# Honours Analysis - Week 1 - Real Numbers

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# 1 Real Numbers

## 1.1 Properties of the Real Numbers

- $\mathbb{R}$ is an **ordered** set
  - $\forall a, b \in \mathbb{R}$ there are only 3 possibilities:
    1. $a < b$
    2. $a = b$
    3. $a > b$
- $\mathbb{R}$ is **complete**
  - $\forall A \subset \mathbb{R}$ if $A$ is **non-empty** and **bounded**, then $A$ has an **infimum** and a **supremum**
  - this follows from the **Completeness Theorem**
- $\mathbb{R}$ has the **Archimidean Property**
  - this means that there is no **largest** or **smallest** real. In other words:

$$\forall a, b \in \mathbb{R} : a > 0,\ \exists n \in \mathbb{N} : b < na$$

# 2 Nested Intervals

## 2.1 Definition: Nested Intervals

- **What are nested intervals?**
  - consider a **sequence of intervals** $(I_n)_{n \in \mathbb{N}}$
  - they form a set of **nested intervals** if:

$$I_1 \supset I_2 \supset \dots$$

  - for example, $I_n = [0, \frac{1}{n}]$ represents the sequence of nested intervals:

$$[0, 1], \left[0, \frac{1}{2}\right], \left[0, \frac{1}{3}\right], \dots$$

## 2.2 Theorem: The Nested Interval Property

*Let $(I_n)_{n\in\mathbb{N}}$ be a nested sequence of:*

- ***non-empty***,
- ***closed***,
- ***bounded***

*intervals. Define the set $E$ as:*

$$E = \bigcap_{n\in\mathbb{N}} I_n = \{x \in \mathbb{R} \ : \ \forall n \in \mathbb{N}, x \in I_n\}$$

*Then:*

1. *$E$ is **non-empty** ($|E| \geq 1$)*
2. *if $\lim_{n\to\infty} \lambda(I_n) = 0$, then $|E| = 1$, where $\lambda(I_n)$ denotes the length of the interval*

*The conditions for the theorem are very important:*

- *if $\exists n \in \mathbb{N} \ : \ I_n = \emptyset$, then trivially, the intersection of any set with the empty set is in itself empty, so $E = \emptyset$*
- *if $I_n$ is an* ***open*** *interval, the Nested Interval Property might not hold*
  - *consider $I_n = (1, \frac{1}{n})$. Intuitively, $\bigcap_{n \in \mathbb{N}} (1, \frac{1}{n})$ might seem non-empty.*
  - *lets assume that $\exists x \in \bigcap_{n \in \mathbb{N}} (1, \frac{1}{n})$. However, this would mean that $\forall n \in \mathbb{N}, x < \frac{1}{n}$, which violates the Archimidean Property (it would mean that there exists a real number which is infinitesimally small)*
  - *an alternative view. Consider an arbitrary element $x \in (0, \frac{1}{n})$. By the Archimidean property, we can* ***always*** *find an interval $(0, \frac{1}{m}), m > n$, such that $x \notin (0, \frac{1}{m})$.*
  - *overall what this means is that:*

$$\bigcap_{n \in \mathbb{N}} \left(1, \frac{1}{n}\right) = \emptyset$$

  - *see this Reddit thread on the topic*
- *if $I_n$ is an* ***unbounded*** *interval, the Nested Interval Property might not hold*
  - *consider $I_n = [n, \infty)$. By similar arguments to the above case, any element of $\bigcap_{n \in \mathbb{N}} [n, \infty]$ would need to be larger than any $n$, which again violates the Archimidean Property (or rather, the Archimidean Property contradicts this claim)*
  - *see this stackexchange post on the topic*

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*Proof.* Consider this alternative proof if the following seems too complicated or over the top.

We first need to prove a Lemma:

*If $A$ and $B$ are* ***non-empty*** *subsets of $\mathbb{R}$, such that:*

$$\forall a \in A, \ b \in B, \ a \leq b$$

*Then:*

1. *$supA \leq infB$*
2. *if $\forall \varepsilon > 0, \exists a \in A, \ b \in B \ : \ |b - a| < \varepsilon$, then $supA = infB$*

Firstly, since $A$ and $B$ are non-empty, $supA$ and $infB$ exist. We claim that $infB$ is an upper bound of $A$. If this is the case, since $supA$ is the *least upper bound* of $A$, then it follows from the definition that:

$$supA \leq infB$$

as required. Thus, we seek to prove that:

$$\forall a \in A, infB \geq a$$

We proceed by contradiction, assuming that $\exists a \in A \ : \ a > infB$. Since $infB$ is the greatest lower bound, then $infB + \varepsilon, \varepsilon > 0$ is **not** a lower bound of $B$. Consider $\varepsilon = \frac{a-infB}{2}$. Then, since $infB + \frac{a-infB}{2}$ is **not** a lower bound, $\exists b \in B \ : \ b < infB + \frac{a-infB}{2}$. But this then means that:

$$infB < b < infB + \frac{a - infB}{2} < a$$

which implies that $b < a$, which contradicts the assumption of the Lemma. Thus, it must be the case that $infB$ is an upper bound for $A$, as required.

Now, for the second part. Since $supA \leq infB$, there are 2 possibilities:

$$supA < infB$$

or

$$supA = infB$$

Assume that $supA < infB$. Then:

$$infB - supA > 0$$

In particular, let $\varepsilon' = infB - supA > 0$. Now, by the assumption,

$$\forall \varepsilon > 0, \exists a \in A, \ b \in B \ : \ |b - a| < \varepsilon$$

Thus, choose $a, b$, such that:

$$|b - a| < \varepsilon'$$

Then, from the properties of $supA$ and $infB$:

$$a \leq supA < infB \leq b$$

From which it follows that:

$$infB - supA \leq b - a < \varepsilon'$$

But this is a contradiction since we have defined $\varepsilon' = infB - supA$ but we get that $infB - supA < \varepsilon'$. In other words, it must be the case that:

$$infB - supA = 0 \implies supA = infB$$

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Now that this is done, we prove the Nested Interval Property.

Consider an interval $I_n$ satisfying the conditions of the theorem. Then, $\exists a_n, b_n \in \mathbb{R}$ such that $a_n$ is the lower bound of $I_n$ and $b_n$ is its upper bound. Considering $n \in \mathbb{N}$, it follows that:

$$a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_k \leq \ldots \leq b_k \leq \ldots \leq b_3 \leq b_2 \leq b_1$$

from the fact that $I_n \supset I_{n+1}$ and $I_n \supset I_k, \ \forall n < k$.

Now, define the sets:

$$A = \{a_1, a_2, \dots\}$$

$$B = \{b_1, b_2, \dots\}$$

Notice that $A$ and $B$ satisfy the conditions of the Lemma, so from the first property of the Lemma, $supA \leq infB$, and so:

$$a_1 \leq a_2 \leq a_3 \leq \dots \leq a_k \leq \dots \leq supA \leq infB \leq \dots \leq b_k \leq \dots \leq b_3 \leq b_2 \leq b_1$$

Moreover, it follows that:

$$E = \bigcap_{n \in \mathbb{N}} I_n = [supA, infB]$$

and clearly $[supA, infB]$ is non-empty, as $[supA, infB] \subset I_k, \ \forall k \in \mathbb{N}$.

Now, suppose that $\lim_{n \to \infty} \lambda(I_n) = 0$. Then, in particular, it follows that:

$$\lim_{n \to \infty} (b_n - a_n) = 0$$

But by the definition of a limit, it follows that, $\forall \varepsilon > 0$:

$$|b_n - a_n| < \varepsilon$$

This is precisely the condition required for the second property of the Lemma. Overall, it implies that $supA = infB$, which means that $[supA, infB]$ is a degenerate interval, and only contains one element, as required. □

# 3 Compactness

## 3.1 Covers and Subcovers

Consider a closed interval/set $X$.

- **What is the cover of the set?**
  - a **cover** of $X$ is a collection of sets whose union has $X$ as a subset
- **What is the subcover of the set?**
  - a subset of a cover of $X$, which still covers $X$

## 3.2 Theorem: Compactness of a Closed Interval

*Let $E = [a, b]$ be a closed interval with $a, b \in \mathbb{R}$. Consider an arbitrary collection of* ***open*** *intervals, $(I_\alpha)_{\alpha \in \mathbb{A}}$, such that they cover $E$:*

$$E \subset \bigcup_{\alpha \in \mathbb{A}} I_\alpha$$

*Then, there exists a* ***finite*** *set of indices $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \mathbb{A}$, such that they form a* ***finite subcover*** *of $E$:*

$$E \subset \bigcup_{i=1}^{n} I_{\alpha_i}$$

*[Theorem 1.2]*

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*Proof: Compactness of a Closed Interval.* Consider a set of open intervals, $(I_\alpha)_{\alpha \in \mathbb{A}}$, such that they form a cover of $E = [a, b]$. Define a subset of $E$:

$$S = \{x \mid x \in [a, b], [a, x] \textit{ can be covered by finitely many sets from } (I_\alpha)_{\alpha \in \mathbb{A}}\}$$

Notice, if we prove that $b \in S$, then this means that $[a, b]$ can be covered by finitely many open sets, proving our theorem.

Firstly, $S \neq \emptyset$, as $a \in S$. Indeed, $a \in [a, b] \subset \bigcup_{\alpha \in \mathbb{A}} I_\alpha$, so there exists *at least one* open interval, call it $I_{\alpha_1}$, such that $a \in I_{\alpha_1}$. Thus, the degenerate interval $[a, a] = \{a\}$ can be covered by a finite (1) number of intervals from $(I_\alpha)_{\alpha \in \mathbb{A}}$, so $a \in S$.

Secondly, $S$ is bounded above by $b$, since any element from $E$ is at most $b$. Since $S$ is non-empty and bounded above, $supS$ exists, and $supS \leq b$.

Thirdly, we claim that $supS = b$. We proceed by contradiction. Since $E$ is closed, $\exists c \in S \ : \ c = supS$. Now, assume that $c < b$. As in the case for $a$, if $c \in [a, b] \subset \bigcup_{\alpha \in \mathbb{A}} I_\alpha$, then there exists some open interval, call it $I_{\alpha'}$, such that $c \in I_{\alpha'}$. Moreover, since $I_{\alpha'}$ is an open interval:

$$\exists \delta > 0 \ : \ (c - \delta, c + \delta) \subset I_{\alpha'}$$

By the approximation property of the supremum, we know that:

$$\exists s \in S \ : \ s \in (c - \delta, c]$$

By definition of $S$, there exists a finite set of intervals which cover the interval $[a, s]$:

$$[a, s] \subset \bigcup_{i=1}^{n} I_{\alpha_i}$$

Moreover, $s \in (c-\delta, c+\delta) \subset I_{\alpha'}$, so:

$$[a, c+\delta) \subset I_{\alpha'} \cup \bigcup_{i=1}^{n} I_{\alpha_i}$$

But this is a contradiction: there are many points between $c$ and $c+\delta$ which belong to $S$, but this contradicts the fact that $c$ is the supremum of $S$. Thus, our assumption that $c < b$ must be false, and so:

$$c = supS = b$$

Now, since we have shown that $c$ is in $S$, and $b = c$, then $b \in S$ (using the same cover, $I_{\alpha'} \cup \bigcup_{i=1}^{n} I_{\alpha_i}$). Thus, there exists a finite subcovering of the closed interval $[a, b]$, as required.

□

# 4 Sequences in $\mathbb{R}$

## 4.1 Convergence of a Sequence

- **When does a sequence converge?**
  - let $(x_n)_{n\in\mathbb{N}}$ be a sequence
  - we say it **converges** to $a \in \mathbb{R}$ if:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ :\ n \geq N \implies |x_n - a| < \varepsilon$$

  and we write:

$$\lim_{n\to\infty} x_n = a$$
$$x_n \to a$$

- **What is the limit of a sequence?**
  - the value to which a sequence converges
- **What does it mean if a sequence diverges?**
  - if the sequence does not converge to any real number, then it is said to **diverge**

# 5 Exercises

1. **Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, and $a \in \mathbb{R}$. Suppose that $a_n \to a$. Show that:**

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \to a$$

*Proof.* Since $a_n$ converges, in particular:

1. $a_n$ is *bounded*: $\forall n \in \mathbb{N},\ \exists M \in \mathbb{Z}^+\ :\ |a_n| < M$
2. $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ : \forall\ n \geq N \implies |a_n - a| < \frac{\varepsilon}{2}$

We claim that $\frac{a_1+a_2+\ldots+a_n}{n} \to a$ so, by the definition of the limit, we require that:

$$\forall \varepsilon > 0,\ \exists N^* \in \mathbb{N}\ :\ \forall n \geq N^* \implies \left|\frac{a_1 + a_2 + \ldots + a_n}{n} - a\right| < \varepsilon$$

Now, let:

$$N^* = \max\left\{N, \frac{4M(N-1)}{\varepsilon}\right\}$$

and let $n \geq N^*$.

Using the triangle inequality, we can split the LHS into 2 summations: since we are considering $n \geq N^*$, in particular $n \geq N$, so we can have one summation with terms $(a_i)_{1 \leq i < N}$, and another one with terms $(a_i)_{N \leq i \leq n}$:

$$\begin{aligned}
\left|\frac{a_1 + a_2 + \ldots + a_n}{n} - a\right| &= \frac{1}{n}\left|(a_1 - a) + (a_2 - a) + \ldots + (a_n - a)\right| \\
&\leq \frac{1}{n}|a_1 - a| + |a_2 - a| + \ldots + |a_n - a| \\
&= \frac{1}{n}\left(\sum_{i=1}^{N-1}|a_i - a| + \sum_{i=N}^{n}|a_i - a|\right)
\end{aligned}$$

Now, since $a_n$ is bounded, its limit is also bounded, so $\exists M \in \mathbb{Z}^+$:

$$\forall i \in \mathbb{N},\ |a_i| < M \quad \text{and} \quad |a| < M$$

For $i < N$, the largest possible value of $|a_i - a|$ must be $2M$ (for example if $a_i = M, a = -M$). This also follows from the triangle inequality ($|a_i - a| < |a_i| + |a| < M + M$). Thus:

$$\sum_{i=1}^{N-1}|a_i - a| \leq 2M(N-1)$$

For $i \geq N$, we can impose a tighter bound, as we know that $\forall i \geq N,\ |a_i - a| < \frac{\varepsilon}{2}$, so:

$$\sum_{i=N}^{n}|a_i - a| < \frac{\varepsilon(n-N+1)}{2}$$

Thus, it follows that:

$$\frac{1}{n}\left(\sum_{i=1}^{N-1}|a_i - a| + \sum_{i=N}^{n}|a_i - a|\right) < \frac{2M(N-1)}{n} + \frac{\varepsilon(n-N+1)}{2n}$$

Since $n \geq N^*$, then either $n \geq N \geq \frac{4M(N-1)}{\varepsilon}$ or $n \geq \frac{4M(N-1)}{\varepsilon} \geq N$.

Since $n \geq \frac{4M(N-1)}{\varepsilon}$, then:

$$\frac{2M(N-1)}{n} \leq \frac{2M(N-1)}{\frac{4M(N-1)}{\varepsilon}} = \frac{\varepsilon}{2}$$

Moreover, since $n - N + 1 \leq n \implies \frac{n-N+1}{n} \leq 1$ then:

$$\frac{\varepsilon(n-N+1)}{2n} \leq \frac{\varepsilon}{2}$$

By choosing $n \geq N^*$, we thus ensure that:

$$\begin{aligned}\left|\frac{a_1 + a_2 + \ldots + a_n}{n} - a\right| &= \frac{1}{n}\left|(a_1 - a) + (a_2 - a) + \ldots + (a_n - a)\right| \\ &\leq \frac{1}{n}\left(\sum_{i=1}^{N-1}|a_i - a| + \sum_{i=N}^{n}|a_i - a|\right) \\ &< \frac{2M(N-1)}{n} + \frac{\varepsilon(n-N+1)}{n} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

Thus, $\forall \varepsilon > 0,\ \exists N^* \in \mathbb{N}$ such that for all $n \geq N^*$:

$$\left|\frac{a_1 + a_2 + \ldots + a_n}{n} - a\right| < \varepsilon$$

By the definition of the limit, it follows that:

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \to a$$

□

2. (a) **Suppose that $(a_n)$ is a convergent sequence of real numbers, and let $A$ be the set:**

$$A = \{a_1, a_2, \ldots, a_n, \ldots\}$$

**Prove that $A$ has a maximum or a minimum or both.**

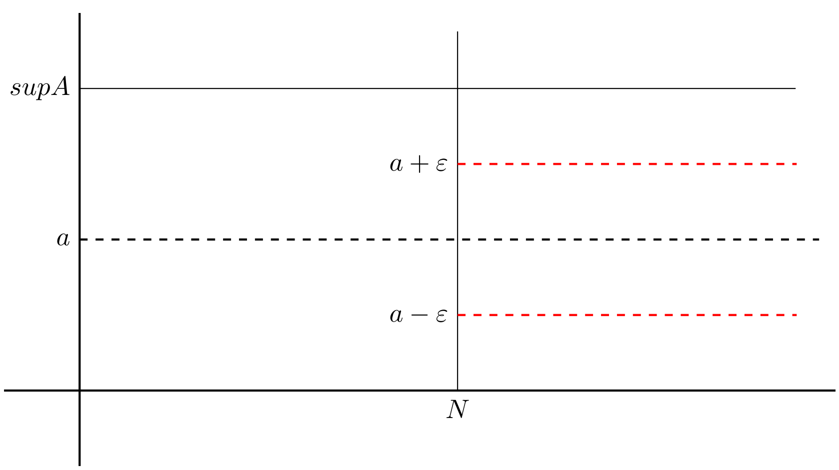
Firstly, since $a_n$ converges, it is a bounded sequence, and so $A$ is also bounded. Since $A$ is non-empty, it follows that $supA$ and $infA$ exist.

Secondly, $a_n$ converges, say to $a \in \mathbb{R}$. Then, one of the 3 possibilities must be true:

1. $infA = a = supA$
2. $a < supA$
3. $infA < a$

If $infA = a = supA$, then it means that the sequence is constant, and so both the maximum and minimum exist, and are the same.

Assume that $a < supA$. Then we have the following situation:



$\forall n \geq N$, $a_n$ are within $\varepsilon$ of the limit. What this means is that $supA$ must be within the first $N-1$ terms. To ensure this, pick:

$$\varepsilon = \frac{supA - a}{2}$$

Then, we are certain that $\exists N \in \mathbb{N}$ such that if $n \geq N$:

$$|a_n - a| < \frac{supA - a}{2}$$

Then:

$$a - \frac{supA - a}{2} < a_n < a + \frac{supA - a}{2}$$

So it follows that:

$$supA = \max\{a_1, a_2, \ldots, a_{N-1}\}$$

and thus the maximum exists.

Using a similar argument, if $infA < a$, then:

$$infA = \min\{a_1, a_2, \ldots, a_{N-1}\}$$

(b) **Give an example of a bounded sequence which has neither a maximum nor a minimum**

$$a_n = (-1)^n \left(1 - \frac{1}{n}\right)$$

The sequence is clearly bounded by -1 and 1, but it never reaches these values, so it has neither a maximum nor a minimum.

3. **Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, and let $a \in \mathbb{R}$. Suppose that every subsequence of $a_n$ has a sub-subsequence that converges to $a$. Prove that $a_n \to a$.**

We proceed by contradiction. Assume that $a_n$ diverges. Then:

$$\exists \varepsilon > 0 \;:\; \forall N \in \mathbb{N},\ \exists n \geq N \;:\; |a_n - a| \geq \varepsilon$$

Clearly, we can produce a subsequence of terms in $a_n$, such that $|a_{n_k} - a| \geq \varepsilon$:

$$n_1 \geq N \implies |a_{n_1} - a| \geq \varepsilon$$

$$n_2 \geq N = n_1 + 1 \implies |a_{n_2} - a| \geq \varepsilon$$

In general, we can obtain $a_{n_k}$ by setting $N = a_{n_{k-1}}$. Thus, we ensure that:

$$n_1 < n_2 < \ldots < n_k \ldots$$

and we obtain the subsequence:

$$a_{n_1}, a_{n_2}, \ldots, a_{n_k}, \ldots$$

Thus, we have found a subsequence, such that:

$$|a_{n_k} - a| \geq \varepsilon$$

By the assumption of the question, we can define a sub-subsequence, such that:

$$|a_{n_{k_l}} - a| < \varepsilon$$

But this is clearly a contradiction, as every term of our subsequence differs from $a$ at least by $\varepsilon$, so no subsequence of it can possibly satisfy $|a_{n_{k_l}} - a| < \varepsilon$. Thus, our initial assumption was wrong, and it must be the case that:

$$a_n \to a$$