $\forall k \leq N$, each a_k in $\sum_{k=1}^{n} a_k$ will cancel with some $a_{z(i)}$ in $\sum_{i=1}^{n} a_{z(i)}$. Thus:

$$\begin{aligned} |s_n - t_n| &= \left| \sum_{k=1}^{n} a_k - \sum_{i=1}^{n} a_{z(i)} \right| \\ &= \left| \sum_{k=N+1}^{n} a_k - \sum_{i \in S_n \setminus \{1,2,\ldots,N\}}^{n} a_{z(i)} \right| \\ &\leq \sum_{k=N+1}^{n} |a_k| - \sum_{i \in S_n \setminus \{1,2,\ldots,N\}}^{n} |a_{z(i)}| \end{aligned}$$

But notice, both sums must contain (non-repeated) elements with index greater than $N+1$, so it follows that:

$$|s_n - t_n| \leq \sum_{k=N+1}^{n} |a_k| - \sum_{i \in S_n \setminus \{1,2,\ldots,N\}}^{n} |a_{z(i)}| < 2 \sum_{k=N+1}^{\infty} |a_k| < 2\varepsilon$$

Hence, since s_n is convergent, it follows that t_n is also convergent, and in fact, they converge to the same value.

□

4.4 Conditionally Convergent Series

From the Properties of Absolute Convergence, Part 2, we might think that it is not anything important: after all, addition is commutative, so the order in which we add terms shouldn't matter. However, if we have an infinite sum, this is not the case. In fact:

> *Let $S = \sum_{k=1}^{\infty} a_k$ be a conditionally convergent series. Then, there exists a bijection $z : \mathbb{N} \to \mathbb{N}$ such that:*
>
> 1. *$\forall r \in \mathbb{R}$, $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, and its sum is r*
> 2. *the series $\sum_{k=1}^{\infty} a_{z(k)}$ diverges to $+\infty$*
> 3. *the series $\sum_{k=1}^{\infty} a_{z(k)}$ diverges to $-\infty$*
> 4. *the partial sums of $\sum_{k=1}^{\infty} a_{z(k)}$ oscillate between any 2 real numbers*
>
> *[Theorem 1.9]*

To see a nice example, see this video by patrickJMT. Moreover, the reason is the Riemann Series Theorem.

5 Exercises

1. (a) **Let $a_n = \sqrt{n}$. Show that** $|a_n - a_{n-1}| \to 0$

$$
\begin{aligned}
\sqrt{n} - \sqrt{n-1} &= \frac{(\sqrt{n} - \sqrt{n-1})(\sqrt{n} + \sqrt{n-1})}{\sqrt{n} + \sqrt{n-1}} \\
&= \frac{n - (n-1)}{\sqrt{n} + \sqrt{n-1}} \\
&= \frac{1}{\sqrt{n} + \sqrt{n-1}} \\
\therefore \qquad \lim_{n\to\infty} |a_n - a_{n-1}| &= \lim_{n\to\infty} \frac{1}{\sqrt{n} + \sqrt{n-1}} = 0
\end{aligned}
$$

(b) **Is (a_n) Cauchy?**

Clearly not, as $\lim_{n\to\infty} \sqrt{n} = \infty$

(c) **If b_n is such that**:

$$
|b_n - b_{n-1}| < \frac{1}{2^n}
$$

show that b_n is a Cauchy sequence.

Let $n < m$. Then:

$$
\begin{aligned}
|a_m - a_n| &= |a_m - a_{m-1} + a_{m-1} - \ldots - a_{n+1} + a_{n+1} - a_n| \\
&\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \ldots + |a_{n+1} - a_n| \\
&\leq \frac{1}{2^m} + \frac{1}{2^{m-1}} + \ldots + \frac{1}{2^{n+1}} \\
&\leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} \\
&= \frac{\frac{1}{2^{n+1}}}{\frac{1}{2}} \\
&= \frac{1}{2^n}
\end{aligned}
$$

Now, pick N such that $\forall n \geq N$, $\frac{1}{2^n} < \varepsilon$, and thus, we have shown that $|a_m - a_n| < \varepsilon$, $\forall m, n \in \mathbb{N}$, so a_n must be Cauchy.

2. **Let (a_n) be a sequence of real numbers. If $\sum_n a_n^2$ converges, show that $\sum_n \frac{a_n}{n}$ converges. Is the converse true?**

Recall the Cauchy-Schwarz inequality:

$$
\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)
$$

Using $b_n = \frac{1}{n}$, it follows that:

$$
\sum_n \left|\frac{a_n}{n}\right| \leq \sqrt{\left(\sum_{i=1}^{n} a_n^2\right)}\sqrt{\left(\sum_{i=1}^{n} \frac{1}{n^2}\right)}
$$

Since the product to the right is finite, it follows that $\frac{a_n}{n}$ gives an absolutely convergent series, so in particular $\sum_n \frac{a_n}{n}$ converges.

The converse is not true. For example, if $a_n = \frac{1}{\sqrt{n}}$.

3. **Prove that the sequence:**

$$a_n = \sum_{k=1}^{n-1} \frac{1}{k} - \log n$$

converges

Firstly, notice the sequence is monotone:

$$a_{n+1} - a_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) - \sum_{k=1}^{n-1} \frac{1}{k} + \log(n) = \frac{1}{n} - \log\left(1 + \frac{1}{n}\right)$$

Using the MVT, it can be easily checked that $x > \log(1+x)$, so it follows that a_n is monotonically increasing.

We can also prove that the sequence is bounded. Firstly:

$$\frac{1}{k} < \int_{k-1}^{k} \frac{1}{x} dx = \log k - \log(k-1)$$

Thus:

$$a_n < 1+(\log 2-\log 1)+(\log 3-\log 2)+\ldots+(\log(n-1)-\log(n-2))+\log(n) = 1+\log(n-1)-\log(n) \leq 1$$

So the sequence is both monotonically increasing and bounded. Thus, it is convergent.

6 Workshop

1. **Prove that every convergent sequence is bounded.**

Consider a convergent sequence (x_n) such that $x_n \to x$. By definition $\forall \varepsilon > 0$ we can find a $N \in \mathbb{N}$ such that if $n \geq N$, then:

$$|x_n - x| < \varepsilon$$

In particular, if we pick $\varepsilon = 1$, then:

$$|x_n - x| < 1 \iff -1 - x < 1 - x < x_n < 1 + x$$

So letting $M = |1 + x|$, we have that if $n \geq N$, $|x_n| < M$.

Look out! Here I assumed that this was sufficient, but I ignored the possibility of x_n with $n < N$. It could be possible that such x_n don't satisfy this inequality!

Hence, we can say that $\forall n \in \mathbb{N}, |x_n| \leq C$, where:

$$C = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, M\}$$

2. **True or false?**

   (a) **If a sequence (x_n) converges, then the sequence $\left(\frac{x_n}{n}\right)$ is also convergent.**
   This is true.

   > *To prove this, there are 2 possibilities:*
   >
   > - *From the solutions: bound the sequence $\frac{x_n}{n}$ by 2 convergent sequences, and apply the squeeze theorem to show that $\frac{x_n}{n}$ converges*
   > - *Self: intuitively, we should see that $\frac{x_n}{n} \to 0$, so we can try to prove this from definition of the limit (and we can use the fact that x_n converges, so it is bounded)*

   Since x_n converges, it is bounded, say that $|x_n| \leq M$. Then, it follows that:

   $$\frac{-M}{n} \leq \frac{x_n}{n} \leq \frac{M}{n}$$

   Since M is a constant:

   $$\lim_{n\to\infty} \frac{M}{n} = \lim_{n\to\infty} -\frac{M}{n} = 0$$

   So it follows that $\frac{x_n}{n}$ converges by the squeeze theorem, and:

   $$\frac{x_n}{n} \to 0$$

   (b) **If a sequence (x_n) does not converge, then the sequence $\left(\frac{x_n}{n}\right)$ does not converge.**
   False.
   From the notes, $x_n = (-1)^n$ diverges, but $\frac{(-1)^n}{n} \to 0$.
   From self, $x_n = n$ diverges, but $\frac{n}{n} = 1 \to 1$

   (c) **If the sequence (x_n) is convergent and (y_n) is bounded, then $(x_n y_n)$ is convergent.**
   False. Consider $x_n = 1$ and $y_n = (-1)^n$.

   (d) **If the sequence (x_n) is convergent to zero, and (y_n) is a sequence such that $y_n > 0, \forall n \in \mathbb{N}$, then $(x_n y_n)$ is convergent.**
   False. Consider $x_n = \frac{1}{n}$ and $y_n = n^2$.

3. **If $a > 0$ show that $a^{\frac{1}{n}} \to 1$ as $n \to \infty$**

   > *It is clear that proving this from definition can be quite hard. The* ***Monoton Convergence Theorem*** *is our friend.*
   > *In my initial attempt, I tried to use the MCT, alongside the fact that such sequences will converge to the infimum of their set. However, I couldn't find a way to show that 1 is the infimum of the set (admittedly, being quite rusty).*
   > *The solutions exploit a much neater treat, by showing that the limit is equal to its square.*

We consider 2 cases.

If $a \geq 1$, then clearly:

$$a^{\frac{1}{n}} \geq 1^{\frac{1}{n}} = 1$$

So if $a \geq 1$, $a^{\frac{1}{n}}$ is bounded below by 1. Now, consider:

$$\frac{a^{\frac{1}{n+1}}}{a^{\frac{1}{n}}} = a^{\frac{1}{n(n+1)}}$$

Now, since $a \geq 1$ and $\frac{1}{n(n+1)} < 1$, it follows that $a^{\frac{1}{n(n+1)}} \leq 1$, so in particular $a^{\frac{1}{n+1}} \leq a^{\frac{1}{n}}$. Hence, when $a \leq 1$, $a^{\frac{1}{n}}$ is monotone decreasing and bounded below, so by the Monotone Convergence Theorem, it converges.

Alternatively, if $0 < a < 1$, then:

$$a^{\frac{1}{n}} < 1^{\frac{1}{n}} = 1$$

So if $0 < a < 1$, $a^{\frac{1}{n}}$ is bounded above by 1. Now, consider:

$$\frac{a^{\frac{1}{n+1}}}{a^{\frac{1}{n}}} = a^{\frac{1}{n(n+1)}}$$

Now, since $a < 1$ and $\frac{1}{n(n+1)} < 1$, it follows that $a^{\frac{1}{n(n+1)}} > 1$, so in particular $a^{\frac{1}{n+1}} > a^{\frac{1}{n}}$. Hence, when $a < 1$, $a^{\frac{1}{n}}$ is monotone increasing and bounded above, so by the Monotone Convergence Theorem, it converges.

Thus, $\forall a > 0$, the sequence $a^{\frac{1}{n}}$ converges. Say:

$$\lim_{n\to\infty} a^{\frac{1}{n}} = L$$

Then, by properties of limits:

$$\left(\lim_{n\to\infty} a^{\frac{1}{n}}\right)^2 = L^2 \implies \lim_{n\to\infty} a^{\frac{2}{n}} = L^2$$

But notice that:

$$\lim_{n\to\infty} a^{\frac{2}{n}} = \lim_{n\to\infty} (a^2)^{\frac{1}{n}}$$

Since a is an arbitrary positive constant, and since a^2 is also an arbitrary positive constant, so we must have that:

$$L = L^2$$

This is only positive if $L = 0$ or $L = 1$. Since we have shown that the sequence is bounded below/above by 1, it must be the case that $L = 1$, as required.

For this I also thought that using logs would be very helpful, but I'm unsure whether we are allowed to use them.

4. **Suppose that (x_n) is a Cauchy sequence, such that x_n is an integer $\forall n \in \mathbb{N}$. Prove that (x_n) is eventually constant; that is, $\exists a \in \mathbb{Z}, N \in \mathbb{N}$ such that $x_n = a$ for all $n \geq N$**

Since x_n is Cauchy, then $\forall \varepsilon > 0$, we have $N \in \mathbb{N}$ such that with $m > n$ and $m, n \geq N$ we have:

$$|x_m - x_n| < \varepsilon$$

In particular, the above is true with $\varepsilon = 1$:

$$|x_m - x_n| < 1$$

But since $x_m, x_n \in \mathbb{Z}$:

$$|x_m - x_n| < 1 \implies |x_m - x_n| = 0$$

(distance between integers is always an integer, and the only non-negative integer below 1 is 0)
Hence, it follows that $\forall n \geq N, x_n = a \in \mathbb{Z}$, as required.

5. **Let (x_n) be a sequence of real numbers and suppose that there is $a \in (0,1)$ such that:**

$$|x_{n+1} - x_n| \leq a^n, \qquad \forall n \in \mathbb{N}$$

Prove that the sequence (x_n) is convergent to some $x \in \mathbb{R}$

> *The intuition here is to exploit the fact that if a sequence is Cauchy it is convergent.*

Consider $m, n \in \mathbb{N}$ with $m > n$. Then:

$$\begin{aligned}
|x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} + \ldots - x_{n+1} + x_{n+1} - x_n| \\
&\leq \sum_{k=n}^{m-1} |x_{k+1} - x_k| \\
&\leq \sum_{k=n}^{m-1} a^k \\
&\leq \sum_{k=n}^{\infty} a^k
\end{aligned}$$

This is an infinite geometric series, with first term a^n and common ratio 1. Hence, we have that:

$$|x_m - x_n| \leq \frac{a^n}{1-a}$$

Notice, since $a \in (0,1)$, it is always possible ot pick an N such that $\forall \varepsilon > 0$, we have:

$$\frac{a^N}{1-a} < \varepsilon$$

Hence, we ensure that $\forall \varepsilon > 0$, we have said N, such that if $m, n \geq N$, then:

$$|x_m - x_n| < \varepsilon$$

Therefore, x_n is a Cauchy sequence, and so it converges, as required.

6. **Let $E \subset \mathbb{R}$. A point $a \in \mathbb{R}$ is called a *cluster point* of E if $E \cap (a - r, a + r)$ contains infinitely many points $\forall r > 0$. Prove that every bounded infinite subset of $\mathbb{R}$ has at least one cluster point.**

> *Notice, this hints at using the Bolzano-Weierstrass theorem; in fact, from its prove, we already construct subsets of E of length $\frac{b-a}{2^n}$ which have infinitely many elements.*

Consider a sequence e_n of points in E. Since all the points lie in E, and E is a bounded subset of $\mathbb{R}$, it follows by Bolzano-Weierstrass that e_n has a convergent subsequence e_{n_k}. Say that:

$$e_{n_k} \to a \in E$$

Now, let $r \in \mathbb{R}$, and consider the interval $(a-r, a+r)$. By the proof of the Bolzano Weierstrass Theorem, e_{n_k} is constructed by taking points from infinite subintervals of E, with $e_{n_k} \in I_k$, where I_k is an interval of length $\frac{b-a}{2^k}$ with infinitely many points. It is then easy to see that by making k large enough, we can create an interval with infinitely many points I_k such that $I_k \subseteq (a-r, a+r)$. Thus, it means that $E \cap (a-r, a+r)$ will contain infinitely many points, so in particular, a must be a cluster point.

For the nitty-gritty details, I include the original proof I submitted as part of the assignment.

Let $E \subset \mathbb{R}$ be a bounded, infinite subset of $\mathbb{R}$. Since E is bounded, define an interval $I_0 = [a, b]$, such that a is a lower bound for E, and b is an upper bound for E.

Now consider the 2 following intervals:

$$I' = \left[a, \frac{a+b}{2}\right]$$

$$I'' = \left[\frac{a+b}{2}, b\right]$$

At least one of I and I'' must contain infinitely many terms from E (otherwise, E would be a finite set). Let I_1 be the set out of I' and I'' with infinitely many elements of E (if both have infinitely many elements of I_0, then just define $I_1 = I'$).

Notice, like I_0, I_1 is a closed, bounded interval, which contains infinitely many terms from E. Moreover, it is also the case that $I_0 \supset I_1$. We can continue sub-dividing the interval I_0 as to produce intervals which are closed, bounded and contain infinitely many terms from E: if we have one such interval I_k, then we know I_k is bounded, so $I_k = [a_k, b_k]$, and $b_k - a_k = \frac{\lambda(I_{k-1})}{2}$, where λ denotes the length of the interval. We can then split I_k into $I'_k = \left[a_k, \frac{a_k+b_k}{2}\right]$ and $I''_k = \left[\frac{a_k+b_k}{2}, b_k\right]$, and set I_{k+1} to be the interval out of I'_k and I''_k which contains infinitely many terms from E. Thus, we are capable of constructing an infinite sequence of nested, bounded, closed intervals with infinitely many terms from E:

$$(I_n)_{n\in\mathbb{N}}$$

which also satisfies:

$$I_0 \supset I_1 \supset \ldots \supset I_k \supset I_{k+1} \supset \ldots$$

Moreover, the length of some interval k is given by:

$$\lambda(I_k) = \frac{b-a}{2^k}$$

since at each step we are halving the length of I_0.

Since we have nested, closed, bounded, non-empty intervals, and $\lambda(I_n) \to 0$, it then follows that $\exists x \in \mathbb{R}$, such that:

$$x = \bigcap_{kn=0}^{\infty} I_n$$

We claim that x is a cluster point for E. Indeed, pick any $r \in \mathbb{R}$, and consider the open interval $(x-r, x+r)$. For any r, we can always find a $k \in \mathbb{N}$ such that $\frac{b-a}{2^k} < r$ and in particular, $I_k \subset (x-r, x+r)$.

This is because $\frac{b-a}{2^k}$ can be made arbitrarily small by increasing k, so the length of an interval I_k can be made arbitrarily small. Moreover, $x \in I_k$, so we can always ensure that there exists a k such that $I_k \subset (x-r, x+r)$. Since I_k contains infinitely many elements from E by construction, it follows that $(x-r, x+r)$ contains infinitely many elements of E, and so, $E \cap (x-r, x+r)$ also contains infinitely many elements of E, for any $r > 0$. Hence, by definition, it must be the case that x is a cluster point of E. Overall, we have shown that for any arbitrary, bounded, infinte subset of the real numbers, we can always find at least one cluster point.

7. **A subset E of $\mathbb{R}$ is called *sequentially compact* if and only if every sequence $(x_n) \subset E$ has a convergent subsequence whose limit belongs to E.**

   (a) **Prove that every closed bounded interval is sequentially compact.**

   Consider a closed, bounded interval $E = [a, b]$ where $a, b \in \mathbb{R}, a < b$. Consider a sequence, such that $\forall n \in \mathbb{N}, x_n \in E$. Clearly, x_n is bounded, since:

   $$a \leq x_n \leq b, \qquad \forall n \in \mathbb{N}$$

   Thus, Bolzano-Weierstrass applies, and x_n has a convergent subsequence $x_{n_k} \to c$. By construction:

   $$a \leq x_{n_k} \leq b \implies \lim_{n\to\infty} a \leq \lim_{n\to\infty} x_{n_k} \leq \lim_{n\to\infty} b$$

   So it follows that:

   $$a \leq c \leq b$$

   so the limit of x_{n_k} belongs in E. Thus, E is sequentially compact.

   (b) **Prove that there exist bounded intervals that are not sequentially compact.**

   For example, consider $x_n = \frac{1}{n}$ over the interval $(0, 1)$. This is clearly bounded, and each term x_n belongs to the interval, but the subsequence $x_{n_k} = x_n$ is such that:

   $$\frac{1}{n} \to 0 \notin (0, 1)$$

   (c) **Prove that there are closed intervals that are not sequentially compact.**

   For example, consider $x_n = n$ over the interval $[1, \infty)$. This is a closed interval, and each element of x_n is in it, but any subsequence of x_n will diverge.

8. **Using the definition of $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$, find them for $x_n = 2 + (-1)^n$**

   Recall the definitions:

   $$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup_{k\geq n} x_k \right)$$

   $$\liminf n \to \infty x_n = \lim_{n\to\infty} \left(\inf_{k\geq n} x_k \right)$$

   Notice, the sequence only contains 2 terms:

   $$x_n = \{1, 3\}$$

   So it is clear that:

   $$\sup_{k\geq n} x_k = 3$$

   $$\sup_{k\geq n} x_k = 1$$

   So then:

   $$\limsup_{n\to\infty} x_n = 3$$

   $$\liminf_{n\to\infty} x_n = 1$$