Honours Algebra - Week 8 - Inner Product Spaces and Orthogonality

Antonio León Villares

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Contents

Based on the notes by Iain Gordon, Sections 5.1 - 5.2

1 Introducing Inner Product Spaces

1.1 Real Inner Product Spaces

- **What is an inner product?**
  - let V be a **vector space** over $\mathbb{R}$
  - an **inner product** on V is a **mapping**:

    $$(*,*) : V \times V \to \mathbb{R}$$

    such that:

    1. $(\lambda \underline{x} + \mu \underline{y}, \underline{z}) = \lambda(\underline{x}, \underline{z}) + \mu(\underline{y}, \underline{z})$
    2. $(\underline{x}, \underline{y}) = (\underline{y}, \underline{x})$
    3. $(\underline{x}, \underline{x}) \geq 0$, and $(\underline{x}, \underline{x}) = 0 \iff \underline{x} = \underline{0}$

- **What is a real inner product space?**
  - **real vector space** endowed with an **inner product**

- **How are inner products related to multilinear forms?**
  - recall how we defined **symmetric bilinear forms**:

A ***bilinear form*** *is a* ***mapping****:*

$$H : U \times V \to W$$

where U, V, W *are* ***F-Vector Spaces****, satisfying:*

$$H(u_1 + u_2, v) = H(u_1, v) + H(u_2, v)$$

$$H(\lambda u, v) = \lambda H(u, v)$$

$$H(u, v_1 + 1_2) = H(u, v_1) + H(u, v_2)$$

$$H(u, \lambda v) = \lambda H(u, v)$$

A ***bilinear form*** *is* ***symmetric*** *if:*

$$H(u, v) = H(v, u), \qquad \forall u, v \in U$$

  - definition of **inner product** isn't explicit about linearity of the second entry, but this follows by using the **symmetric property** (2):

    $$(\underline{x}, \lambda \underline{y} + \mu \underline{z}) = (\lambda \underline{y} + \mu \underline{z}, \underline{x}) = \lambda(\underline{y}, \underline{x}) + \mu(\underline{z}, \underline{x}) = \lambda(\underline{x}, \underline{y}) + \mu(\underline{x}, \underline{z})$$

  - hence, an **inner product** is a **symmetric bilinear form**, which is **positive definite** (from condition 3)

1.1.1 Examples

- Consider a mapping:
$$(\underline{x}, \underline{y}) = x_1 + y_1 + 2x_2y_2$$
This is **not** an inner product: it fails property 1. Indeed:
$$((0,0),(1,0)) = 0 + 1 + 2(0)(0) = 1$$
However, notice that if $\lambda = 0$, property 1 tells us that:
$$(\lambda\underline{x}, \underline{y}) = \lambda(\underline{x}, \underline{y}) = 0 \implies (\underline{0}, \underline{y}) = 0$$
independently of the inner product used.

- Consider a mapping:
$$(\underline{x}, \underline{y}) = x_1y_2 + 2x_2y_1$$
This is **not** an inner product: it fails property 2. Indeed:
$$((1,0),(0,1)) = (1)(1) + 2(0)(0) = 1$$
but:
$$((0,1),(1,0)) = (0)(0) + 2(1)(1) = 2$$
So:
$$((1,0),(0,1)) \neq ((0,1),(1,0))$$

- Consider a mapping:
$$(\underline{x}, \underline{y}) = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 3x_2y_2$$
This is **not** an inner product: it fails property 3. This is typically the hardest property to verify. Typically, it is shown by putting the inner product in terms of squares, since these satisfy the property.

  However, consider:
$$\begin{aligned}(\underline{x}, \underline{x}) &= x_1^2 + 2x_1x_2 + 2x_2x_1 + 3x_2^2 \\ &= x_1^2 + 4x_1x_2 + 3x_2 \\ &= (x_1 + 2x_2)^2 - x_2^2\end{aligned}$$
But now notice, if we pick:
$$\underline{x} = (-2x_2, x_2)$$
We get that:
$$(\underline{x}, \underline{x}) = -x_2^2 < 0, \forall x_2 \in \mathbb{R} \setminus \{0\}$$

- Consider a mapping:
$$(\underline{x}, \underline{y}) = x_1y_1 + x_1y_2 + x_2y_1 + 3x_2y_2$$
This **is** a inner product. We check each property:

  1.
$$\begin{aligned}&(\lambda\underline{x} + \mu\underline{y}, \underline{z}) \\ =&(\lambda x_1 + \mu y_1)z_1 + (\lambda x_1 + \mu y_1)z_2 + (\lambda x_2 + \mu y_2)z_1 + 3(\lambda x_2 + \mu y_2)z_2 \\ =&\lambda x_1z_1 + \lambda x_1z_2 + \lambda x_2z_1 + \lambda 3x_2z_2 + \mu y_1z_1 + \mu y_1z_2 + \mu y_2z_1 + \mu 3y_2z_2 \\ =&\lambda(\underline{x}, \underline{z}) + \mu(\underline{y}, \underline{z})\end{aligned}$$

2.

$$\begin{aligned}(\underline{x}, \underline{y}) &= x_1y_1 + x_1y_2 + x_2y_1 + 3x_2y_2 \\ &= y_1x_1 + y_2x_1 + y_1x_2 + 3y_2x_2 \\ &= y_1x_1 + y_1x_2 + y_2x_1 + 3y_2x_2 \\ &= (\underline{y}, \underline{x})\end{aligned}$$

3.

$$\begin{aligned}(\underline{x}, \underline{x}) &= x_1x_1 + x_1x_2 + x_2x_1 + 3x_2x_2 \\ &= x_1^2 + 2x_1x_2 + 3x_2^2 \\ &= (x_1 + x_2)^2 + 2x_2^2 \\ &\geq 0\end{aligned}$$

If $x_1 = x_2 = 0$, this is 0, so indeed, $(\underline{x}, \underline{x}) \geq 0$, with equality when $\underline{x} = \underline{0}$

- the **dot product**:

$$\underline{x} \cdot \underline{y} = \sum_{i=1}^{n} x_i y_i = \underline{x}^T \cdot \underline{y}$$

is an inner product

1.1.2 Exercises (TODO)

1. **Confirm the following:**
   (a) **The following is an inner product:**

$$(\underline{x}, \underline{y}) = x_1y_1 + 4x_2y_2$$

   (b) **The following is an inner product:**

$$(\underline{x}, \underline{y}) = 2x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$$

   (c) **The following is not an inner product:**

$$(\underline{x}, \underline{y}) = x_1y_1 + 2x_1y_2 + 2x_2y_1 + x_2y_2$$

   (d) **The following is an inner product, for** $a, b \in \mathbb{R},\ a < b,\ P, Q \in \mathbb{R}[X]_{<n}$**:**

$$(P, Q) = \int_a^b P(X)Q(X)dX$$

1.2 Complex Inner Product Spaces

- **How are complex inner products defined?**
  - let V be a **vector space** over $\mathbb{C}$
  - an **inner product** on V is a **mapping**:

$$(*, *) : V \times V \to \mathbb{C}$$

    such that:
    1. $(\lambda\underline{x} + \mu\underline{y}, \underline{z}) = \lambda(\underline{x}, \underline{z}) + \mu(\underline{y}, \underline{z})$

2. $(\underline{x}, \underline{y}) = \overline{(\underline{y}, \underline{x})}$
3. $(\underline{x}, \underline{x}) \geq 0$, and $(\underline{x}, \underline{x}) = 0 \iff \underline{x} = \underline{0}$

  - here $\bar{z}$ denotes the **complex conjugate**

- **How does the notion of $(\underline{x}, \underline{x}) \geq 0$ make any sense for complex numbers?**
  - the **complex inner product** always maps to the reals
  - this follows from the second property:

$$(\underline{x}, \underline{x}) = \overline{(\underline{x}, \underline{x})} \iff (\underline{x}, \underline{x}) \in \mathbb{R}$$

- **What is a complex inner product space?**
  - a **complex vector space** endowed with an **inner product**
- **Is the complex inner product a symmetric bilinear form?**
  - no: it fails linearity in the second element:

$$\begin{aligned} &(\underline{z}, \lambda\underline{x} + \mu\underline{y}) \\ =&\overline{(\lambda\underline{x} + \mu\underline{y}, \underline{z})} \\ =&\overline{\lambda(\underline{x}, \underline{z}) + \mu(\underline{y}, \underline{z})} \\ =&\bar{\lambda}(\underline{z}, \underline{x}) + \bar{\mu}(\underline{z}, \underline{y}) \end{aligned}$$

- **How can we define the complex inner product?**
  - it is **sesquilinear** (it is linear in the first element, and **skew-linear** in the second element, since $f(\lambda\underline{v}_1) = \bar{\lambda}f(\underline{v}_1)$)
  - it is **hermitian** (since $(\underline{x}, \underline{y}) = \overline{(\underline{y}, \underline{x})}$)
  - it is **positive definite** (property 3)
- **What is a real Euclidean vector space?**
  - finite dimensional real inner product space
- **What is a pre-Hilbert space/unitary space?**
  - a complex inner product space
- **What is a finite dimensional Hilbert space?**
  - a finite dimensional complex inner product space

1.2.1 Examples

- Consider the mapping:

$$((\underline{x}), \underline{y}) = x_1y_1 + x_1y_2 + x_2y_1 + 3x_2y_2$$

  This is **not** a complex inner product. It fails property 2:

$$((1,0),(i,0)) = (1)(i) + (1)(0) + (0)(i) + 3(0)(0) = i$$

$$\overline{((i,0),(1,0))} = ((-i,0),(1,0)) = (-i)(1) + (-i)(0) + (0)(1) + 3(0)(0) = -i$$

  So $((1,0),(i,0)) \neq \overline{((i,0),(1,0))}$. It also fails property 3:

$$((i,0),(i,0)) = i^2 = -1 < 0$$

- Consider the mapping:
$$(\underline{x}, \underline{y}) = x_1\bar{y}_1 + x_1\bar{y}_2 + x_2\bar{y}_1 + 3x_2\bar{y}_2$$
We show this satisfies the third property of the inner product (it also satisfies the first 2, but it is less interesting):
$$\begin{aligned}(\underline{x}, \underline{x}) &= x_1\bar{x}_1 + x_1\bar{x}_2 + x_2\bar{x}_1 + 3x_2\bar{x}_2 \\ &= (x_1 + x_2)\overline{(x_1 + x_2)} + 2x_2\bar{x}_2 \\ &= |x_1 + x_2|^2 + 2|x|^2 \\ &\geq 0\end{aligned}$$

- Consider the vector space $V = Mat(n; \mathbb{C})$, and the mapping:
$$(A, B) = Tr(A^T \bar{B})$$
where $\bar{B}$ is the matrix B with all the entries changed by applying the complex conjugate.

  Then, (A, B) is a inner product.

  1. $$\begin{aligned}(\lambda A + \mu A', B) &= Tr((\lambda A + \mu A')^T \bar{B}) \\ &= Tr(\lambda A^T \bar{B} + \mu A'^T \bar{B}) \\ &= Tr(\lambda A^T \bar{B}) + Tr(\mu A'^T \bar{B}) \\ &= \lambda Tr(A^T \bar{B}) + \mu Tr(A'^T \bar{B}) \\ &= \lambda(A, B) + \mu(A', B)\end{aligned}$$

  2. $$\begin{aligned}(A, B) &= Tr(A^T \bar{B}) \\ &= Tr((\bar{B}^T A)^T) \\ &= Tr(\bar{B}^T A) \\ &= Tr(\overline{B^T \bar{A}}) \\ &= \overline{Tr(B^T \bar{A})} \\ &= \overline{(B, A)}\end{aligned}$$
     where we use the fact that $Tr(A) = Tr(A^T)$, since taking the transpose doesn't change the diagonal elements.

3.
$$\begin{aligned}(A,A) &= Tr(A^T\bar{A})\\ &= Tr(A^T\bar{A})\\ &= \sum_{i=1}^{n}(A^T\bar{A})_{ii}\\ &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n}(A^T)_{ij}(\bar{A})_{ji}\right)\\ &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n}(A)_{ji}(\bar{A})_{ji}\right)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}|(A)_{ji}|^2\\ &\geq 0\end{aligned}$$

with equality holding **if and only if** $A_{ji} = 0 \iff A$ is the 0 matrix, as required.

1.2.2 Exercises (TODO)

1. **Confirm the following:**
   (a) **On $\mathbb{C}$, this is a inner product:**
   $$(\underline{z}, \underline{w}) = z_1\bar{w}_1 + 4z_2\bar{w}_2$$
   (b) **Let $V = C_{\mathbb{C}}[a,b]$ be the vector space of all continuous complex calud functions defined on $[a,b]$, with $a < b$. The mapping:**
   $$(f,g) = \int_a^b f(t)\overline{g}(t)dt$$
   is a inner product.

We can use the inner product as to bring geometric interpretations to vectors.

1.3 Inner Products, Geometry and Orthogonality

- **How is the length of a vector defined?**
  - the **length** or **inner product norm** is:
  $$\|\underline{v}\| = \sqrt{(\underline{v}, \underline{v})}$$
- **What is a unit vector?**
  - a vector with $\|\underline{v}\| = 1$
- **When are 2 vectors orthogonal?**
  - whenever $(\underline{v}, \underline{w}) = 0$
  - we write:
  $$\underline{v} \perp \underline{w}$$
- **What are orthogonal sets?**
  - sets such that:
  $$V \perp W \iff \forall \underline{v} \in V, \underline{w} \in W, \quad \underline{v} \perp \underline{w}$$

1.3.1 Examples

- Inner products allow us to rediscover the Pythgaorean Theorem.

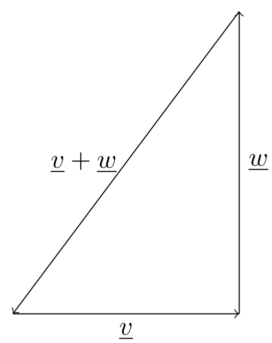
  If $\underline{v}, \underline{w}$ are **orthogonal**, then:

$$\|\underline{v} + \underline{w}\|^2 = (\underline{v} + \underline{w}, \underline{v} + \underline{w}) = (\underline{v}, \underline{v}) + (\underline{v}, \underline{w}) + (\underline{w}, \underline{v}) + (\underline{w}, \underline{w})$$

  Since $\underline{v} \perp \underline{w}$, then $(\underline{w}, \underline{v}) = (\underline{v}, \underline{w}) = 0$, so:

$$\|\underline{v} + \underline{w}\|^2 = (\underline{v}, \underline{v}) + (\underline{w}, \underline{w}) = \|\underline{v}\| + \|\underline{w}\|$$

  As a diagram:



1.3.2 Exercises (TODO)

1. **Show that in an inner product space V we have:**

$$\|\lambda \underline{v}\| = |\lambda| \|\underline{v}\|$$

   with $\lambda \in \mathbb{R}$ **or** $\lambda \in \mathbb{C}$

1.4 Remark: Orthonormal Families

- **What is an orthonormal family?**
  - a **family** of vectors $\{\underline{v}_i\}_{i \in [1,n]}$ such that:

$$\forall i \in [1, n],\ \|\underline{v}_i\| = 1 \qquad and \qquad \forall i, j \in [1, n],\ i \neq j,\ (\underline{v}_i, \underline{v}_j) = 0$$

  - alternatively,

$$(\underline{v}_i, \underline{v}_j) = \delta_{ij}$$

- **What is an orthonormal basis?**
  - a **basis** which is an **orthonormal family**

Let V be a ***inner product space****, with a* ***orthonormal basis*** $\{\underline{v}_i\}$*.*
Then, we can write $\underline{w} \in V$ via:

$$\underline{w} = \sum_{i¡j=1}^{n} \lambda_i \underline{v}_i$$

If we take the inner product with respect to $\underline{v}_i$:

$$\begin{aligned}(\underline{w}, \underline{v}_i) &= \left(\sum_{j=1}^{n} \lambda_j \underline{v}_j, \underline{v}_i \right) \\ &= \sum_{j=1}^{n} \lambda_j \left(\underline{v}_i, \underline{v}_i \right) \\ &= \lambda_i\end{aligned}$$

since the summands are 0, except when $i = j$.
Thus, this gives us the coefficients to use when defining linear combinations:

$$\underline{w} = \sum_{i¡j=1}^{n} (\underline{w}, \underline{v}_i) \underline{v}_i$$

[Remark 5.1.9]

1.5 Theorem: Existence of Orthonormal Basis

Every ***finite*** *dimensional* ***inner product space*** *has a* ***orthonormal basis****. [Theorem 5.1.10]*

Proof. Consider a inner product space V over a field F (with $F = \mathbb{R}$ or $F = \mathbb{C}$).
We prove this theorem by induction on $n = dim_F(V)$.

(1) **Base Case: n = 0**

In this case, we have $V = \{\underline{0}\}$, whose only linearly independent set is $\{\}$, which is certainly orthonormal.

(2) **Inductive Hypothesis: n = k**

Assume true for $n = k$: any vector space of dimension $dim_F(V) = k > 0$ has an orthonormal basis.

(3) **Inductive Step: n = k+1**

Now consider a vector space with $dim_F(V) = k+1$. Then, we can find a vector $\underline{v} \in V$. Scaling it into a unit vector, define:

$$\underline{v}_1 = \frac{\underline{v}}{\|\underline{v}\|}$$

Now, consider the linear mapping:

$$\phi : V \to F$$

defined by:

$$\phi(\underline{w}) = (\underline{w}, \underline{v}_1)$$

Now recall the Rank-Nullity Theorem:

$$dim(V) = dim(im(\phi)) + dim(ker(\phi)) \implies dim(ker(\phi)) = dim(V) - dim(im(\phi))$$

Notice that:

$$\phi(\underline{v}_1) = (\underline{v}_1, \underline{v}_1) = 1$$

so in particular a basis for $im(\phi)$ is $\{\underline{v}_1\}$ so it follows that $dim(im(\phi)) = 1$ (since F can be generated by $\langle 1 \rangle$). Thus:

$$dim(ker(\phi)) = k + 1 - 1 = k$$

The inductive hypothesis thus applies to $ker(\phi)$ (since $ker(\phi)$ is a subspace of V, it is also an inner product space), meaning that $ker(\phi)$ has an orthonormal basis of k vectors.
But now, recall that if $\underline{w} \in ker(\phi)$, then:

$$\phi(\underline{w}) = \underline{0} \iff (\underline{w}, \underline{v}_1) = 0$$

So in particular, $\forall \underline{w} \in ker(\phi),\ \underline{w} \perp \underline{v}_1$. Hence, an **orthonormal basis** for V is given by:

$$ker(\phi) \cup \{\underline{v}_1\}$$

□

2 Orthogonal Complements and Projections

2.1 Proposition: Orthogonal and Complementary Sets

- **What is an orthogonal set?**
  - consider a inner product space V, with $T \subseteq V$ an arbitrary subset.
  - the **orthogonal** to T is the set:

$$T^{\perp} = \{\underline{v} \mid \underline{v} \perp \underline{t}, \forall \underline{t} \in T, \underline{v} \in V\}$$

Let V be an ***inner product space****, with finite dimensional subspace U. Then, $U, U^{\perp}$ are complementary:*

$$V = U \oplus U^{\perp}$$

Proof. For this, we make use of what we proved in week 1. That is 2 subspaces are complementary if:

$$V = U + U^{\perp} \qquad U \cap U^{\perp} = 0$$

Consider $\underline{v} \in U \cap U^{\perp}$. This means that $\underline{v}$ is orthogonal to itself, so:

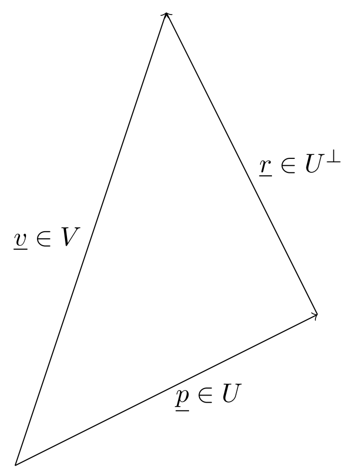
$$(\underline{v}, \underline{v}) = 0$$

But the properties of the inner product say that this is only possible if $\underline{v} = \underline{0}$, as required.

Now we need to show that $\forall \underline{v} \in V$ we can find $\underline{p} \in U, \underline{r} \in U^{\perp}$ such that:

$$\underline{v} = \underline{p} + \underline{r}$$

Intuitively this makes sense:



Now, since U is a subset of V, it is a inner product space, so it has an orthonormal basis, say:

$$\underline{v}_1, \ldots, \underline{v}_n$$

such that:

$$\underline{p} = \sum_{i=1}^{n} \lambda_i \underline{v}_i$$

Since $\underline{p}$ is a projection of $\underline{v}$ onto U, we have that $\lambda_i = (\underline{v}, \underline{v}_i)$.

In other words, it is sufficient to show that:

$$\underline{r} = \underline{v} - \underline{p} = \underline{v} - \sum_{i=1}^{n} \lambda_i \underline{v}_i$$

or alternatively, that $\underline{r} = \underline{v} - \sum_{i=1}^{n} \lambda_i \underline{v}_i$ is perpendicular to any of the basis elements $\underline{v}_j$

Thus, we compute:

$$
\begin{aligned}
(\underline{r}, \underline{v}_j) &= \left(\underline{v} - \sum_{i=1}^{n} \lambda_i \underline{v}_i, \underline{v}_j\right) \\
&= (\underline{v}, \underline{v}_j) - \left(\sum_{i=1}^{n} \lambda_i \underline{v}_i, \underline{v}_j\right) \\
&= (\underline{v}, \underline{v}_j) - \sum_{i=1}^{n} \lambda_i \left(\underline{v}_i, \underline{v}_j\right) \\
&= (\underline{v}, \underline{v}_j) - \sum_{i=1}^{n} \lambda_i \delta_{ij} \\
&= (\underline{v}, \underline{v}_j) - \lambda_j \\
&= 0
\end{aligned}
$$

so as required, $\underline{r}$ is orthogonal to each of the basis vectors, so it is orthogonal to any element in U.

□

2.2 Proposition: Orthogonal Projections

Let U be a ***finite*** *dimensional* ***subspace*** *of the* ***inner product space*** *V. Define π_U as the* ***orthogonal projection*** *from V to U as the map:*

$$\pi_U : V \to V$$

defined by:

$$\pi_U(\underline{v}) = \pi_U(\underline{p} + \underline{r}) = \underline{p}$$

Then:

1. *π_U is a* ***linear mapping*** *and:*

$$im(\pi_U) = U \qquad ker(\pi_U) = U^{\perp}$$

2. *if $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is an* ***orthonormal basis*** *of U, then:*

$$\pi_U(\underline{v}) = \sum_{i=1}^{n} (\underline{v}, \underline{v}_i)\underline{v}_i$$

3. *$\pi_U^2 = \pi_U$ (its* ***idempotent)***

[Proposition 5.2.4]

In general, if we have 2 vectors $\underline{u}, \underline{v}$, and the inner product is the dot product. we can define the projection of $\underline{v}$ onto $\underline{u}$ via:

$$proj_{\underline{u}}(\underline{v}) = \frac{(\underline{u} \cdot \underline{v}}{\|\underline{u}\|^2}\underline{u}$$

This exploits the fact that:

$$\underline{u} \cdot \underline{v} = \|\underline{u}\|\|\underline{v}\|\cos(\theta)$$

2.2.1 Examples

Consider the vector space:

$$V = \mathbb{R}[X]_{<4}$$

with inner product:

$$(P, Q) = \int_0^1 P(x)Q(X)dX$$

Consider the subset:

$$T = \{1, X\}$$

Then its **orthogonal complement** is the set:

$$T^{\perp} = \left\{A \mid \int_0^1 AdX = 0 \ \wedge \ \int_0^1 AXdX = 0, \ A \in V\right\}$$

So for example, consider:

$$A(X) = aX^3 + bX^2 + cX + d$$

Then:

$$\int_0^1 AdX = \int_0^1 aX^3 + bX^2 + cX + ddX = \frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d$$

$$\int_0^1 AXdX = \int_0^1 aX^4 + bX^3 + cX^2 + dXdX = \frac{a}{5} + \frac{b}{4} + \frac{c}{3} + \frac{d}{2}$$

Thus, we require that:

$$\frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = 0 = \frac{a}{5} + \frac{b}{4} + \frac{c}{3} + \frac{d}{2}$$

This system has infinitely many solutions, so we can parametrise the solution, using $a = s, b = t$, such that:

$$\frac{s}{4} + \frac{t}{3} + \frac{c}{2} + d = 0 = \frac{s}{5} + \frac{t}{4} + \frac{c}{3} + \frac{d}{2}$$

We then just need to solve for c, d. The first equation says that:

$$d = -\frac{s}{4} - \frac{t}{3} - \frac{c}{2}$$

whilst the second one says that:

$$d = -\frac{2s}{5} - \frac{t}{2} - \frac{2c}{3}$$

So:

$$
\begin{aligned}
&-\frac{s}{4}-\frac{t}{3}-\frac{c}{2}=-\frac{2s}{5}-\frac{t}{2}-\frac{2c}{3} \\
\implies &\frac{2c}{3}-\frac{c}{2}=-\frac{2s}{5}-\frac{t}{2}+\frac{s}{4}+\frac{t}{3} \\
\implies &\frac{c}{6}=-\frac{3s}{20}-\frac{t}{6} \\
\implies &c=-\frac{18s}{20}-t \\
\implies &c=-\frac{9s}{10}-t
\end{aligned}
$$

So then:

$$
\begin{aligned}
&d=-\frac{s}{4}-\frac{t}{3}-\frac{c}{2} \\
\implies &d=-\frac{s}{4}-\frac{t}{3}+\frac{9s}{20}+\frac{t}{2} \\
\implies &d=\frac{4s}{20}+\frac{t}{6} \\
\implies &d=\frac{s}{5}+\frac{t}{6}
\end{aligned}
$$

Hence, if for example we pick $s=1, t=0$, we'd get that:

$$A(X)=X^3-\frac{9}{10}X+\frac{1}{5}\in T^{\perp}$$

2.3 Theorem: The Cauchy Schwarz Inequality

*Let $\underline{v}, \underline{w}$ be vectors in an **inner product space**. Then:*

$$|(\underline{v},\underline{w})|\leq\|\underline{v}\|\|\underline{w}\|$$

*with equality **if and only if** $\underline{v}$ and $\underline{w}$ are **linearly dependent**.*

Proof. If $\underline{w}=0$, then $(\underline{v},\underline{w})=0=\|\underline{v}\|\|\underline{w}\|$, as expected, since $\underline{w}$ is linearly dependent with all vectors.

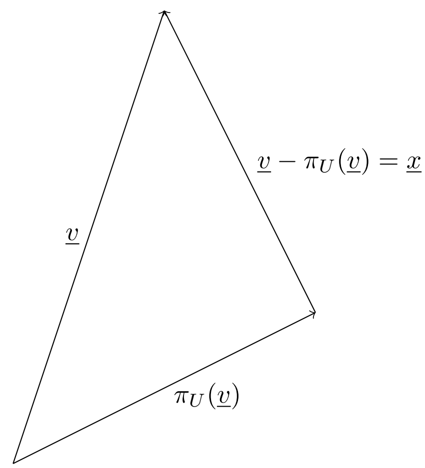
Thus, consider non-zero $\underline{w}$. In particular, consider the subspace generated by $\underline{w}$:

$$U=\langle\underline{w}\rangle$$

and define:

$$\underline{x}=\underline{v}-\pi_U(\underline{v})$$

It is the case that $\underline{x}\perp\underline{U}$. This is easy to see diagrammatically:



Then, we can apply the Pythgorean Theorem:

$$\|\underline{v}\|^2 = \|\underline{x} + \pi_U(\underline{v}\|^2 = \|\underline{x}\|^2 + \|\pi_U(\underline{v})\|^2$$

Notice, $\{\underline{w}\}$ is an orthogonal basis for U, so $\{\underline{w}/\|\underline{w}\|\}$ is an **orthonormal basis**. Call $\underline{w}' = \underline{w}/\|\underline{w}\|$. It follows from (2.2) that:

$$\pi_U(\underline{v}) = (\underline{v}, \underline{w}')\,\underline{w}'$$

So then:

$$\begin{aligned}\|\pi_U(\underline{v})\|^2 &= (\|\,(\underline{v}, \underline{w}')\,\underline{w}'\|)^2 \\ &= |(\underline{v}, \underline{w}')|^2\|\underline{w}'\|^2 \\ &= \left|\left(\underline{v}, \frac{\underline{w}}{\|\underline{w}\|}\right)\right|^2 \left\|\frac{\underline{w}}{\|\underline{w}\|}\right\|^2 \\ &= \frac{|(\underline{v}, \underline{w}|^2\|\underline{w}\|^2}{\|\underline{w}\|^4} \\ &= \frac{|(\underline{v}, \underline{w})|^2}{\|\underline{w}\|^2}\end{aligned}$$

Thus:

$$\|\underline{v}\|^2 = \|\underline{x}\|^2 + \|\pi_U(\underline{v})\|^2 = \|\underline{x}\|^2 + \frac{|(\underline{v}, \underline{w})|^2}{\|\underline{w}\|^2} \geq \frac{|(\underline{v}, \underline{w})|^2}{\|\underline{w}\|^2}$$

Equality only occurs when $\underline{x} = 0$; that is, when $\underline{v} = \pi_U(\underline{v}) \implies \underline{v} \in U \implies \underline{v} = \lambda\underline{w}$. Multiplying through by $\|\underline{w}\|^2$:

$$\|\underline{v}\|^2\|\underline{w}\|^2 \geq |(\underline{v}, \underline{w})|^2$$

Finally, taking the square root of both sides:

$$\|\underline{v}\|\|\underline{w}\| \geq |(\underline{v}, \underline{w})|$$

□

Clasically the Cauchy-Schwarz inequality is presented as:

$$\sum_{i=1}^{n} x_i y_i \leq \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}$$

where the inner product uses the standard dot product.

2.4 Inner Products for Geometry: The Angle

- **What is the angle between 2 vectors?**
  - notice, from Cauchy-Schwarz:

$$\|\underline{v}\|\|\underline{w}\| \geq |(\underline{v}, \underline{w})| \implies \frac{|(\underline{v}, \underline{w}|}{\|\underline{v}\|\|\underline{w}\|} \leq 1 \implies -1 \leq \frac{(\underline{v}, \underline{w}}{\|\underline{v}\|\|\underline{w}\|} \leq 1$$

  - thus, we can find some unique $\theta \in [0, \pi]$ such that:

$$\cos(\theta) = \frac{(\underline{v}, \underline{w})}{\|\underline{v}\|\|\underline{w}\|}$$

  - this θ can be defined as the **angle** between $\underline{v}$ and $\underline{w}$

2.5 Corollary: Properties of the Norm

*The **norm** on an **inner product space** V satisfies the follow, $\forall \underline{v}, \underline{w} \in V,\ \lambda \in F$:*

1. $\|\underline{v}\| \geq 0$ *and* $\|\underline{v}\| = 0 \iff \underline{v} = \underline{0}$
2. $\|\lambda \underline{v}\| = |\lambda|\|\underline{v}\|$
3. $\|\underline{v} + \underline{w}\| \leq \|\underline{v}\| + \|\underline{w}\|$

Proof. 1. Follows directly from axiomso f an inner product:

$$(\underline{v}, \underline{v}) \geq 0 \implies \|\underline{v}\|^2 \geq 0 \implies \|\underline{v}\| \geq 0$$

2. This was an exercise
3. We compute:

$$\|\underline{v} + \underline{w}\|^2 = (\underline{v} + \underline{w}, \underline{v} + \underline{w}) = \|\underline{v}^2\| + \|\underline{w}\|^2 + 2Re((\underline{v}, \underline{w}))$$

We have that:

$$2Re((\underline{v}, \underline{w})) \leq 2|(\underline{v}, \underline{w})|$$

So by Cauchy-Schwarz:

$$2Re((\underline{v}, \underline{w})) \leq 2\|\underline{v}\|\|\underline{w}\|$$

Thus:

$$\|\underline{v} + \underline{w}\|^2 \leq \|\underline{v}^2\| + \|\underline{w}\|^2 + 2\|\underline{v}\|\|\underline{w}\| = (\|\underline{v}\| + \|\underline{w}\|)^2$$

The result follows by taking the square root

□

2.6 Theorem: The Gram-Schmidt Process

Let:

$$\underline{v}_1, \ldots, \underline{v}_k$$

be a ***linearly independent*** *set of vectors in an* ***inner product*** *space V. Then, there exists an* ***orthonormal family:***

$$\underline{w}_1, \ldots, \underline{w}_k$$

such that:

$$\underline{w}_i \in \underline{v}_i + \langle \underline{v}_{i-1}, \ldots, \underline{v}_1 \rangle, \qquad i \in [1, k]$$

That is, each $\underline{w}_i$ is composed by considering the sum of $\underline{v}_i$ and an element of the $i-1$th dimensional subspace generated by the $\underline{v}_1, \ldots, \underline{v}_{i-1}$.

Proof. We know that we can decompose any $\underline{v}_i$ using complementary (orthogonal) subspaces:

$$\underline{v}_i = \underline{p}_i + \underline{r}_i$$

where:

$$\underline{p}_i = \pi_U(\underline{v}_i), \qquad U = \langle \underline{v}_1, \ldots, \underline{v}_{i-1} \rangle$$

and $\underline{r}_i \in U^{\perp}$.

We can then just define:

$$\underline{w}_i = \frac{\underline{r}_i}{\|\underline{r}_i\|}$$

In this way, we ensure that anything spanned by $\underline{v}_1, \ldots, \underline{v}_k$ is also spanned by $\underline{w}_1, \ldots, \underline{w}_k$, since each $\underline{w}_i$ is a linear combination of $\underline{v}_i$ and $\underline{p}_i$ (and each $\underline{p}_i$ is projected into the subspace spanned by $\underline{v}_1, \ldots, \underline{v}_{i-1}$). $\square$

2.6.1 Worked Example: Applying the Gram-Schmidt Process to Construct an Orthonormal Basis

The Gram-Schmidt Process

Let $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ be a basis for a subspace W of $\mathbb{R}^n$ and define the following:

$$\mathbf{v}_1 = \mathbf{x}_1, \qquad W_1 = \operatorname{span}(\mathbf{x}_1)$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1, \qquad W_2 = \operatorname{span}(\mathbf{x}_1, \mathbf{x}_2)$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2, \qquad W_3 = \operatorname{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$$

$$\vdots$$

$$\mathbf{v}_k = \mathbf{x}_k - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_k}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_k}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \cdots - \left(\frac{\mathbf{v}_{k-1} \cdot \mathbf{x}_k}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}}\right)\mathbf{v}_{k-1}, \qquad W_k = \operatorname{span}(\mathbf{x}_1, \dots, \mathbf{x}_k)$$

Then for each $i = 1, \dots, k$, $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$ is an orthogonal basis for W_i. In particular, $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis for W.

Consider $\mathbb{R}^4$ endowed with the standard dot product. Consider a subspace V with basis:

$$\underline{v}_1 = (1,1,0,0) \quad \underline{v}_2 = (1,0,1,1) \quad \underline{v}_3 = (1,0,0,1)$$

We use Gram-Schmidt to construct an orthonormal basis $\{\underline{w}_1, \underline{w}_2, \underline{w}_3\}$.

(1) $\underline{w}_1$

The simplest step: we just convert $\underline{v}_1$ into a unit vector

$$\underline{w}_1 = \frac{1}{\sqrt{2}}(1,1,0,0)$$

(2) $\underline{w}_2$

Any element in the space spanned by $\underline{v}_1$ can be written as $\lambda\underline{v}_1$, so we seek λ such that:

$$\underline{w}_2' = \underline{v}_2 - \lambda\underline{v}_1 = (1-\lambda, -\lambda, 1, 1)$$

is orthogonal to $\underline{w}_1$. Computing the dot product:

$$(\underline{w}_2', \underline{w}_1) = \frac{1}{\sqrt{2}}(1-\lambda-\lambda) = 0 \implies \lambda = \frac{1}{2}$$

Thus:

$$\underline{w}_2' = \left(\frac{1}{2}, -\frac{1}{2}, 1, 1\right) \implies \underline{w}_2 = \frac{1}{\sqrt{10}}(1,-1,2,2)$$

(3) $\underline{w}_3$

In a similar vein:

$$\underline{w}_3' = \underline{v}_3 - \lambda_1 \underline{w}_1 - \lambda_2 \underline{w}_2$$

(whilst we could use the formula, it is good to do it in this way, since it is general, and so, more didactic for future cases).

We now need to compute 2 dot products, ensuring that they are 0:

$$(\underline{w}_3', \underline{w}_1) = (\underline{v}_3, \underline{w}_1) - \lambda_1 = \frac{2}{\sqrt{2}} - \lambda_1 \implies \lambda_1 = \frac{1}{\sqrt{2}}$$

(here I used the fact that $(\underline{w}_1, \underline{w}_1) = 1$ and $(\underline{w}_1, \underline{w}_2) = 0$)

$$(\underline{w}_3', \underline{w}_2) = (\underline{v}_3, \underline{w}_2) - \lambda_2 = \frac{3}{\sqrt{10}} - \lambda_2 \implies \lambda_2 = \frac{3}{\sqrt{10}}$$

Thus:

$$\underline{w}_3' = \frac{1}{5}(1, -1, -3, 2) \implies \underline{w}_3 = \frac{1}{\sqrt{15}}(1, -1, -3, 2)$$

3 Workshop

1. **True or false. Let $h : Mat(2; \mathbb{C}) \to Mat(2; \mathbb{C})$ be the mapping given by the conjugate transpose:**

$$h(M) = \overline{M}^T$$

   Then h is a ring homomorphism.

   This is false. It fails linearity of multiplication. If $A, B \in Mat(2; \mathbb{C})$ then:

$$h(AB) = (\overline{AB})^T = (\bar{A}\bar{B})^T = \bar{B}^T \bar{A}^T = h(B)h(A) \neq h(A)h(B)$$

   This is an example of a **ring antihomomorphism**, which reverses the order of multiplication.

2. **Which of the following are inner products on $\mathbb{R}^2$?**

> *For this, when disproving I showed that they failed the inner product axioms in general (using variables); however, in the solutions, specific counterexamples are used, which I think is faster, so I will use that.*
> *In terms of coming up with counterexamples, you can just intuitively see which of the properties can be failed (i.e symmetry will fail if there are non-symmetric terms; linearity might fail if the constants are affected by some non-linear function, etc...).*

   (a) $(\underline{x}, \underline{y}) = x_1^2 y_1^2 + x_2 y_2$

   The presence of squares throughout indicates that sesquilinearity might fail. For example, consider $\underline{x} = 2(1, 0)^T = (2, 0)$ and $\underline{y} = (1, 0)$. Then:

$$(\underline{x}, \underline{y}) = 2^2 \times 1^2 = 4 \neq 2 \times (1^2 \times 1^2)$$

   (b) $(\underline{x}, \underline{y}) = x_1 y_2 - x_2 y_1$

   The terms are not symmetric, so this might fail symmetry. Indded, with $\underline{x} = (1, 0)^T, \underline{y} = (0, 1)^T$:

$$(\underline{x}, \underline{y}) = 1 - 0 = 1$$

$$(\underline{y}, \underline{x}) = 0 - 1 = -1$$

   So:

$$(\underline{x}, \underline{y}) \neq (\underline{y}, \underline{x})$$

(c) $(\underline{x}, \underline{y}) = x_1y_1 + x_1y_2 + x_2y_1 - 3x_2y_2$
This one will fail positivity. For example, with $\underline{x} = (0,1)^T$:

$$(\underline{x}, \underline{x}) = -3 < 0$$

(d) $(\underline{x}, \underline{y}) = x_1y_1$
This also fails positivity, since if $\underline{x} = (1,0)^T$:

$$(\underline{x}, \underline{x}) = 0$$

but $\underline{x} \neq 0$.

(e) $(\underline{x}, \underline{y}) = 3x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2$
This one will be an inner product.

(1) **Sesquilinearity**

$$\begin{aligned}
(\lambda\underline{x} + \underline{y}, \underline{z}) &= 3(\lambda x_1 + y_1)z_1 - (\lambda x_1 + y_1)z_2 - (\lambda x_2 + y_2)z_1 + (\lambda x_2 + y_2)z_2 \\
&= \lambda(3x_1z_1 - x_1z_2 - x_2z_1 + x_2z_2) + 3y_1z_1 - y_1z_2 - y_2z_1 + y_2z_2 \\
&= \lambda(\underline{x}, \underline{z}) + (\underline{y}, \underline{z})
\end{aligned}$$

(2) **Symmetry**

$$\begin{aligned}
(\underline{x}, \underline{y}) &= 3x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2 \\
&= 3y_1x_1 - y_2x_1 - y_1x_2 + y_2x_2 \\
&= (\underline{y}, \underline{x})
\end{aligned}$$

(3) **Positivity**

Solutions:
We have:

$$(\underline{x}, \underline{x}) = 3x_1^2 - 2x_1x_2 + x_2^2 = 2x_1^2 + (x_1^2 - 2x_1x_2 + x_2^2) = 2x_1^2 + (x_1 - x_2)^2 \geq 0$$

with equality whenever $x_1 = x_2 = 0$.
Self:
We directly complete the square:

$$\begin{aligned}
(\underline{x}, \underline{x}) &= 3x_1^2 - 2x_1x_2 + x_2^2 \\
&= 3\left(x_1^2 - \frac{2}{3}x_1x_2 + \frac{1}{3}x_2^2\right) \\
&= 3\left(\left(x_1 - \frac{1}{3}x_2\right)^2 - \frac{1}{9}x_2^2 + \frac{1}{3}x_2^2\right) \\
&= 3\left(x_1 - \frac{1}{3}x_2\right)^2 - \frac{1}{3}x_2^2 + x_2^2 \\
&= 3\left(x_1 - \frac{1}{3}x_2\right)^2 + \frac{2}{3}x_2^2 \\
&\geq 0
\end{aligned}$$

with equality whenever $x_1 = x_2 = 0$.

3. **Consider $\mathbb{R}^3$ equipped with the usual inner product. By giving a basis, describe explicitly the elements $(x, y, z)^T$ of the following subspaces:**

   (a) $\{(1, 2, 1)^T\}^\perp$

   A basis for this is a basis for elements $(x, y, z)^T$ which are orthogonal to $(1, 2, 1)^T$. Since we operate over an inner product space, such vector must satisfy:

$$x + 2y + z = 0 \implies z = -x - 2y$$

   so any element in $\{(1, 2, 1)^T\}^\perp$ has the form $(x, y, -x - 2y$. Thus, a basis is:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\}$$

   (b) $\{(1, 2, 1)^T, (0, 2, 0)\}^\perp$

   Again, any element $(x, y, z)^T \in \{(1, 2, 1)^T, (0, 2, 0)\}^\perp$ must satisfy:

$$x + 2y + z = 0 \qquad 2y = 0$$

   which means that:

$$x + z = 0 \implies x = -z$$

   Hence, a basis is:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

   (c) $\{(x, y, z)^T \mid x^2 + y^2 + z^2 = 1\}^\perp$

   Notice, we are seeking an orthogonal complement to all the points in a sphere. In particular, we seek a collection of vectors which will all be perpendicular to the standard basis vectors (since these are part of the unit sphere). However, in $\mathbb{R}^3$ the only such vector will be the 0 vector. Hence:

$$\{(x, y, z)^T \mid x^2 + y^2 + z^2 = 1\}^\perp = \{\underline{0}\}$$

4. **Cauchy's Inequality asserts:**

$$\sum_{i=1}^{n} x_i y_i \leq \sqrt{\sum_{i=1}^{n} x_i^2} \sqrt{\sum_{i=1}^{n} y_i^2}$$

   (a) **Prove this by induction on n. The case $n = 1$ is trivial, so begin with $n = 2$.**

   > *For this, I failed to notice that when asked to use induction, you should use induction **as much as possible**.*

   (1) **Base Case:** $n = 2$

   We want to show that:

$$x_1 y_1 + x_2 y_2 \leq \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}$$

Consider:

$$(x_1y_2 - x_2y_1)^2 \geq 0 \implies x_1^2y_2^2 + x_2^2y_1^2 \geq 2x_1y_1x_2y_2$$

Notice, assuming that $x_1y_1 + x_2y_2 \geq 0$ (we can assume this, as otherwise the inequality is trivially true), the Cauchy-Schwarz inequality is equivalent to saying that:

$$(x_1y_1 + x_2y_2)^2 \leq (x_1^2 + x_2^2)(y_1^2 + y_2^2)$$

If we expand both sides:

$$x_1^2y_1^2 + x_2^2y_2^2 + 2x_1y_1x_2y_2 \leq x_1^2y_1^2 + x_2^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2$$

which is equivalent to saying that:

$$x_1^2y_2^2 + x_2^2y_1^2 \geq 2x_1y_1x_2y_2$$

which we have shown above is true. Thus, the Cauchy-Schwarz inequality is true when $n = 2$.

(2) Inductive Hypothesis

Assume that the Cauchy-Schwarz is true $\forall n \in [1, k]$, such that:

$$\sum_{i=1}^{k} x_iy_i \leq \sqrt{\sum_{i=1}^{k} x_i^2}\sqrt{\sum_{i=1}^{k} y_i^2}$$

(3) Inductive Step: $n = k + 1$

Now, assume $n = k + 1$. Then:

$$\begin{aligned}\sum_{i=1}^{k+1} x_iy_i &= \sum_{i=1}^{k} x_iy_i + x_{k+1}y_{k+1} \\ &\leq \sqrt{\sum_{i=1}^{k} x_i^2}\sqrt{\sum_{i=1}^{k} y_i^2} + x_{k+1}y_{k+1}\end{aligned}$$

where we have used the inductive hypothesis with $n = k$.

But now notice, we have a sum of 2 products, so in particular the inductive hypothesis with $n = 2$ applies again, and so:

$$\begin{aligned}\sum_{i=1}^{k+1} x_iy_i &\leq \sqrt{\sum_{i=1}^{k} x_i^2}\sqrt{\sum_{i=1}^{k} y_i^2} + x_{k+1}y_{k+1} \\ &\leq \sqrt{\left(\sqrt{\sum_{i=1}^{k} x_i^2}\right)^2 + x_{k+1}^2}\sqrt{\left(\sqrt{\sum_{i=1}^{k} y_i^2}\right)^2 + y_{k+1}^2} \\ &\leq \sqrt{\sum_{i=1}^{k} x_i^2 + x_{k+1}^2}\sqrt{\sum_{i=1}^{k} y_i^2 + y_{k+1}^2} \\ &\leq \sqrt{\sum_{i=1}^{k+1} x_i^2}\sqrt{\sum_{i=1}^{k+1} y_i^2}\end{aligned}$$

as required.

(b) **Let t be a variable and consider the quadratic polynomial in t:**

$$\sum_{i=1}^{n}(x_i t + y_i)^2$$

Use the fact that this is always positive to prove the Cauchy-Schwarz inequality.

> *This is such a simple, elegant and sleek proof, I am sad I couldn't come up with this by myself.*

Notice:

$$(x_i t + y_i)^2 = t^2 x_i^2 + 2t x_i y_i + y_i^2$$

So:

$$\sum_{i=1}^{n}(x_i t + y_i)^2 = t^2\left(\sum_{i=1}^{n} x_i^2\right) + t\left(2\sum_{i=1}^{n} x_i y_i\right) + \left(\sum_{i=1}^{n} y_i^2\right)$$

Hence, if we let:

$$a = \sum_{i=1}^{n} x_i^2 \qquad b = 2\sum_{i=1}^{n} x_i y_i \qquad c = \sum_{i=1}^{n} y_i^2$$

We get:

$$\sum_{i=1}^{n}(x_i t + y_i)^2 = at^2 + bt + c \geq 0$$

Because of this, it follows that either $\sum_{i=1}^{n}(x_i t + y_i)^2$ has 0 as a repeated root, or it has 2 imaginary roots. In particular this means that by the discriminant:

$$b^2 - 4ac \leq 0$$

So:

$$4\left(\sum_{i=1}^{n} x_i y_i\right) - 4\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 \leq 0 \implies \sum_{i=1}^{n} x_i y_i \leq \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}$$

as required.

5. **Let $T : V \to V$ be an endomorphism of a finite-dimensional inner product space. Let T^* be the adjoint of T**

(a) **Show that T^*T is self-adjoint**

Let $\underline{v}, \underline{w} \in V$. Then, using the fact that T is adjoint to T^*:

$$\begin{aligned}(T^*T\underline{v}, \underline{w}) &= (T^*(T\underline{v}), \underline{w}) \\ &= (T\underline{v}, T\underline{w}) \\ &= (\underline{v}, T^*T\underline{w})\end{aligned}$$

so it follows that:

$$(T^*T)^* = T^*T$$

and so, T^*T is self-adjoint.

(b) **Show that if $T^*T = 0$, then $T = 0$**

> *This question exemplifies the act that if you are working over an inner product space,* ***the inner product space axioms are extremely useful****.*

Assume that $T^*T = 0$. Then, $\forall \underline{v} \in V$ we have:

$$(T^*T\underline{v}, \underline{v}) = (T\underline{v}, T\underline{v})$$

But since

$$T^*T = 0$$

, then $T^*T\underline{v} = \underline{0}$, so:

$$(T\underline{v}, T\underline{v}) = (\underline{0}, \underline{v}) = 0$$

(where we have used sesquilinearity, to "factor" the 0 out)
But by properties of inner product,

$$(T\underline{v}, T\underline{v}) = 0 \iff T\underline{v} = \underline{0}$$

But since this will be true $\forall \underline{v} \in V$, this can only be true if $T = 0$, as required.

6. **Begin by making sure that you can see how to deduce from Cauchy's inequality the fact that:**

$$\sum_{k=1}^{\infty} x_k^2 < \infty \qquad \sum_{k=1}^{\infty} y_k^2 < \infty$$

together imply that:

$$\sum_{k=1}^{\infty} |x_k y_k| < \infty$$

Now, prove this without using the Cauchy-Schwarz inequality:

(a) **Can you find a C such that $\forall x, y \in \mathbb{R}$:**

$$xy \leq C(x^2 + y^2)$$

(b) Now, apply this to $x = |x_k|$ and $y = |y_k|$ and sum over all k. Do you see a new inequality looking different from Cauchy's inequality?

7. **Calculate the inequality you just produced with normalised variables:**

$$\hat{x}_j = \frac{x_j}{(\sum_{k=1}^{\infty} x_k^2)^{\frac{1}{2}}}$$

$$\hat{y}_j = \frac{x_j}{(\sum_{k=1}^{\infty} y_k^2)^{\frac{1}{2}}}$$

Can you find a new interesting inequality out of this?

8. **Let's go back to Cauchy's inequality. Precisely when do we get an equality between the 2 sides?**

9. **Has the notation that you've used been a pain? Lots of sums, infinities, and so on? Can you find a concise say to write key statements down? What are the important properties of numbers in the notation that you are using? Have you just invented an axiomatic system?**